LOCALLY BICOMPLEX CONVEX MODULE AND THEIR APPLICATIONS

STANZIN KUNGA AND ADITI SHARMA

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU AND KASHMIR, INDIA.

stanzinkunga19@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU AND KASHMIR, INDIA.

aditi.sharmaro@gmail.com

ABSTRACT. Let $X$ be a locally $\mathbb{B}C$ convex module and $L(X)$ be the family of all continuous bicomplex linear operators on $X$. In this paper, we study some concepts of $D$-valued seminorms on locally $\mathbb{B}C$ convex module. Further, we study the bicomplex version of $C_{0}$ and $(C_{0}, 1)$ semigroup. The work of this paper is inspired by the work in [2] and [6].

Key words and phrases: Hyperbolic modules; Locally $\mathbb{B}C$-convex modules; $C_{0}$ and $(C_{0}, 1)$ semigroup.

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1. Introduction

Bicomplex numbers have been studied for quite a long time and lot of work has been done in this area. The work on bicomplex numbers probably begin with the work of Italian school of Segre, Spampinato and Scorza Dragoni. At present bicomplex analysis is an active area of research and many paper are being published in this direction. In [15], G. B. Price present the most comprehensive and detail on bicomplex analysis. The Recent book [1] and [13] give the most systematic developments of bicomplex analysis and bicomplex functional analysis. Topological vector spaces are one of the basic structures investigated in functional analysis. The bicomplex version of topological vector spaces was introduced in [10]. For the study of topological vector spaces we refer the reader to [3], [5], [14], [16] and [17] and references therein provide more information on these applications.

Some results on (C₀, 1) and C₀ semigroup were proposed in paper [2] for locally convex space. Now, we summarize some basic properties of Bicomplex numbers. For the details about bicomplex numbers we can refer to [1], [7], [13], [15], [18], [19].

The set of bicomplex numbers is denoted by BC and is defined as the commutative ring whose elements are of the form \( z = z_1 + jz_2 \), where \( z_1 = x_1 + iy_1 \in \mathbb{C}(i) \) and \( z_2 = x_2 + iy_2 \in \mathbb{C}(i) \) are complex numbers with imaginary units \( i \) and \( j \) respectively. Note that \( i^2 = j^2 = −1 \).

The set \( \mathbb{D} \) of hyperbolic numbers is defined as

\[
\mathbb{D} = \{ \alpha = \alpha_1 + k\alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R} \text{ with } k \neq \mathbb{R} \},
\]

where \( k \) is hyperbolic unit such that \( k^2 = 1 \).

Since bicomplex numbers are defined as the pair of two complex numbers connected through another imaginary unit, there are several notions of conjugations. Let \( Z = z_1 + jz_2 \in \mathbb{B}C \). Then the following three conjugations can be defined in \( \mathbb{B}C \):

(i) \( \overline{Z} = \overline{z_1} + j\overline{z_2} \), (ii) \( Z^\dagger = z_1 − jz_2 \), (iii) \( Z^* = \overline{z_1} − j\overline{z_2} \), where \( \overline{z_1}, \overline{z_2} \) denote the usual complex conjugates to \( z_1, z_2 \) in \( \mathbb{C}(i) \).

For bicomplex numbers we have three possible moduli which are defined as follows: (i) \( |\overline{Z}|_j^2 = Z\overline{Z} \), (ii) \( |Z|^i_2 = Z\overline{Z}^\dagger \), (iii) \( |Z|^k_2 = Z\overline{Z}^* \).

The hyperbolic numbers \( e \) and \( e^\dagger \) are defined as

\[
e = \frac{1 + k}{2} \quad \text{and} \quad e^\dagger = \frac{1 − k}{2}.
\]

Here, \( e \) and \( e^\dagger \) form a pair of idempotents such that their product is zero and sum is equal to 1. Therefore these are the zero divisors and we denote the set of zero divisors of \( \mathbb{B}C \) by \( \mathbb{NC} \) i.e.,

\[
\mathbb{NC} = \{ Z \mid Z \neq 0, \ z_1^2 + z_2^2 = 0 \}.
\]

Any bicomplex number \( Z = z_1 + jz_2 \) can be uniquely written as

\[
Z = \beta_1 e + \beta_2 e^\dagger,
\]

where \( \beta_1 = z_1 − iz_2 \) and \( \beta_2 = z_1 + iz_2 \in \mathbb{C}(i) \). Formulae (1.1) is called the idempotent decomposition representation of a bicomplex number \( Z \).

A hyperbolic number \( \alpha = \gamma_1 + k\gamma_2 \) can be written as

\[
\alpha = \alpha_1 e + \alpha_2 e^\dagger,
\]

where \( \alpha_1, \alpha_2 \in \mathbb{R} \).
where \( \alpha_1 = \gamma_1 + \gamma_2, \alpha_2 = \gamma_1 - \gamma_2 \) are real numbers, we say that \( \alpha \) is positive if \( \alpha_1 \geq 0 \) and \( \alpha_2 \geq 0 \). Thus, the set of positive hyperbolic numbers \( \mathbb{D}^+ \) is given by
\[
\mathbb{D}^+ = \{ \alpha = \alpha_1 e + \alpha_2 e^\dagger : \alpha_1 \geq 0, \alpha_2 \geq 0 \}.
\]

For \( P, Q \in \mathbb{D} \), (set of hyperbolic numbers) we define a relation \( \leq' \) on \( \mathbb{D} \) by \( P \leq' Q \iff Q - P \in \mathbb{D}^+ \). This relation is reflexive, anti-symmetric as well as transitive and hence defines a partial order on \( \mathbb{D} \), (cf. [1]).

A \( \mathbb{B} \mathbb{C} \)-module (or \( \mathbb{D} \)-module) \( B \) can be written as
\[
B = eB_1 + e^\dagger B_2,
\]
where \( B_1 = eB \) and \( B_2 = e^\dagger B \) are \( \mathbb{C}(i) \)-vector (or \( \mathbb{R} \)-vector) spaces. The bicomplex modules were introduced in [11], [21]. In this paper, we extend the results of paper [2].

2. \( \mathbb{D} \)-VALUED SEMINORM ON LOCALLY \( \mathbb{B} \mathbb{C} \) CONVEX MODULE

In this section, we study some properties of topological vector spaces with \( \mathbb{B} \mathbb{C} \) scalars. For details on topological vector spaces, we refer to [3], [9], [10].

**Definition 2.1.** [11] Let \( X \) be \( \mathbb{B} \mathbb{C} \)-module and \( \tau \) be a Hausdorff topology on \( X \) such that the operations
(i) \( + : X \times X \to X \) and
(ii) \( \cdot : \mathbb{B} \mathbb{C} \times X \to X \)
are continuous. Then the pair \( (X, \tau) \) is called a topological \( \mathbb{B} \mathbb{C} \)-module.

**Remark 2.1.** Let \( (X, \tau) \) be a topological \( \mathbb{B} \mathbb{C} \)-module. Write
\[
X = X_1 e + X_2 e^\dagger
\]
where \( X_1 = eX \) and \( X_2 = e^\dagger X \) are \( \mathbb{C}(i) \)-vector spaces. Then \( \tau_1 = \{ e_1 G : G \in \tau \} \) is a Hausdorff on \( X_1 \) for \( l = 1, 2 \).

