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SHARP INEQUALITIES BETWEEN HÖLDER AND STOLARSKY MEANS OF TWO POSITIVE NUMBERS

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ABSTRACT. Given any index of the Stolarsky means, we find the greatest and least indexes of the Hölder means, such that for any two positive numbers, the Stolarsky mean with the given index is bounded from below and above by the Hölder means with those indexes, of the two positive numbers. Finally, we present a geometric application of this inequality involving the Fermat-Torricelli point of a triangle.

Key words and phrases: Hölder means; Stolarsky means; Monotone functions; Jensen inequality; Hermite–Hadamard inequality.

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1. INTRODUCTION

The Hölder and Stolarsky means of two positive numbers a and b, with $a < b$, are obtained by taking a probability measure μ, whose support contains the set $\{a, b\}$ and is contained in the interval $[a, b]$, integrating the function $x \mapsto x^p$, for some $p \in [-\infty, \infty]$, with respect to that probability measure, and then taking the $1/p$ power of that integral. Of course, this definition does not make sense for $p \in \{-\infty, 0, \infty\}$, but it can be made rigorous by the process of taking a limit. In other words, for $p \geq 1$, these two means are L^p-norms of the function $f(x) = x$, with respect to certain probability measures μ, whose supports S satisfy the condition $\{a, b\} \subseteq S \subseteq [a, b]$. There is another way to understand these means, namely through the mean value theorem for integrals. Thus, for any $p \in \mathbf{R} \setminus \{0\}$, there exists $c \in [a, b]$, such that given a probability measure μ supported on $[a, b]$, we have:

$$c^p = \frac{1}{b-a} \int_a^b x^p d\mu(x). \tag{1.1}$$

Since this number c is unique, we define it to be the (μ, p)-mean of a and b.
For the Hölder means, the probability measure μ_H is chosen to be the simplest possible symmetric one, with respect to the midpoint of $[a, b]$, whose support satisfies the above conditions, namely

$$\mu_H := \frac{1}{2}\delta_a + \frac{1}{2}\delta_b, \tag{1.2}$$

where δ_a and δ_b denote the Dirac probability measures that concentrate all the mass at the point a and b, respectively.
In opposition to the way that the Hölder means are defined, by concentrating all the mass in a symmetric way at the margins a and b of the interval $[a, b]$, the Stolarsky means are defined using the probability measure μ_S that spreads the mass uniformly along the entire interval $[a, b]$, namely:

$$d\mu_S := \frac{1}{b-a} 1_{[a,b]} dx, \tag{1.3}$$

where $1_{[a,b]}$ is the characteristic function of the interval $[a, b]$, and dx the Lebesgue measure.
Since both family of means, Hölder and Stolarsky, are defined via probability measures μ, that means $\mu([a, b]) = 1$, Lyapunov inequality (which is derived from Hölder inequality) implies that if we increase the index p, then the p-mean of a and b, increases, too. Let $H_p(a, b)$ and $S_p(a, b)$ be the p-Hölder mean and p-Stolarsky mean, respectively, of a and b, for $p \in [-\infty, \infty]$. It is natural to ask the following question:

Question 1. *Given a number $n \in [-\infty, \infty]$, what are the greatest $p(n)$ and the least $q(n)$ in $[-\infty, \infty]$ such that, for all a and b positive numbers, we have:*

$$H_{p(n)}(a, b) \leq S_n(a, b) \leq H_{q(n)}(a, b)? \tag{1.4}$$

We are not the first people to ask and answer this question. It is known for example, that, for $n = 0$, $S_0(a, b)$ becomes the logarithmic mean of a and b, and the answer was given for the first time in [8], and later on in [6]. For $n = 0$, the answer is $p(0) = 0$ and $q(0) = 1/3$, that means, for all $0 < a < b$, we have:

$$\sqrt{ab} \leq \frac{b-a}{\ln(b) - \ln(a)} \leq \left(\frac{a^{1/3} + b^{1/3}}{2}\right)^3. \tag{1.5}$$

A weaker inequality, for all $0 < a < b$, we have:

$$\sqrt{ab} \leq \frac{b-a}{\ln(b)-\ln(a)} \leq \frac{a+b}{2}, \tag{1.6}$$

was obtained in [3], [9], and [11] (actually, in [3], $q = 1$ was improved to $q = 1/2$). For other inequalities concerning the Stolarsky means see also [4].
The paper is structured as follows:
In section 2, we review the definitions of the Hölder and Stolarsky means of two positive numbers. In section 3, we introduce some mathematical language and vocabulary that will be useful later in treating many cases of our proof in a unitary way. We formulate the main result of this paper in section 4, and prove it in section 5. Finally, in section 6, we present an application of these inequalities, by formulating two geometric inequalities involving the Fermat-Torricelli point of a triangle, in which the measure of each angle is at most $120°$.