**Example 2.1.** Every \( \mathbb{B} \mathbb{C} \)-module with \( \mathbb{D} \)-valued norm (or real-valued norm) is a topological \( \mathbb{B} \mathbb{C} \)-module.

Let \( X \) be a \( \mathbb{B} \mathbb{C} \)-module. Then \( X = X_1 e + X_2 e^\dagger \) and \( p : X \to \mathbb{D} \) be a \( \mathbb{D} \)-valued seminorm on \( X \).
\[
p(x) = (p_1 e + p_2 e^\dagger)(x_1 e + x_2 e^\dagger) = p_1(xe)e + p_2(xe^\dagger)e^\dagger,
\]
where \( p_1, p_2 : X \to \mathbb{R} \) are real seminorms on \( X_1 \) and \( X_2 \) respectively.

**Definition 2.2.** [11] A topological \( \mathbb{B} \mathbb{C} \)-module \( (X, \tau) \) is a locally bicomplex convex (or \( \mathbb{B} \mathbb{C} \)-convex) module if it has a neighbourhood base at 0 of \( \mathbb{B} \mathbb{C} \)-convex sets.

**Definition 2.3.** A family of \( \mathbb{D} \)-valued seminorm \( P \) on locally \( \mathbb{B} \mathbb{C} \) convex module \( X \) is said to be saturated if \( \max_{1 \leq \alpha \leq n} p_\alpha \in P \) where \( p_\alpha \in P \) (1 \( \leq \alpha \leq n \)).

Let \( X \) be a locally \( \mathbb{B} \mathbb{C} \) convex module and \( P = \{ p_\alpha : \alpha \in I \} \) be a saturated family of continuous \( \mathbb{D} \)-valued seminorms on \( X \). We set
\[
\overline{B}_{p_\alpha}(0, \epsilon) = \{ x \in X : p_\alpha(x) \leq' \epsilon \}, \ \epsilon >' 0, \alpha \in I \text{ and } p_\alpha \in P.
\]
Then, \( \cup_{p_\alpha} = \{ \overline{B}_{p_\alpha}(0, \epsilon) : \alpha \in I, \epsilon >' 0 \} \) forms the neighbourhood base at origin for the topology of \( X \).

Let \( \overline{B}_{p_\alpha}(0) = \{ x \in X : p_\alpha(x) \leq' 1 \} \) and \( B_{p_\alpha}(0) = \{ x \in X : p_\alpha(x) <' 1 \} \). Then \( \overline{B}_{p_\alpha}(0) \) and \( B_{p_\alpha}(0) \) are \( \mathbb{B} \mathbb{C} \)-convex, \( \mathbb{B} \mathbb{C} \)-balanced and \( \mathbb{B} \mathbb{C} \)-absorbing set[11].
Theorem 2.1. Suppose $X$ and $Y$ are locally $\mathbb{B}C$ convex-module, whose topologies defined by two families of $\mathbb{D}$-valued seminorm: $\mathcal{P}_1$ on $X$ and $\mathcal{P}_2$ on $Y$. For a $\mathbb{B}C$-linear operator $T : X \to Y$, the following are equivalent:

(i) $T$ is continuous.
(ii) $T$ is continuous at $0$.
(iii) For every $q \in \mathcal{P}_2$, $\exists p_1, p_2, \ldots, p_n \in \mathcal{P}_1$ such that $\sup \{q(Tx) : x \in B_{p_1}(0) \cap \cdots \cap B_{p_n}(0)\} < t' \infty$.
(iv) For every $q \in \mathcal{P}_2$, $\exists p_1, p_2, \ldots, p_n \in \mathcal{P}_1$ and $t_1, t_2, \ldots, t_n \geq 0$ such that $q(Tx) \leq t_1 p_1(x) + t_2 p_2(x) + \cdots + t_n p_n(x), \forall x \in X$.

Proof. (i) $\Leftrightarrow$ (ii) trivial.
(iii) $\Rightarrow$ (iv) Assume that $T$ is continuous at 0 and let $q$ is a $\mathbb{D}$-valued seminorm. Since $q$ is continuous, the set $B_q(0) = \{y \in Y : q(y) < t' 1\}$ is an (open) neighbourhood at 0 in $Y$. Since $T$ is continuous at 0, the preimage $\mathcal{N} = T^{-1}B_q(0)$, which is given by $\mathcal{N} = \{x \in X : q(Tx) < t' 1\}$ is a neighbourhood of 0 in $X$, then there exist $p_1, \ldots, p_n \in \mathcal{P}_1$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n > 0$ such that $B_{p_1}(0, \epsilon_1) \cap B_{p_2}(0, \epsilon_2) \cap \cdots \cap B_{p_n}(0, \epsilon_n) \subset \mathcal{N}$, where $B_p(0, \epsilon) = \{x \in X : p(x) < t' \epsilon\}$.

Let $\epsilon = \min \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Suppose $x \in B_p(0) \subset B_{p_n}(0)$.

Then $\exists \epsilon_1 \in B_p(0, \epsilon) \cap \cdots \cap B_{p_n}(0, \epsilon) \subset \mathcal{N} \Rightarrow \exists x \in \mathcal{N}$, for $x \in B_p(0)$. So, we get $q(T(\epsilon x)) < t' 1 \Rightarrow \sup \{q(Tx) : x \in B(p)\} \leq \epsilon^{-1} < \infty$.

i.e, $q(T) = \sup \{q(Tx) : x \in B(p)\} < \infty$.

(iv) $\Rightarrow$ (i) Assume condition (iii) i.e, $q(T) = \sup \{q(Tx) : x \in B_p(0)\} < \infty$.

Let $t = \sup \{q(Tx) : x \in B_{p_1}(0) \cap \cdots \cap B_{p_n}(0)\}$. To prove (iv), we show that $q(Tx) \leq t p_1(x) + \cdots + t p_2(x)$.

Take $x \in X$, and then for every $\epsilon > 0$, the vector $x_\epsilon = \frac{x}{p_1(x) + \cdots + p_n(x) + \epsilon}$ satisfies $p(x_\epsilon) = \frac{p(x)}{p_1(x) + \cdots + p_n(x) + \epsilon} < t' 1$.

So, $x_\epsilon \in B_{p_1}(0) \cap \cdots \cap B_{p_n}(0)$, by condition (ii), it follows that $q(Tx_\epsilon) \leq \sup q(Tx) = t$. Then, we get

$$q(Tx) = q(Tx_\epsilon)[p_1(x) + \cdots + p_n(x) + \epsilon] \leq t [p_1(x) + \cdots + p_n(x) + \epsilon].$$

Therefore, the inequality $q(Tx) \leq t q(x)$ holds, for $\epsilon > 0$.

Definition 2.4. Let $X$ be a locally $\mathbb{B}C$ convex module. Then we say that a $\mathbb{B}C$ linear operator $T : X \to X$ is bounded with respect to $\mathbb{P}$, if for any $p_n \in \mathbb{P}$ there exists $\lambda$ (depend on $\alpha, T$) with

\[0 \leq q(Tx_1 - T x) \leq t_1 p_1(x_1 - x) + \cdots + t_n p_n(x_1 - x)\]
\[ \lambda >' 0 \text{ such that} \]

\[ p_\alpha(Tx) \le' \lambda p_\alpha(x), \forall x \in X \text{ and } \alpha \in I. \]  

We know that \( \overline{B}_{p_\alpha}(0) = \{ x \in X : p_\alpha(x) \le' 1 \} \). Then (2.3) means \( T\overline{B}_{p_\alpha}(0) \subset \lambda \overline{B}_{p_\alpha}(0) \), \( \forall \alpha \in I. L_I(X) \) denote the set which consists of all bounded operators with respect to \( D \).