2. HÖLDER AND STOLARSKY MEANS FOR TWO POSITIVE NUMBERS

In this section we review the definitions of the Hölder and Stolarsky means of two positive numbers. Let a and b be two positive numbers. Without loss of generality we may assume that $a \leq b$. For any $p \in [-\infty, \infty]$, we define, the p-Hölder mean of a and b, as:

$$H_p(a,b) \;:=\; \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p} & \text{if} \quad p \in \mathbb{R}\setminus\{0\} \\ \lim_{p\to 0}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \sqrt{ab} & \text{if} \quad p = 0 \\ \lim_{p\to\infty}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \max\{a,b\} = b & \text{if} \quad p = \infty \\ \lim_{p\to-\infty}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \min\{a,b\} = a & \text{if} \quad p = -\infty \end{cases}. \tag{2.1}$$

Let us observe that for all $p \in \mathbb{R}\setminus\{0\}$, we have:

$$\begin{aligned} H_p(a,b) &= \left(\int_{\mathbb{R}} |x|^p d\mu_H(x)\right)^{1/p} \\ &= \| f \|_p, \end{aligned} \tag{2.2}$$

where $\mu_H := (1/2)\delta_a + (1/2)\delta_b$ and $f(x) := x$. Here, for any point $c \in \mathbb{R}$, δ_c denotes the Dirac delta measure concentrated at c. Since μ_H is a probability measure, Lyapunov inequality implies that the function $p \mapsto H_p(a,b)$ is an increasing function. Thus, as p increases from $-\infty$ to ∞, the p-Hölder mean of a and b takes in a continuous and increasing way all real numbers in between a and b. If $a < b$ and $p < q$, then $H_p(a,b) < H_q(a,b)$; that means in this case the function $p \mapsto H_p(a,b)$ is also injective.
For $0 < a < b$, we define the p-Stolarsky mean of a and b, as:

$$S_p(a,b) \;:=\; \begin{cases} \left(\frac{b^n-a^n}{n(b-a)}\right)^{1/(n-1)} & \text{if} \quad n \in \mathbb{R}\setminus\{0,1\} \\ \lim_{n\to 0}\left(\frac{b^n-a^n}{n(b-a)}\right)^{1/(n-1)} = \frac{b-a}{\ln(b)-\ln(a)} & \text{if} \quad n = 0 \\ \lim_{n\to 1}\left(\frac{b^n-a^n}{n(b-a)}\right)^{1/(n-1)} = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} & \text{if} \quad n = 1 \\ \lim_{n\to\infty}\left(\frac{b^n-a^n}{n(b-a)}\right)^{1/(n-1)} = \max\{a,b\} = b & \text{if} \quad n = \infty \\ \lim_{n\to-\infty}\left(\frac{b^n-a^n}{n(b-a)}\right)^{1/(n-1)} = \min\{a,b\} = a & \text{if} \quad n = -\infty \end{cases}. \tag{2.3}$$

If $a = b$, then for all $p \in [-\infty, \infty]$, we define:

$$\begin{aligned} S_p(a,a) &:= \lim_{b\to a^+} S_p(a,b) \\ &= a. \end{aligned} \tag{2.4}$$

In fact, for all $0 < a < b$, and all $n \in \mathbb{R} \setminus \{1\}$, we have:

$$\begin{aligned} S_n(a, b) &= \left(\int_{\mathbb{R}} |x|^{n-1} d\mu_S(x)\right)^{1/(n-1)} \\ &= \| f \|_{n-1}, \end{aligned} \tag{2.5}$$

where $d\mu_S/dx := [1/(b-a)]1_{[a,b]}$. Here dx denotes the Lebesgue measure on $\mathbb{R}$ and $1_{[a,b]}$ the characteristic function of the interval $[a, b]$, and $f(x) = x$. Since μ_S is also a probability measure on $\mathbb{R}$, Lyapunov inequality implies again that the function $n \mapsto S_n(a, b)$ is increasing.

3. MATHEMATICAL LANGUAGE VOCABULARY

In this section we organize our thoughts, in order to prove our main theorem in an efficient way. We introduce the following notations.

Notation . We denote by:

- "$(-1)er$" the comparative adjective-conjunction group "less than" or "smaller than",
- "$(-1)ereq$" the expression "less than or equal to",
- "$(0)er$" the comparative adjective-conjunction group "equal to",
- "$(0)ereq$" the comparative adjective-conjunction group "equal to",
- "$(+1)er$" the comparative adjective-conjunction group "greater than" or "bigger than",
- "$(+1)ereq$" the expression "greater than or equal to".

Notation . We denote by:

- "$(-1)creasing$" the adjective "decreasing",
- "$(0)creasing$" the adjective "constant",
- "$(+1)creasing$" the adjective "increasing".