Since each \( \overline{B}_{p_\alpha}(0) \) is \( \mathbb{B}\mathbb{C} \)-balanced, so if for any \( \lambda \in \mathbb{B}\mathbb{C} \) with \( |\lambda| \le' 1 \), then \( \lambda \overline{B}_{p_\alpha}(0) \subset \overline{B}_{p_\alpha}(0) \). It follows that \( T\overline{B}_{p_\alpha}(0) \subset \lambda \overline{B}_{p_\alpha}(0) \subset \overline{B}_{p_\alpha}(0) \). Thus, \( T\overline{B}_{p_\alpha}(0) \subset \overline{B}_{p_\alpha}(0) \) and

\[ p_\alpha(Tx) \le' |\lambda| \lambda p_\alpha(x) \le' p_\alpha(x), \text{ for } |\lambda| \le' 1 \]

\[ p_\alpha(Tx) \le' 1. \]

**Remark 2.2.** We can say that the family of \( \mathbb{B}\mathbb{C} \)-linear Operator \( T : X \to X \) such that there exist a hyperbolic number \( \lambda \) (depending on \( \alpha \) and \( T \)) with \( T\overline{B}_{p_\alpha}(0) \subset \lambda \overline{B}_{p_\alpha}(0), \forall \alpha \in I \) is identical with \( L_I(X) \).

Furthermore, with addition defined pointwise and multiplication by composition, \( L_I(X) \) becomes a \( \mathbb{B}\mathbb{C} \)-algebra. For each \( p_\alpha \in \{ p_\alpha : \alpha \in I \} \), the mapping \( p_\alpha : L_I(X) \to \mathbb{D} \) defined as

\[ p_\alpha(T) = \inf_{\mathbb{D}} \{ \lambda : p_\alpha(Tx) \le' \lambda p_\alpha(x), \forall x \in X \} \]

\[ = \sup_{x \in X} \{ p_\alpha(Tx) : p_\alpha(x) \le' 1 \} \]

is also a \( \mathbb{D} \) valued seminorm.

Thus, \( p_\alpha(T) = \inf_{\mathbb{D}} \{ \lambda : p_\alpha(Tx) \le' \lambda p_\alpha(x) \} \).

Setting \( T = eT_1 + e^tT_2 \) and \( \lambda = e\lambda_1 + e^t\lambda_2 \) with \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \).

\[ p_\alpha(T) = p_\alpha(eT_1 + e^tT_2) \]

\[ = \inf_{\mathbb{D}} \{ \lambda : p_\alpha(T) \le' \lambda p_\alpha(x) \} \]

\[ = \inf_{\mathbb{D}} \{ e\lambda_1 + e^t\lambda_2 \} \]

such that

\[ p_\alpha(eT_1x_1 + e^tT_2x_2) \le' (e\lambda_1 + e^t\lambda_2)(e\alpha_1(x_1) + e^t\alpha_1(x_2)) \]

\[ \Rightarrow P_\alpha(T) = e\inf\lambda_1 + e^t\inf\lambda_2 \]

\[ such \ that \ p_{\alpha_1}(T_1x_1) \le \lambda_1p_{\alpha_1}(x_1) \ and \ p_{\alpha_2}(T_2x_2) \le \lambda_2p_{\alpha_2}(x_2). \]

Thus, the hyperbolic semi-norm of \( T \) can also be defined as

\[ p_\alpha(T) = e.p_{\alpha_1}(T_1) + e^t.p_{\alpha_2}(T_2) \text{ where } p_{\alpha,l}(T_l) = \inf \{ \lambda_l : p_{\alpha,l}(Tx) \le \lambda_l p_{\alpha,l}(x) \}, l = 1, 2. \]

**Remark 2.3.** Let \( X \) be a locally \( \mathbb{B}\mathbb{C} \) convex module. A bicomplex linear Operators \( T \) and \( S \) are belongs to \( L_I(X) \). Then there exist \( \mathbb{C}(i) \) linear (or \( \mathbb{R} \)-linear) Operator \( T_i \) and \( S_i \) on \( L_I(X), l = 1, 2 \) such that \( T = T_1e + T_2e^t \) and \( S = S_1e + S_2e^t \), \( TS = T_1S_1e + T_2S_2e^t \) and
\[ p_\alpha = p_{\alpha,1}e + p_{\alpha,2}e^\dagger. \]

\[
\begin{align*}
p_\alpha(TS) &= p_{\alpha,1}(T_1S_1)e + p_{\alpha,2}(T_2S_2)e^\dagger \\
&\leq p_{\alpha,1}(T_1)e p_{\alpha,1}(S_1)e + p_{\alpha,2}(T_2)e^\dagger p_{\alpha,2}(S_2)e^\dagger \\
&= [(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(T_1e + T_2e^\dagger)][(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(S_1e + S_2e^\dagger)] \\
&= p_\alpha(T)p_\alpha(S) \\
p_\alpha(Tx) &= p_{\alpha,1}(T_1x_1)e + p_{\alpha,2}(T_2x_2)e^\dagger \\
&\leq p_{\alpha,1}(T_1)(x_1)e + p_{\alpha,2}(T_2)e^\dagger p_{\alpha,2}(x_2)e^\dagger \\
&= [(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(T_1e + T_2e^\dagger)][(p_{\alpha,1}e + p_{\alpha,2}e^\dagger)(x_1e + x_2e^\dagger)] \\
&= p_\alpha(T)p_\alpha(x)
\end{align*}
\]

The family \( \{p_\alpha : \alpha \in I\} \) of \( \mathbb{D} \)-valued seminorms on \( L_I(X) \) defined the topology on \( L_I(X) \). Under this topology, \( L_I(X) \) becomes Hausdorff locally multiplicatively \( \mathbb{B}\mathbb{C} \)-convex topological algebra. Then \( L_I(X) \) is complete if locally \( \mathbb{B}\mathbb{C} \) convex module \( X \) is complete iff \( X_1 \) and \( X_2 \) are complete locally convex spaces.

**Definition 2.5.** A \( \mathbb{B}\mathbb{C} \)-algebra \( L_I(X) \) is said to be locally multiplicatively \( \mathbb{B}\mathbb{C} \)-convex if it has a neighbourhood base \( U_{p_\alpha} \) at 0 such that each \( B_{p_\alpha}(0) \subset \mathbb{B}_p(0) \) is \( \mathbb{B}\mathbb{C} \)-convex and \( \mathbb{B}\mathbb{C} \)-balanced (i.e, \( \lambda B_{p_\alpha}(0) \subset B_{p_\alpha}(0) \) for \( |\lambda| \leq 1 \) and satisfies \( B_{p_\alpha}(0) \subset \mathbb{B}_{p_\alpha}(0) \).

**Theorem 2.2.** Let \( X \) be a \( \mathbb{B}\mathbb{C} \)-module and \( p_\alpha \) be a continuous \( \mathbb{D} \) valued seminorm on \( X \). Then \( N_\alpha = p_\alpha^{-1}(0) = \{x \in X : p_\alpha(x) = 0\} \) is a submodule of \( X \).