Notation . We denote by:

- "$(-1)vex$" the expression "strictly concave" or "strictly concave downward",
- "$(0)vex$" the adjective "linear",
- "$(+1)vex$" the expression "strictly convex" or "strictly concave upward".

Notation . We denote by:

- "$(-1)est$" the adjective "the least" or "the smallest",
- "$(+1)est$" the adjective "the greatest" or "the biggest".

Notation . We denote by:

- "$(-1)mum$" the "infimum",
- "$(+1)mum$" the "supremum".

We consider the following two monoids with respect to the multiplication operation:

$$M_{-,+} := \{-1, 1\} \tag{3.1}$$

and

$$M_{-,0,+} := \{-1, 0, 1\}. \tag{3.2}$$

We also consider the sets of order relations, words, or expressions:

$$A_{rel} = \{<, =, >\}, \tag{3.3}$$

$$A_{releq} = \{\leq, =, \geq\}, \tag{3.4}$$

$$\begin{aligned} A_{er} &= \{less\ than, equal\ to, greater\ than\} \\ &= \{(-1)er, (0)er, (+1)er\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} A_{creasing} &= \{decreasing, constant, increasing\} \\ &= \{(-1)creasing, (0)creasing, (+1)creasing\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} A_{vex} &= \{strictly\ concave, linear, stricty\ convex\} \\ &= \{(-1)vex, (0)vex, (+1)vex\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} A_{est} &= \{the\ least, the\ greatest\} \\ &= \{(-1)est, (+1)est\}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} A_{mum} &= \{infimum, supremum\} \\ &= \{(-1)mum, (+1)mum\}. \end{aligned} \tag{3.9}$$

We define the action of the monoid $M_{-,0,+}$ on the set A_{rel}, $(m, x) \mapsto mx$, by the following table:

$\cdot$	$<$	$=$	$>$
-1	$>$	$=$	$<$
0	$=$	$=$	$=$
$+1$	$<$	$=$	$>$

To make sure that no confusion arises later on, in this paper, we put a box around the product (action) of an element from a monoid and an element from the set upon which that monoid is acting. Thus, for example, we write:

$$\boxed{(-1) >} \;=\; < . \tag{3.10}$$

It is not hard to see that this is indeed an action of the monoid $(M_{-,0,+}, \cdot)$ on the set A_{rel}, which means:

- $\forall m_1, m_2 \in M_{-,0,+}$ and $x \in A_{rel}$, we have:

$$\boxed{m_1 \boxed{m_2 x}} \;=\; \boxed{(m_1 m_2)\, x} \tag{3.11}$$

- $\forall x \in A_{rel}$, we have:

$$\boxed{(+1)x} \;=\; x. \tag{3.12}$$

We define the action of the monoid $M_{-,0,+}$ on the set A_{releq}, $(m, x) \mapsto mx = m \cdot x$, by the following table:

$\cdot$	$\leq$	$=$	$\geq$
-1	$\geq$	$=$	$\leq$
0	$=$	$=$	$=$
$+1$	$\geq$	$=$	$\leq$

We also define the actions of the monoid $(M_{-,0,+}, \cdot)$ on the sets A_{er}, $A_{creasing}$, and A_{vex}, in the following way:
For all ϵ, $\delta \in \{-1, 0, +1\}$ and each $suffix \in \{er, creasing, vex\}$,

$$\epsilon\,((\delta)suffix) \;:=\; (\epsilon \cdot \delta)\, suffix. \tag{3.13}$$

Finally, we define the actions of the monoid $(M_{-,+}, \cdot)$ on the sets A_{est} and A_{mum}, in the following way:
For all ϵ, $\delta \in \{-1, +1\}$ and each $suffix \in \{est, mum\}$,

$$\epsilon\,((\delta)suffix) \;:=\; (\epsilon \cdot \delta)\, suffix. \tag{3.14}$$

With these notations and actions, we can restate Jensen inequality, in the following way:

Proposition 3.1. *(Jensen inequality) Let $I \subseteq \mathbb{R}$ be an interval, and $\varphi : I \to \mathbb{R}$ an $(\epsilon)vex$ function, for some $\epsilon \in \{-1, 0, +1\}$. Let $L : I \to \mathbb{R}$ be a linear function. We assume that the graphs of φ and L intersect at two distinct points (x_1, y_1) and (x_1, y_2). Then for all $x \in I$, we have:*

$$\varphi(x) \;\; \boxed{sgn(\epsilon)sgn(x - x_1)sgn(x - x_2) \cdot >} \;\; L(x). \tag{3.15}$$