**Proof.** If \( x, y \in X \) and \( a, b \in \mathbb{B} \). Then

\[ 0 \leq p_\alpha(ax + by) \leq |a|_k p_\alpha(x) + |b|_k p_\alpha(y) = 0. \]

\[ \Rightarrow ax + by \in N_\alpha. \]

Hence, \( N_\alpha = p_\alpha^{-1}(0) \) is a submodule of \( X \).

**Definition 2.6.** [12] Let \( X \) be a \( \mathbb{B}\mathbb{C} \)-module and \( N_\alpha = p_\alpha^{-1}(0) \) be a submodule of \( X \). We write

\[ X = X_1e + X_2e^\dagger, \]

where \( X_1 \) and \( X_2 \) are complex linear spaces and \( N_\alpha = N_{\alpha,1}e + N_{\alpha,2}e^\dagger. \) Then \( N_{\alpha,1} \) and \( N_{\alpha,2} \) are complex linear subspace of \( X_1 \) and \( X_2 \) respectively, so that \( \frac{X_l}{N_{\alpha,l}}, l = 1, 2 \) are quotient space over the Complex field.

Consider the set \( \frac{X}{N_\alpha} = \{x + N_\alpha : x \in X\}, \) where \( x_\alpha = x + N_\alpha \) is the coset of \( x \) in \( \frac{X}{N_\alpha}. \) Let \( x, y \in X, a \in \mathbb{B} \).

\[(i) \ (N_\alpha + x) + (N_\alpha + y) = (N_\alpha + x + y), \]
\[(ii) \ a(N_\alpha + x) = N_\alpha + ax. \]

With the operations defined above \( \frac{X}{N_\alpha} \) form a module over \( \mathbb{B} \mathbb{C} \) and is called \( \mathbb{B}\mathbb{C} \) quotient module.

Further, \( N_\alpha + x = (N_{\alpha,1} + x)e + (N_{\alpha,2} + x)e^\dagger \) for any \( x \in X \), so one can write that

\[
\frac{X}{N_\alpha} = \frac{X_1e + X_2e^\dagger}{N_{\alpha,1}e + N_{\alpha,2}e^\dagger} = \frac{X_1}{N_{\alpha,1}e} + \frac{X_2}{N_{\alpha,2}e^\dagger}.
\]
Let $X$ be a locally $\mathbb{B}\mathbb{C}$ convex module and $p_\alpha$ is a $\mathbb{D}$-valued seminorm on $X$. If $N_\alpha$ is the kernel of the $\mathbb{D}$-valued seminorm, then $\frac{X}{N_\alpha}$ is $\mathbb{B}\mathbb{C}$-normed linear module and the $\mathbb{D}$-valued norm on $\frac{X}{N_\alpha}$ is defined as

$$\|x_\alpha\|_\mathbb{D} = \|x + N_\alpha\|_\mathbb{D} = p_\alpha(x) \text{ for each } x \in X,$$

where $x_\alpha$ is the coset of $x$ in the $\mathbb{B}\mathbb{C}$-quotient module $\frac{X}{N_\alpha}$.

**Remark 2.4.**

$$\|x_\alpha\|_\mathbb{D} \begin{array}{c}= \|x + N_\alpha\|_\mathbb{D} \\= \|x + N_{\alpha,1}\|_1 \|e\| + \|x + N_{\alpha,2}\|_2 e^\dagger \\= p_{\alpha,1}(x)e + p_{\alpha,2}(x)e^\dagger \\= p_\alpha(x). \end{array}$$

Thus, $\|\cdot\|_\mathbb{D}$ is a $\mathbb{D}$-valued norm defined on $\mathbb{B}\mathbb{C}$-quotient module $\frac{X}{N_\alpha}$.

**Theorem 2.3.** $\|\cdot\|_\mathbb{D}$ is a $\mathbb{D}$-valued norm defined on $\mathbb{B}\mathbb{C}$-quotient module $\frac{X}{N_\alpha} \Leftrightarrow \|\cdot\|_1$ and $\|\cdot\|_2$ are real-valued norm on the quotient spaces $\frac{X}{N_{\alpha,1}}$ and $\frac{X}{N_{\alpha,2}}$.

**Proof.** We have $x = x_1e + x_2e^\dagger$ and $N_\alpha = N_{\alpha,1}e + N_{\alpha,2}e^\dagger$.

(i) 

$$\|x + N_\alpha\|_\mathbb{D} = 0 \Rightarrow x + N_\alpha = 0$$

$\Leftrightarrow \|x_1 + N_{\alpha,1}\|_1 e + \|x_2 + N_{\alpha,2}\|_2 e^\dagger = 0 \Rightarrow (x_1 + N_{\alpha,1})e + (x_2 + N_{\alpha,2})e^\dagger = 0$ and $\|x_2 + N_{\alpha,2}\|_2 = 0 \Rightarrow (x_2 + N_{\alpha,2}) = 0$.

(ii) 

$$\|ax_\alpha\|_\mathbb{D} = |a| \|x_\alpha\|_\mathbb{D}$$

$\Leftrightarrow \|a_1(x_1 + N_{\alpha,1})\|_1 e + \|a_2(x_2 + N_{\alpha,2})\|_2 e^\dagger = |a_1| \|x_1 + N_{\alpha,1}\|_1 e + |a_2| \|x_2 + N_{\alpha,2}\|_2 e^\dagger$ and $\|a_2(x_2 + N_{\alpha,2})\|_2 = |a_2| \|x_2 + N_{\alpha,2}\|_2$

(iii) 

$$\|(x + N_\alpha) + (y + N_\alpha)\|_\mathbb{D} \leq \|x + N_\alpha\|_\mathbb{D} + \|y + N_\alpha\|_\mathbb{D}$$

$\Leftrightarrow \|((x_1 + N_{\alpha,1}) + (y_1 + N_{\alpha,1})) e + \|(x_2 + N_{\alpha,2}) + (y_2 + N_{\alpha,2})\|_2 e^\dagger$ 

$\leq \|x_1 + N_{\alpha,1}\|_1 e + \|y_1 + N_{\alpha,1}\|_1 e + \|x_2 + N_{\alpha,2}\|_2 e^\dagger + \|y_2 + N_{\alpha,2}\|_2 e^\dagger$ and $\|(x_2 + N_{\alpha,2}) + (y_2 + N_{\alpha,2})\|_2 e^\dagger \leq \|x_2 + N_{\alpha,2}\|_2 + \|y_2 + N_{\alpha,2}\|_2$

The set $L(X_\alpha)$ denote the set of all $\mathbb{B}\mathbb{C}$-linear Operator i.e., $L(X_\alpha) = \{T/T : X_\alpha \rightarrow X_\alpha\}$. Let $X_\alpha$ is a $\mathbb{B}\mathbb{C}$-normed module. We can write $X_\alpha = X_{\alpha,1}e + X_{\alpha,2}e^\dagger$, where $X_{\alpha,1} = eX_\alpha$ and $X_{\alpha,2} = e^\dagger X_\alpha$ are normed linear space.