Proof. We have three cases:
Case 1. If $\epsilon = +1$, then φ is convex and $sgn(\epsilon) = +1$. Thus, the portion(s) of the graph of φ, corresponding to values of x outside the interval $[x_1, x_2]$, is(are) above the corresponding portion(s) of the secant line joining the points $(x_1, \varphi(x_1))$ and $(x_2, \varphi(x_2))$ which is the graph of the linear function L, the graph of φ coincides with the graph of L at the points x_1 and x_2, and the portion of the graph of φ corresponding to the values of x inside the interval (x_1, x_2) is below the corresponding graph of L. We can also observe that $x \in I \setminus [x_1, x_2]$ is equivalent to $sgn(x - x_1)sgn(x - x_2) > 0$, $x \in \{x_1, x_2\}$ means $sgn(x - x_1)sgn(x - x_2) = 0$, and $x \in (x_1, x_2)$ is the same as $sgn(x - x_1)sgn(x - x_2) < 0$. Remembering that $\boxed{(+1) \cdot >} \; = >$, $\boxed{0 \cdot >} \; = =$, and $\boxed{(-1) \cdot >} \; = <$ inequality (3.15) follows.
Case 2. If $\epsilon = 0$, then φ is linear and $sgn(\epsilon) = 0$. Since the graphs of both φ and L are straight lines, and these two lines have two distinct common points (x_1, y_1) and (x_2, y_2), the two graphs must coincide. Thus (3.15) becomes the equality:

$$\varphi(x) \;=\; L(x), \tag{3.16}$$

for all $x \in I$, which agrees with the definition $\boxed{0 \cdot >} \; = =$.
Case 3. If $\epsilon = -1$, then the proof is similar to the proof of Case 1. ∎

4. SHARP INEQUALITY ABOUT THE HÖLDER AND STOLARSKY MEANS

In this section we answer the following question:
Question: Given a number $n \in [-\infty, \infty]$, find the greatest number $p = p(n)$ and the least number $q = q(n)$, such that for all positive numbers a and b, we have:

$$H_p(a, b) \le S_n(a, b) \le H_q(a, b). \tag{4.1}$$

To answer this question we need the following lemma, for an inequality between two functions, whose graphs touch at one point, to hold, see [12].

Lemma 4.1. *Let $I \subseteq \mathbb{R}$ be an interval, and let $\mathring{I} := \{x \in I \mid \exists r > 0, (x - r, x + r) \subset I\}$ be the set of the interior points of I. Suppose f and g are two real valued functions, such that:*

1. *$f(x) \le g(x)$, for all $x \in I$.*
2. *f and g are continuous on I.*
3. *f and g are twice differentiable on $\mathring{I}$.*
4. *There exists $x_0 \in \mathring{I}$, such that $f(x_0) = g(x_0)$.*
5. *f'' is continuous at x_0.*

Then, we must have $f'(x_0) = g'(x_0)$ and $f''(x_0) \le g''(x_0)$.

We present below a proof slightly different from the one from [12].

Proof. Let $h(x) := f(x) - g(x)$. For all $x \in I$, we have:

$$\begin{aligned} h(x) &\leq 0 \\ &= h(x_0). \end{aligned} \tag{4.2}$$

Therefore, h has an absolute maximum value at x_0. Since x_0 is a point in the interior of I, Fermat theorem implies $h'(x_0) = 0$. That means, $f'(x_0) = g'(x_0)$.
Since $x_0 \in \mathring{I}$, there exists $\delta > 0$, such that $(x_0 - \delta, x_0 + \delta) \subset I$. Applying Taylor formula with Lagrange remainder, for each $t \in (0, \delta)$, there exists $\xi_t \in (x_0, x_0 + t)$, such that:

$$h(x_0 + t) = h(x_0) + h'(x_0)\,t + \frac{1}{2}h''(\xi_t)\,t^2. \tag{4.3}$$

Since $h(x_0 + t) \leq 0$, $h(x_0) = 0$, and $h'(x_0) = 0$, we conclude from the last equation that $h''(\xi_t) \leq 0$, for all $t \in (0, \delta)$. Letting $t \to 0^+$, since $x_0 < \xi_t < x_0 + t$, we conclude that $\xi_t \to x_0$. Because h'' is continuous at x_0, we obtain:

$$\begin{aligned} h''(x_0) &= \lim_{t \to 0^+} h''(\xi_t) \\ &\leq 0. \end{aligned} \tag{4.4}$$

The last inequality is equivalent to $f''(x_0) \leq g''(x_0)$. ∎

We will also need the following proposition.

Proposition 4.2. *The function* $L : (0, \infty) \to (0, \infty)$*, defined by:*

$$\begin{cases} \frac{x-1}{\ln(x)} & \text{if} \quad x \neq 1 \\ \lim_{x \to 1} \frac{x-1}{\ln(x)} = 1 & \text{if} \quad x = 1 \end{cases} \tag{4.5}$$

is increasing and concave on $(0, \infty)$*. In fact,* $L(x)$ *is the logarithmic mean of* 1 *and* x*, for all* $x > 0$*.*

Proof. Indeed, L is continuous on $(0, \infty)$. Its derivative is:

$$\begin{aligned} L'(x) &= \frac{\ln(x) - (x-1)/x}{\ln^2(x)} \\ &= \frac{x\ln(x) - x + 1}{x\ln^2(x)}, \end{aligned} \tag{4.6}$$

for all $x > 0$, $x \neq 1$. Since we have:

$$\begin{aligned} \lim_{x \to 1} L'(x) &= \lim_{x \to 1} \frac{1}{x} \cdot \lim_{x \to 1} \frac{x\ln(x) - x + 1}{\ln^2(x)} \\ &= 1 \cdot \lim_{x \to 1} \frac{(x\ln(x) - x + 1)'}{\left(\ln^2(x)\right)'} \text{ by L'Hôpital rule} \\ &= \lim_{x \to 1} \frac{\ln(x)}{2\ln(x)/x} \\ &= \lim_{x \to 1} \frac{x}{2} \\ &= \frac{1}{2}, \end{aligned} \tag{4.7}$$

a consequence of Lagrange Mean Value Theorem implies that L is differentiable at 1, and:

$$L'(1) = \frac{1}{2}. \tag{4.8}$$

For the same reason L is twice differentiable on $(0, \infty)$. The only differentiability problem is at $x = 1$, but in fact, L is analytic on the interval $(0, 2)$, which is a neighborhood of 1.
The derivative of L is:

$$\begin{aligned} L'(x) &= \frac{\ln(x) - (x-1)/x}{\ln^2(x)} \\ &= \frac{\ln(x) - 1 + (1/x)}{\ln^2(x)} \\ &= \frac{(1/x) - 1 - \ln(1/x)}{\ln^2(x)} \\ &= \frac{s - 1 - \ln(s)}{\ln^2(s)} \quad \text{for } s := 1/x \\ &\geq 0, \end{aligned} \tag{4.9}$$

since for all $s > 0$, $s \neq 1$, we have $\ln(s) < s + 1$, due to the fact that the function $s \mapsto \ln(s)$ is concave on $(0, \infty)$, and so its graph is below its tangent line at the point $(1, 0)$.
Thus L is increasing on $(0, \infty)$.
Let us compute now the second derivative of L. Since:

$$\begin{aligned} L'(x) &= \frac{\ln(x) - 1 + (1/x)}{\ln^2(x)} \\ &= \frac{1}{\ln(x)} - \frac{1}{\ln^2(x)} + \frac{1}{x\ln^2(x)}, \end{aligned} \tag{4.10}$$

we have:

$$\begin{aligned} L''(x) &= -\frac{1}{x\ln^2(x)} + \frac{2}{x\ln^3(x)} - \frac{1}{x^2\ln^2(x)} - \frac{2}{x^2\ln^3(x)} \\ &= \frac{2x - x\ln(x) - \ln(x) - 2}{x^2\ln^3(x)} \\ &= \frac{2x - x\ln(x) - \ln(x) - 2}{\ln(x)} \cdot \frac{1}{x^2\ln^2(x)}, \end{aligned} \tag{4.11}$$

for all $x > 0$, $x \neq 1$. Since $1/(x^2\ln^2(x)) > 0$, in order to show that $L''(x) < 0$, for $x \neq 1$, we need to show that the function:

$$u(x) := 2x - x\ln(x) - \ln(x) - 2 \tag{4.12}$$

is positive for $0 < x < 1$, and negative for $x > 1$.
This can be done in the following way:

$$\begin{aligned} u(x) &= (x-1)(x+1)\left[\frac{2}{x+1} - \frac{1}{x-1}\int_1^x \frac{1}{t}dt\right] \\ &= (x-1)(x+1)\left[\frac{1}{(x+1)/2} - \frac{1}{x-1}\int_1^x \frac{1}{t}dt\right]. \end{aligned} \tag{4.13}$$

Since the function $t \mapsto 1/t$ is convex on $(0, \infty)$, if $x > 1$, then Hermite-Hadamard inequality tells us that the average value of this function, $1/(x-1) \cdot \int_1^x (1/t)dt$, on the interval $[1, x]$, is greater than or equal to the value of the function, $2/(x+1)$, at the midpoint, $(x+1)/2$, of the interval $[1, x]$. Because the factors $x - 1$ and $x + 1$ are both positive, we can see that $u(x) < 0$,

for all $x > 1$.
If $0 < x < 1$, we can write:

$$u(x) \;=\; (x-1)(x+1)\left[\frac{1}{(x+1)/2} - \frac{1}{1-x}\int_x^1 \frac{1}{t}dt\right], \tag{4.14}$$

and Hermite-Hadamard inequality and the fact that the factor $x - 1$ is now negative imply that $u(x) > 0$.
Thus L is concave. ∎