**Definition 2.7.** Let $X_\alpha$ be a $\mathbb{B}\mathbb{C}$-module with $\mathbb{D}$-valued norm. Let $T_\alpha : X_\alpha \rightarrow X_\alpha$ be a map such that

$$T_\alpha(ax_\alpha + by_\alpha) = aT_\alpha(x_\alpha) + bT_\alpha(y_\alpha), \ \forall x_\alpha, y_\alpha \in X_\alpha, \forall a, b \in \mathbb{B}\mathbb{C}.$$
Then we say that $T_{\alpha}$ is a $\mathbb{B}\mathbb{C}$-linear Operator on $X_\alpha$.
The idempotent decomposition of the operator is given as (see[4])

$$T_{\alpha} = T_{\alpha,1}e + T_{\alpha,2}e^\dagger,$$

where $T_{\alpha,1} : eX_\alpha \to eX_\alpha$ and $T_{\alpha,2} : e^\dagger X_\alpha \to e^\dagger X_\alpha$ are the linear operators.

Let $x, y \in X$ and $a, b \in \mathbb{B}\mathbb{C}$. Then $\pi_\alpha$ is a mapping from locally $\mathbb{B}\mathbb{C}$ convex module $X$ onto $X_{N_\alpha} = X_\alpha$ as

$$\pi_\alpha(x) = x_\alpha = x + N_\alpha.$$

Thus, $\pi_\alpha$ is $\mathbb{B}\mathbb{C}$ homomorphism on $X_{N_\alpha}$.

**Remark 2.5.** For any $X$, $\pi_\alpha(x) = x_\alpha = x + N_\alpha$.

$$\Rightarrow x + N_\alpha = (x_1 + N_{\alpha,1})e + (x_2 + N_{\alpha,2})e^\dagger = \pi_{\alpha,1}(x)e + \pi_{\alpha,2}(x)e^\dagger.$$

So, we can conclude that

$$\pi_\alpha(x) = \pi_{\alpha,1}(x)e + \pi_{\alpha,2}(x)e^\dagger,$$

where $\pi_{\alpha,l}$ are natural Homomorphism of $X_l$ onto $X_{\alpha,l}$ respt., $l = 1, 2$.

**Definition 2.8.** Let $X_\alpha$ be the $\mathbb{B}\mathbb{C}$-normed module. Define a mapping $\pi_\alpha : X \to X_\alpha$, for each $\alpha$ as $\pi_\alpha(x) = x_\alpha$ for each $x \in X$. Clearly, $\pi_\alpha$ is a continuous and is called a $\mathbb{B}\mathbb{C}$ natural homomorphism of $X$ onto $X_\alpha$.

If $T \in L_I(X)$ implies $T(p_{\alpha}^{-1}(0)) \subset p_{\alpha}^{-1}(0)$.

**Remark 2.6.** Let $X_\alpha$ be a $\mathbb{B}\mathbb{C}$ normed module and $T_{\alpha} = T_{\alpha,1}e + T_{\alpha,2}e^\dagger$ be the operator on $X_\alpha$ defined by $T_{\alpha}x_\alpha = (Tx)_\alpha$, $x_\alpha \in X_\alpha$.

Now

$$T_{\alpha}x_\alpha = (Tx)_\alpha,$$

$$\Rightarrow T_{\alpha,1}x_{\alpha,1}e + T_{\alpha,2}x_{\alpha,2}e^\dagger = (T_1x_1e + T_2x_2e^\dagger)_\alpha,$$

$$\Rightarrow T_{\alpha,1}x_{\alpha,1}e + T_{\alpha,2}x_{\alpha,2}e^\dagger = (T_1x_1)_\alpha e + (T_2x_2)_\alpha e^\dagger,$$

$$\Rightarrow T_{\alpha,1}x_{\alpha,1} = (T_1x_1)_\alpha and T_{\alpha,2}x_{\alpha,2} = (T_2x_2)_\alpha$$

where $T_{\alpha,l}$ and $T_l$ are the operator on $X_{\alpha,l}$ and $X_l$ respt., $l = 1, 2$. 
Proposition 2.4. Let $T \in L_1(X)$, where $X$ is a locally $\mathbb{BC}$-convex module. Then the operator $T_\alpha : X_\alpha \to X_\alpha$, $\alpha \in I$ defined by $T_\alpha x_\alpha = (Tx)_\alpha$, $x_\alpha \in X_\alpha$ is in $L(X_\alpha)$.

Proof. Let $X_\alpha$ be the $\mathbb{BC}$ normed module and $x_\alpha \in X_\alpha$

\[
x_\alpha = y_\alpha
\]

\[
x + N_\alpha = y + N_\alpha \Rightarrow x - y \in N_\alpha
\]
\[
\Leftrightarrow x_1 + N_{\alpha,1} = y_1 + N_{\alpha,1} \quad \text{and} \quad x_2 + N_{\alpha,2} = y_2 + N_{\alpha,2}
\]
\[
\Rightarrow x_1 - y_1 \in N_{\alpha,1} \quad \text{and} \quad x_2 - y_2 \in N_{\alpha,2}.
\]

Then,
\[
T_\alpha (x - y)_\alpha = T_{\alpha,1}(x_1 - y_1)_\alpha e + T_{\alpha,2}(x_2 - y_2)_\alpha e^\dagger
\]
\[
= (T_1(x_1 - y_1)e + T_2(x_2 - y_2)e^\dagger)_\alpha
\]
\[
= (T(x - y))_\alpha = 0
\]

Also,
\[
T_\alpha x_\alpha - T_\alpha y_\alpha = (T_{\alpha,1}x_{\alpha,1} - T_{\alpha,1}y_{\alpha,1}) e + (T_{\alpha,2}x_{\alpha,2} - T_{\alpha,2}y_{\alpha,2}) e^\dagger
\]
\[
= (T_1x_1 - T_1y_1)_\alpha e + (T_2x_2 - T_2y_2)_\alpha e^\dagger
\]
\[
= (T_1(x_1 - y_1))_\alpha e + (T_2(x_2 - y_2))_\alpha e^\dagger
\]
\[
= T_\alpha (x - y)_\alpha = 0.
\]

Thus, $T$ is well defined.

Now, given that $T = T_1e + T_2e^\dagger \in L_1(X)$, so
\[
p_\alpha(Tx) = p_{\alpha,1}(T_1x_1)e + p_{\alpha,2}(T_2x_2)e^\dagger
\]
\[
\leq' \ p_{\alpha,1}(T_1)p_{\alpha,1}(x_1)e + p_{\alpha,2}(T_2)p_{\alpha,2}(x_2)e^\dagger
\]
\[
= p_\alpha(T)p_\alpha(x).
\]

\[
\|T_\alpha x_\alpha\|_D = \|T(x)_\alpha\|_D
\]
\[
= p_\alpha(Tx)
\]
\[
\leq' \ p_\alpha(T)p_\alpha(x)
\]
\[
= p_\alpha(T)\|x_\alpha\|, \forall x_\alpha \in X_\alpha.
\]

Therefore, $T_\alpha$ is bounded on $X_\alpha$.

Next, we will check the linearity of $T_\alpha$. Given that $T \in L_1(X)$ and $a, b \in \mathbb{D}$,
\[
T_\alpha (ax_\alpha + by_\alpha) = (T(ax + by))_\alpha
\]
\[
= (aT(x) + bT(y))_\alpha
\]
\[
= aT_\alpha x_\alpha + bT_\alpha y_\alpha.
\]

Definition 2.9. Let $X_\alpha$ be a $\mathbb{BC}$-normed module and we defined a $\mathbb{BC}$ linear operator $T_\alpha : X_\alpha \to X_\alpha$ by

\[
T_\alpha x_\alpha = (Tx)_\alpha
\]

and
\[
\|T_\alpha x_\alpha\| = \|(Tx)_\alpha\| = p_\alpha(Tx) \leq' \ p_\alpha(T)p_\alpha(x) \leq' \ p_\alpha(x) = \|x_\alpha\|
\]
\(\Rightarrow T_\alpha\) is bounded.