Corollary 4.3. *The following inequalities hold:*

- *For all $x \in (1/2,\, 2)$, we have:*

$$\frac{(x-1)\ln(2)}{\ln(x)} \;>\; \frac{x+1}{3}. \tag{4.15}$$

- *For all $x \in (0,\, 1/2) \cup (2,\, \infty)$, we have:*

$$\frac{(x-1)\ln(2)}{\ln(x)} \;<\; \frac{x+1}{3}. \tag{4.16}$$

- *For all $x \in (0,\, 1/2)$, we have:*

$$\frac{(x-1)\ln(2)}{\ln(x)} \;>\; x. \tag{4.17}$$

- *For all $x \in (1/2,\, \infty)$, we have:*

$$\frac{(x-1)\ln(2)}{\ln(x)} \;<\; x. \tag{4.18}$$

Proof. Let $g(x) := (x-1)\ln(2)/\ln(x)$ and $f(x) := (x+1)/3$. Then we have:

$$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2} \quad \text{and} \quad g(2) = f(2) = 1. \tag{4.19}$$

Since g is strictly concave and f is linear, the graphs of g and f have no other common points except $(1/2,\, 1/2)$ and $(2,\, 1)$. Moreover, on the interval $(1/2,\, 2)$ the graph of g is above the graph of f, while outside of this interval the graph of g is below the graph of f. These facts prove inequalities (4.15) and (4.16).
On the other hand, if we define $i(x) = x$, for all $x \in \mathbb{R}$, then we have:

$$\begin{aligned} g(0) &:= \lim_{x\to 0^+} \frac{(x-1)\ln(2)}{\ln(x)} \\ &= 0 \\ &= i(0) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} g\left(\frac{1}{2}\right) &= \frac{1}{2} \\ &= i\left(\frac{1}{2}\right). \end{aligned} \tag{4.21}$$

Again since g is concave and i is linear, the points $(0,\, 0)$ and $(1/2,\, 1/2)$ are the only common points of the graphs of g and i. Moreover, on the interval $(0,\, 1/2)$ the graph of g is above the graph of i, while on the interval $(1/2,\, \infty)$ the graph of g is below the graph on i. These two facts prove inequalities (4.17) and (4.18). ∎

We present now the answer to our question.

Theorem 4.4. *Let* f, $g : [-\infty, \infty] \to [-\infty, \infty]$ *be the continuous and nondecreasing functions defined by:*

$$f(x) = \frac{x+1}{3} \tag{4.22}$$

and

$$g(x) = \begin{cases} \frac{(x-1)\ln(2)}{\ln(x)} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}. \tag{4.23}$$

Then, for all $n \in [-\infty, \infty]$*, the greatest number* $p = p(n)$ *and the least number* $q = q(n)$*, such that, for all* a *and* b *positive we have:*

$$H_p(a,b) \leq S_n(a,b) \leq H_q(a,b), \tag{4.24}$$

are:

$$p(n) = \min\{f(n), g(n)\} \tag{4.25}$$

and

$$q(n) = \max\{f(n), g(n)\}. \tag{4.26}$$

More precisely, the above statement means:

- *For* $n = \infty$*, we have:*

$$p(\infty) = q(\infty) = \infty. \tag{4.27}$$

- *For* $2 < n < \infty$*, we have:*

$$p(n) = \frac{(n-1)\ln(2)}{\ln(n)} \quad \text{and} \quad q(n) = \frac{n+1}{3}. \tag{4.28}$$

- *For* $n = 2$*, we have:*

$$p(2) = q(2) = 1. \tag{4.29}$$

- *For* $1/2 < n < 2$*, we have:*

$$p(n) = \frac{n+1}{3} \quad \text{and} \quad q(n) = \frac{(n-1)\ln(2)}{\ln(n)}. \tag{4.30}$$

- *For* $n = 1/2$*, we have:*

$$p\left(\frac{1}{2}\right) = q\left(\frac{1}{2}\right) = \frac{1}{2}. \tag{4.31}$$

- *For* $0 < n < 1/2$*, we have:*

$$p(n) = \frac{(n-1)\ln(2)}{\ln(n)} \quad \text{and} \quad q(n) = \frac{n+1}{3}. \tag{4.32}$$

- *For* $-1 < n \leq 0$*, we have:*

$$p(n) = 0 \quad \text{and} \quad q(n) = \frac{n+1}{3}. \tag{4.33}$$

- *For* $n = -1$*, we have:*

$$p(-1) = q(-1) = 0. \tag{4.34}$$

- *For* $-\infty < n < -1$*, we have:*

$$p(n) = \frac{n+1}{3} \quad \text{and} \quad q(n) = 0. \tag{4.35}$$

- *For $n = -\infty$, we have:*

$$p(-\infty) = q(-\infty) = -\infty. \tag{4.36}$$

We restate this theorem using our mathematical language and vocabulary.

Theorem 4.5. *With the notations from the last theorem, for all $n \in [-\infty, \infty]$, the $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2)}$ est number $p = p(n)$ and the $\boxed{(-1)sgn(n+1)sgn(n-1/2)sgn(n-2)}$ est number $q = q(n)$, such that, for all for all a and b positive we have:*

$$H_p(a,b) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \le} \quad S_n(a,b) \tag{4.37}$$

and

$$S_n(a,b) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \le} \quad H_q(a,b) \tag{4.38}$$

are:

$$p(n) \;=\; g(n) \tag{4.39}$$

and

$$q(n) \;=\; f(n). \tag{4.40}$$

To prove this theorem, we are going to study first the order relation between $S_n(a, b)$ and $H_{h(n)}(a, b)$, for an arbitrary function h. Of course, for $a = b$, we have $S_n(a, b) = H_{h(n)}(a, b) = a = b$. So without loss of generality, we may assume that $a < b$. Then for any $\mathcal{R} \in \{<, =, >\}$, we have:

$$\begin{aligned} S_n(a,b)\mathcal{R}H_{h(n)}(a,b) &\Leftrightarrow \frac{1}{b-a}S_n(a,b)\mathcal{R}\frac{1}{b-a}H_{h(n)}(a,b) \\ &\Leftrightarrow S_n\left(\frac{a}{b-a},\frac{b}{b-a}\right)\mathcal{R}H_{h(n)}\left(\frac{a}{b-a},\frac{b}{b-a}\right). \end{aligned} \tag{4.41}$$

If we define now:

$$x \;:=\; \frac{a}{b-a}, \tag{4.42}$$

then

$$\frac{b}{b-a} \;=\; x+1, \tag{4.43}$$

and $x \in (0, \infty)$.
Thus, for a certain $n \in \mathbb{R}$, there exists $\mathcal{R} \in \{<, =, >\}$, such that, for all $0 < a < b$, we have $S_n(a, b)\mathcal{R}H_{h(n)}(a, b)$, if and only if, for all $x \in (0, \infty)$, we have: $S_n(x, x+1)\mathcal{R}H_{h(n)}(x, x+1)$. The last statement is equivalent to, for all $x > 0$, $\ln(S_n(x, x + 1))\mathcal{R} \ln(H_{h(n)}(x, x + 1))$.
We will pursue the following steps:
Step 1. Find the derivative of the function:

$$\begin{aligned} &F_1(x) \\ =\;& \ln\left(S_n\left(x,x+1\right)\right) - \ln\left(H_{h(n)}\left(x,x+1\right)\right) \\ =\;& \ln\left(\left(\frac{(x+1)^n - x^n}{n}\right)^{1/(n-1)}\right) - \ln\left(\left(\frac{(x+1)^{h(n)} + x^{h(n)}}{2}\right)^{1/h(n)}\right) \\ =\;& \frac{1}{n-1}\ln\left(|(x+1)^n - x^n|\right) - \frac{\ln(|n|)}{n-1} - \frac{1}{h(n)}\ln\left(x^{h(n)} + (x+1)^{h(n)}\right) + \frac{\ln(2)}{h(n)}, \end{aligned}$$

for $n \in \mathbb{R} \setminus \{0, 1\}$.
For $n = 0$, we have:

$$\begin{aligned} & F_1(x) \\ = \; & \ln\left(S_0\left(x, x+1\right)\right) - \ln\left(H_{h(0)}\left(x, x+1\right)\right) \\ = \; & \ln\left(\frac{x+1-x}{\ln(x+1)-\ln(x)}\right) - \ln\left(\left(\frac{(x+1)^{h(0)}+x^{h(0)}}{2}\right)^{1/h(0)}\right) \\ = \; & -\ln\left(\ln(x+1)-\ln(x)\right) - \frac{1}{h(0)}\ln\left(x^{h(0)}+(x+1)^{h(0)}\right) + \frac{\ln(2)}{h(0)}. \end{aligned}$$

For $n = 1$, we have:

$$\begin{aligned} & F_1(x) \\ = \; & \ln\left(S_1\left(x, x+1\right)\right) - \ln\left(H_{h(1)}\left(x, x+1\right)\right) \\ = \; & \ln\left(\frac{1}{e}\frac{(x+1)^{x+1}}{x^x}\right) - \ln\left(\left(\frac{(x+1)^{h(1)}+x^{h(1)}}{2}\right)^{1/h(1)}\right) \\ = \; & -1 + (x+1)\ln(x+1) - x\ln(x) - \frac{1}{h(1)}\ln\left((x+1)^{h(1)}+x^{h(1)}\right) + \frac{\ln(2)}{h(1)}. \end{aligned}$$

We have:

$$F_1'(x) = \frac{n}{(x+1)^n - x^n} \cdot \frac{(x+1)^{n-1} - x^{n-1}}{n-1} - \frac{(x+1)^{h(n)-1} + x^{h(n)-1}}{(x+1)^{h(n)} + x^{h(n)}}. \tag{4.44}$$