**Remark 2.7.** Thus \(T_\alpha\) is bounded \(\mathbb{B}C\) linear operator.

Further, we can write

\[ T_\alpha = T_{\alpha,1}e + T_{\alpha,2}e^\dagger \]

where \(T_{\alpha,l}\) are bounded linear operator on \(X_{\alpha,l}\), \(l = 1, 2\).

**Definition 2.10.** Let \(\overline{X}_\alpha\) be the completion of \(\mathbb{B}C\) normed module \(X_\alpha\) such that \(\overline{X}_\alpha\) form a \(\mathbb{B}C\) Banach module.

Then \(T_\alpha : \overline{X}_\alpha \to \overline{X}_\alpha\) is a bounded \(\mathbb{B}C\) linear operator.

For \(T \in L_I(X)\), \(T_\alpha\) is the extended form of \(T_\alpha\) such that

\[ \|T_\alpha\|_D = \|T_\alpha\|_D = \sup\{\|T_\alpha x_\alpha\|_D : \|x_\alpha\|_D \leq 1\} = p_\alpha(T) \]

The operator norm on \(T_\alpha\) is

\[ \|T_\alpha\| = \sup\{\|T_\alpha x_\alpha\|_D : \|x_\alpha\|_D \leq 1\} \]

Note that this norm is a hyperbolic norm on \(T_\alpha\).

Hence, we can write

\[ \|T_\alpha\| = \|T_\alpha,1\|_1 e + \|T_\alpha,2\|_2 e^\dagger \]

where \(\|\cdot\|_1\) and \(\|\cdot\|_2\) define the usual norm on \(T_\alpha,1\) and \(T_\alpha,2\) resp.

**Definition 2.11.** (Directed Set) Let \(I\) be a partially ordered set with the order relation \(\geq\), then \(I\) is called as a directed set if for any two elements \(a, b \in I\), \(\exists c \in I\) such that \(c \geq a\) and \(c \geq b\).

Let \((I, \leq')\) be a directed set. Then for \(\beta \geq \alpha\), \(\mathbb{B}C\) operator \(\pi_{\alpha\beta} : X_\beta \to X_\alpha\) defined by \(\pi_{\alpha\beta}(x_\beta) = x_\alpha\) is a continuous \(\mathbb{B}C\) normed module \(X_\beta\) onto \(X_\alpha\). This \(\mathbb{B}C\) operator can be extended to a continuous \(\mathbb{B}C\) Homomorphism \(\pi_{\alpha\beta}\) from the completion \(\overline{X}_\beta\) into \(\overline{X}_\alpha\).

A projective system of \(\mathbb{B}C\) Banach module is a pair \((\overline{X}_\alpha, \pi_{\alpha\beta})\) subject to the following properties:

(i) \((I, \leq')\) is a directed set.

(ii) \((\overline{X}_\alpha)_{\alpha \in I}\) is a family of \(\mathbb{B}C\) Banach module.

(iii) \(\{\pi_{\alpha\beta} : \overline{X}_\beta \to \overline{X}_\alpha, \alpha, \beta \in I, \alpha \leq' \beta\}\) is a family of continuous \(\mathbb{B}C\) Homomorphism such that \(\pi_{\alpha\alpha}\) is the identity operator on \(\overline{X}_\alpha\) \(\forall \alpha \in I\).

(iv) \(\pi_{\alpha\gamma} = \pi_{\alpha\beta} \circ \pi_{\beta\gamma} \forall \alpha, \beta, \gamma \in I\) such that \(\alpha \leq' \beta \leq' \gamma\), and its projective limit is denoted by \(X\) i.e,

\[ X = \lim \left\{ \overline{X}_\alpha \right\} \]

where \(X\) is a complete locally bicomplex convex module.

The projective limit of the projective system is defined to be the submodule of the cartesian product \(\prod \overline{X}_\alpha\) consisting of elements which satisfy \(\pi_{\alpha\beta}(\overline{x}_\beta) = (\overline{x}_\alpha)\) for \(\beta > \alpha\).

**Remark 2.8.** The operator \(\pi_{\alpha\beta}\) can be written as

\[ \pi_{\alpha\beta} = \pi_{\alpha\beta,1}e + \pi_{\alpha\beta,2}e^\dagger, \]

where

\[ \pi_{\alpha\beta,l} : \pi_{\beta,l} \to \pi_{\alpha,l}, l = 1, 2 \]

is an operator.

Let us denote \(\overline{X}_\alpha\) with \(Z_\alpha\).
Definition 2.12. Let \( T_\alpha : D(T_\alpha) \subset Z_\alpha \to Z_\alpha \) be a \( \mathbb{B}\mathbb{C} \)-linear operator from \( D(T_\alpha) \subset Z_\alpha \) into \( Z_\alpha \). Then \( \{T_\alpha : \alpha \in I\} \) is called (saturated) projective family of operators if

\[
\begin{align*}
& \iff T_\alpha(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_\beta x_\beta) \\
& \iff \alpha, \beta \in \alpha, T_{\alpha,1}(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_{\beta,1}x_\beta) \text{ and } T_{\alpha,2}(\pi_{\alpha\beta}x_\beta) = \pi_{\alpha\beta}(T_{\beta,2}x_\beta), \text{ for } x_\beta \in D(T_\beta) \text{ and } \beta \geq \alpha.
\end{align*}
\]

where \( T_{\alpha,1}, T_{\alpha,2} \) are linear operator on \( Z_{\alpha,1}, Z_{\alpha,2} \) respt. and \( T_{\beta,1}, T_{\beta,2} \) are linear operator on \( Z_{\beta,1}, Z_{\beta,2} \) respt.

Definition 2.13. A \( \mathbb{B}\mathbb{C} \) linear operator \( T \) on the projective limit \( D(T) \) of \( D(T_\alpha) : \alpha \in I \) can be defined by \( \pi_\alpha(Tx) = T_\alpha(\pi_\alpha x) \) for \( \alpha \in I \) and the operator \( T \) is called the projective limit of the family of operator \( \{T_\alpha : \alpha \in I\} \).

Let \( T_\alpha \in L(X_\alpha) \) for each \( \alpha \), then \( T \in L_1(X) \). Moreover, the family \( \{T_\alpha : \alpha \in I\} \) associated \( T \in L(X_\alpha) \) above is projective and its limit is \( T \).

Remark 2.9.

\[
\pi_\alpha(Tx) = T_\alpha(\pi_\alpha x) \iff \pi_{\alpha,1}(T_1x_1) = T_{\alpha,1}(\pi_{\alpha,1}x_1) \text{ and } \pi_{\alpha,2}(T_2x_2) = T_{\alpha,2}(\pi_{\alpha,2}x_2)
\]

3. Some Basic Properties of \((C_0, 1)\) Semigroup

The result in this section are generalization of results of [2].

Definition 3.1. Let \( X \) be a locally \( \mathbb{B}\mathbb{C} \) convex module and a family \( \{T(t), t \in \mathbb{D}^+\} \) of bounded \( \mathbb{B}\mathbb{C} \) linear operator in \( X \) is called a \( C_0 \)-semigroup if

(i) \( T(t + s)x = T(t)(T(s)x) \forall \ t, s \in \mathbb{D}^+ \text{ and } x \in X \)

(ii) \( T(0)x = x \forall \ x \in X \).