The above expression for $F_1'(x)$ makes sense even for $n = 0$ and $n = 1$, by the process of passing to a limit, and even though the fraction $1/h(n)$ was not defined for $h(n) = 0$, the formula for $F_1'(x)$ is defined for all $x > 0$, even in the case $h(n) = 0$. Let us define the function:

$$G_1(x) := (n-1)\left[(x+1)^n - x^n\right]\left[(x+1)^{h(n)} + x^{h(n)}\right] F_1'(x), \tag{4.45}$$

for $n \neq 0$ and $n \neq 1$. The function G_1 was defined in such a way that we get rid of all denominators of the fractions from the formula of $F_1'(x)$.
We purposely avoid $n = 0$, since in that case $G_1(x) = 0$, for all $x > 0$, due to the presence of the factor $(x+1)^n - x^n$. We also intentionally avoid $n = 1$, since in that case $G_1(x) = 0$, due to the factor $n - 1$.
Since, for $n = 0$, we have:

$$F_1'(x) = \frac{1}{\ln(x+1) - \ln(x)}\left(\frac{1}{x} - \frac{1}{x+1}\right) - \frac{(x+1)^{h(0)-1} + x^{h(0)-1}}{(x+1)^{h(0)} + x^{h(0)}}, \tag{4.46}$$

in order to get rid of the denominators of $F_1'(x)$, we define:

$$G_1(x) := x(x+1)\left[\ln(x+1) - \ln(x)\right]\left[(x+1)^{h(0)} + x^{h(0)}\right] F_1'(x). \tag{4.47}$$

Since, for $n = 1$, we have:

$$F_1'(x) = \ln(x+1) - \ln(x) - \frac{(x+1)^{h(1)-1} + x^{h(1)-1}}{(x+1)^{h(1)} + x^{h(1)}}, \tag{4.48}$$

in order to get rid of the denominators of $F_1'(x)$, we define:

$$G_1(x) := \left[(x+1)^{h(1)} + x^{h(1)}\right] F_1'(x). \tag{4.49}$$

Then, for all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$\begin{aligned} G_1(x) &= n\left[(x+1)^{n-1} - x^{n-1}\right]\left[x^{h(n)} + (x+1)^{h(n)}\right] \\ &\quad -(n-1)\left[(x+1)^n - x^n\right]\left[x^{h(n)-1} + (x+1)^{h(n)-1}\right] \\ &= (x+1)^{n+h(n)-1} - x^{n+h(n)-1} + n(x+1)^{n-1}x^{h(n)} - nx^{n-1}(x+1)^{h(n)} \\ &\quad -(n-1)(x+1)^n x^{h(n)-1} + (n-1)x^n(x+1)^{h(n)-1} \\ &= x^{n+h(n)-1}\left[\left(\frac{x+1}{x}\right)^{n+h(n)-1} - 1 + n\left(\frac{x+1}{x}\right)^{n-1} - n\left(\frac{x+1}{x}\right)^{h(n)}\right. \\ &\quad \left. -(n-1)\left(\frac{x+1}{x}\right)^{n} + (n-1)\left(\frac{x+1}{x}\right)^{h(n)-1}\right] \\ &= x^{n+h(n)-1}K_1(t), \end{aligned} \tag{4.50}$$

where:

$$t := \frac{x+1}{x} \in (1, \infty), \tag{4.51}$$

and

$$K_1(t) := t^{n+h(n)-1} - 1 + nt^{n-1} - nt^{h(n)} - (n-1)t^n + (n-1)t^{h(n)-1}. \tag{4.52}$$

For $n = 0$, we have:

$$\begin{aligned} G_1(x) &= (x+1)^{h(0)} + x^{h(0)} - x(x+1)\left[(x+1)^{h(0)-1} + x^{h(0)-1}\right]\ln\left(\frac{x+1}{x}\right) \\ &= x^{h(0)+1}\left\{\frac{1}{x}\left[\left(\frac{x+1}{x}\right)^{h(0)} + 1\right]\right. \\ &\quad \left. -\frac{x+1}{x}\left[\left(\frac{x+1}{x}\right)^{h(0)-1} + 1\right]\ln\left(\frac{x+1}{x}\right)\right\} \\ &= x^{h(0)+1}K_1(t), \end{aligned} \tag{4.53}$$

where

$$t := \frac{x+1}{x} \in (1, \infty), \tag{4.54}$$

and

$$K_1(t) := (t-1)\left(t^{h(0)} + 1\right) - t\left(t^{h(0)-1} + 1\right)\ln(t). \tag{4.55}$$

For $n = 1$, we have:

$$\begin{aligned} G_1(x) &= \left[(x+1)^{h(1)} + x^{h(1)}\right]\ln\left(\frac{x+1}{x}\right) - \left[(x+1)^{h(1)-1} + x^{h(1)-1}\right] \\