(iii) \( T(t)x \to x \text{ as } t \to 0 \forall \ x \in X \)

A \( C_0 \) semigroup \( T(t) \to T(t) \) is said to be a \((C_0, 1)\) semigroup if \( T(t) \in L_1(X), \forall \ t \geq 0 \) and, for each \( \alpha \) and \( \delta > 0 \), there exist a positive hyperbolic number \( \lambda = \lambda(\alpha, T(t) : t \in [0, \delta]) \) such that

\[
T(t)\mathcal{B}_{p_\alpha}(0) \subset \lambda\mathcal{B}_{p_\alpha}(0)
\]

or equivalently \( p_\alpha(Tx) \leq \lambda p_\alpha(x) \forall \ 0 \leq t \leq \delta \), where \( \mathcal{B}_{p_\alpha}(0) = \{x \in X : p_\alpha(x) \leq 1\} \). It is also called a \( L_1(X)\)-operator semigroup of class \((C_0, 1)\)

Theorem 3.1. If \( \{T(t), t \in \mathbb{D}^+\} \) is a \((C_0, 1)\)-semi-group in \( X \). Then the family \( \{\overline{T}_\alpha(t), t \in \mathbb{D}^+\} \) is a \( C_0 \) semi-group in the \( \mathbb{B}\mathbb{C}\)-Banach module \( \overline{X}_\alpha \), for each \( \alpha \).

Proof. Here, \( \{T(t), t \in \mathbb{D}^+\} \) is a \((C_0, 1)\)-semi-group in \( X \).

Let \( \overline{T}_{\alpha,1}(t_1) = e^{t_1}T_\alpha(t) \) and \( \overline{T}_{\alpha,2}(t_2) = e^{t_1}T_\alpha(t) \). Then using [2] thm 2.3, P-168, we see that for each \( \alpha \), \( \overline{T}_{\alpha,1}(t_1) \) and \( \overline{T}_{\alpha,2}(t_2) \) are \( C_0 \) semi-group in the Banach space \( \overline{X}_{\alpha,1} \) and \( \overline{X}_{\alpha,2} \) respt. Thus,

\[
\overline{T}_\alpha(t) = e^{t_1}\overline{T}_\alpha(t) + e^{t_2}\overline{T}_\alpha(t)
\]

is \( C_0 \) semi-group in \( \mathbb{B}\mathbb{C}\)-Banach module \( \overline{X}_\alpha \).

Let us denote \( \overline{X}_\alpha \) with \( Z_\alpha \) and \( X \) with \( Z \). Now, Let \( Z_\alpha \) be a \( \mathbb{B}\mathbb{C}\)-Banach module and \( Z \) be complete locally \( \mathbb{B}\mathbb{C} \) convex module. A family \( \{T_\alpha(t) \in L(Z_\alpha) : \alpha \in I, t \geq 0\} \) is called a projective family of \( C_0 \) semigroups on \( \mathbb{B}\mathbb{C}\)-Banach module iff

(i) for each \( t \geq 0 \), \( \{T_\alpha(t) : \alpha \in I\} \) is a projective family.

(ii) for each \( \alpha \), \( \{T_\alpha(t) : t \geq 0\} \) is a \( C_0 \) semi-group on the \( \mathbb{B}\mathbb{C}\)-Banach module \( Z_\alpha \).

The limit of such a family is denoted by \( \{T(t) : t \geq 0\} \).
Theorem 3.2. Let \( \Gamma = \{ T_\alpha(t) : \alpha \in I, t \geq 0 \} \) be a projective family of \( C_0 \)-semigroup on \( \mathbb{B} \mathbb{C} \)-Banach module \( Z_\alpha \). Then the following statements are equivalent:

(i) \( \{ T(t) : t \geq 0 \} \) is a \( (C_0, 1) \)-semigroup in \( Z \iff \{ T_1(t) : t \geq 0 \} \) and \( \{ T_2(t) : t \geq 0 \} \) are \( (C_0, 1) \)-semigroup on \( Z_1 \) and \( Z_2 \) resp.

(ii) \( \{ T(t) : t \geq 0 \} \) be the limit of \( \Gamma \iff \{ T_1(t) : t \geq 0 \} \) and \( \{ T_2(t) : t \geq 0 \} \) are the limit of \( \Gamma_1 \) and \( \Gamma_2 \) resp.,

where \( \Gamma_1 = \{ T_{\alpha,1}(t) : \alpha \in I, t \geq 0 \} \) and \( \Gamma_2 = \{ T_{\alpha,2}(t) : \alpha \in I, t \geq 0 \} \).

Proof. : By using definition \[2,7\] a \( \mathbb{B} \mathbb{C} \) linear operator \( T(t) \) can be written as \( T(t) = T_1(t_1)e + T_2(t_2)e^1 \), where \( T_1(t_1) \) and \( T_2(t_2) \) are linear operator on \( Z_1 \) and \( Z_2 \) resp.

Let \( x \in Z \) with \( x = x_1e + x_2e^1 \) where \( x_1 \in Z_1 \), \( x_2 \in Z_2 \) and \( s \geq 0 \) with \( t = t_1e + t_2e^1 \) and \( s = s_1e + s_2e^1 \) where \( t_1, t_2, s_1, s_2 \geq 0 \).

Then (i)

\[ T(t + s)x = T(t)(T(s)x), \forall x \in Z \]

\[ \iff T_1(t + s)x + T_2(t + s)x e^\dagger = T_1(t)(T_1(s)x)e + T_2(t)(T_2(s)x)e^\dagger \]

\[ \iff T_1(t + s)x = T_1(t)(T_1(s)x) \quad \text{and} \quad T_2(t + s)x = T_2(t)(T_2(s)x). \]

(ii)

\[ T(0)x = x \]

\[ \iff T_1(0)x + T_2(0)x e^\dagger = xe + xe^\dagger \]

\[ \iff T_1(0)x = x \quad \text{and} \quad T_2(0)x = x \forall x_1 \in Z_1 \text{ and } x_2 \in Z_2. \]

(iii)

\[ \lim_{t \to 0^+} T(t)x = x, \quad t \notin \mathbb{N} \mathbb{C} \cup \{0\} \]

\[ \iff \lim_{t_1 \to 0} T_1(t)x + \lim_{t_2 \to 0} T_2(t)x e^\dagger = xe + xe^\dagger \]

\[ \iff \lim_{t_1 \to 0} T_1(t)x = x \quad \text{and} \quad \lim_{t_2 \to 0} T_2(t)x = x. \]

(iv) there exist a +ve hyperbolic number \( \sigma_\alpha = \sigma_{\alpha,1} + \sigma_{\alpha,2} \) where \( \sigma_{\alpha,l}, l = 1, 2 \) is a +ve number such that

\[ p_\alpha(T(t)x) \leq e^{\sigma_{\alpha,l}t}p_\alpha(x). \]

\[ \iff p_{\alpha,1}(T_1(t)x) \leq e^{\sigma_{\alpha,1}t}p_{\alpha,1}(x) \quad \text{and} \quad p_{\alpha,2}(T_2(t)x) \leq e^{\sigma_{\alpha,2}t}p_{\alpha,1}(x) \]

Therefore, \( T_1(t) : t \geq 0 \) are \( (C_0, 1) \) semigroup on \( Z_1 \), \( l = 1, 2 \) resp.

Thus, \( \{ T(t) : t \geq 0 \} \) is a \( (C_0, 1) \)-semigroup in \( Z \iff \{ T_1(t) : t \geq 0 \} \) and \( \{ T_2(t) : t \geq 0 \} \) are \( (C_0, 1) \)-semigroup on \( Z_1 \) and \( Z_2 \) resp.

(2) Suppose that \( \{ T(t) : t \geq 0 \} \) be the limit of \( \Gamma \Rightarrow \pi_\alpha(T(t)x) = T_\alpha(\pi_\alpha x). \)

\[ \pi_\alpha(T(t)x) = T_\alpha(\pi_\alpha x) \]

\[ \Rightarrow \pi_{\alpha,1}(T_1(t)x)e + \pi_{\alpha,2}(T_2(t)x)e^\dagger = T_{\alpha,1}(\pi_{\alpha,1}x)e + T_{\alpha,2}(\pi_{\alpha,2}x)e^\dagger \]

\[ \Rightarrow \pi_{\alpha,1}(T_1(t)x) = T_{\alpha,1}(\pi_{\alpha,1}x) \quad \text{and} \quad \pi_{\alpha,2}(T_2(t)x) = T_{\alpha,2}(\pi_{\alpha,2}x) \]

\[ \Rightarrow \{ T_1(t) : t \geq 0 \} \text{ and } \{ T_2(t) : t \geq 0 \} \text{ are the limit of } \Gamma_1 \text{ and } \Gamma_2 \text{ resp.} \]

Conversely, suppose that \( \{ T_1(t) : t \geq 0 \} \text{ and } \{ T_2(t) : t \geq 0 \} \text{ are the limit of } \Gamma_1 \text{ and } \Gamma_2 \text{ resp.} \)

\[ \Rightarrow \pi_{\alpha,1}(T_1(t)x) = T_{\alpha,1}(\pi_{\alpha,1}x) \text{ and } \pi_{\alpha,2}(T_2(t)x) = T_{\alpha,2}(\pi_{\alpha,2}x) \]

Let \( x = x_1e + x_2e^\dagger \in Z \) and \( \pi_\alpha = \pi_{\alpha,1}e + \pi_{\alpha,2}e^\dagger \) be the natural \( \mathbb{B} \mathbb{C} \) homomorphism of \( Z \) onto
$Z_\alpha$, where $\pi_{\alpha,l}$ is the natural homomorphism of $Z_l$ onto $Z_{\alpha,l}$, $l = 1, 2$. Then

$$\pi_\alpha(T(t)x) = \pi_{\alpha,1}(T_1(t)x)e + \pi_{\alpha,2}(T_2(t)x)e^\dagger = T_{\alpha,1}(\pi_{\alpha,1}x)e + T_{\alpha,2}(\pi_{\alpha,2}x)e^\dagger = T_\alpha(\pi_\alpha x).$$

Thus, $\{T(t) : t \geq t' \geq 0\}$ is the limit of $\Gamma$.

**Definition 3.2.** Let $X$ be a locally $\mathbb{BC}$ convex module and the $\mathbb{BC}$ linear operator $A : D(A) \subset X \to X$ is called the infinitesimal generator of the semigroup $\{T(t) : t \geq 0\}$ if it satisfies

$$Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h}, \forall x \in X, h \notin NC \cup \{0\}.$$

**Remark 3.1.** The $\mathbb{BC}$ linear operator $A$ can be written as

$$Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h} = \lim_{h \to 0^+} \frac{(T_1(h_1)x_1 - x_1)e + (T_2(h_2)x_2 - x_2)e^\dagger}{h_1e + h_2e^\dagger} = \lim_{h_1 \to 0^+} \frac{(T_1(h_1)x_1 - x_1)e + \lim_{h_2 \to 0^+} (T_2(h_2)x_2 - x_2)e^\dagger}{h_1e + h_2e^\dagger} = \lim_{h_1 \to 0^+} \frac{(T_1(h_1)x_1 - x_1)e}{h_1} + \lim_{h_2 \to 0^+} \frac{(T_2(h_2)x_2 - x_2)e^\dagger}{h_2} = A_1x_1e + A_2x_2e^\dagger.$$

where $A_i$ are the infinitesimal generator of the semigroup $\{T(t_l) : t_l \geq 0\}$, $l = 1, 2$ in complex version.

Proof of the following theorem is in similar lines as in [2, theorem 1], so we omit the proof.

**Theorem 3.3.** Let $X$ be a complete locally $\mathbb{BC}$ convex module. Then $(C_0, 1)$ semi-groups on $X$ is in 1-1 correspondence with the projective families of $C_0$ semigroups on $\mathbb{BC}$ Banach module $X_\alpha$. Further, if $A$ is the infinitesimal generator of a $(C_0, 1)$ semigroup and $\{A_\alpha\}$ is the family of generators associated with the corresponding $C_0$ semigroups then $\{A_\alpha\}$ is a projective family and its limit is $A$.

**Definition 3.3.** Let a family $\Gamma = \{T(t) : t \geq 0\}$ be a $C_0$ semi-groups in locally $\mathbb{BC}$-convex module $X$. Then $\Gamma$ is a $(C_0, 1)$ semigroup $\iff$ there exist sets of hyperbolic numbers $\{M_\alpha : M_\alpha = M_{\alpha,1}e + M_{\alpha,2}e^\dagger, \alpha \in I\}$ and $\{\sigma_\alpha : \sigma_{\alpha,1}e + \sigma_{\alpha,2}e^\dagger, \alpha \in I\}$ such that for $\alpha \in I$

$$p_\alpha(T(t)x) \leq M_\alpha e^{\sigma_{\alpha}t}p_\alpha(x), \forall t \in \mathbb{D}^+, x \in X,$$

where $\{M_{\alpha,l} : \alpha \in I\}$ and $\{\sigma_{\alpha,l} : \alpha \in I\}$, $l = 1, 2$ are the sets of real numbers.

**Definition 3.4.** $\{T(t) : t \geq 0\}$ be a family of continuous $\mathbb{BC}$ linear operators on $X$. Then $\{T(t) : t \geq 0\}$ is a $(C_0, 1)$ semigroup $\iff$ if it satisfies the following condition:

(i) $\{T(t) : t \geq 0\}$ is a $C_0$ semigroup in $X$;

(ii) for each continuous $\mathbb{D}$ valued seminorm $p$ on $X$ there exist a positive hyperbolic number $\sigma_p$ and a continuous $\mathbb{D}$ valued seminorm $q$ on $X$ such that $p(T(t)x) \leq e^{\sigma_p t}q(x)$ for all $t \geq 0$ and $x \in X$. 

We can write above inequality as
\[
p(T(t)x) \leq e^{\sigma_p t} q(x)
\]
\[
\iff p_1(T_1(t_1)x_1) \leq e^{\sigma_{p_1} t_1} q_1(x_1) \quad \text{and} \quad p_2(T_2(t_2)x_2) \leq e^{\sigma_{p_2} t_2} q_2(x_2).
\]

4. CONCLUSION

Using the idempotent representation of bicomplex numbers, most of the results on \(C_o\) semigroups of linear operators with complex scalars can be extended to \(C_o\) semigroups of linear operators with bicomplex scalars and can be an interesting area of research.

REFERENCES


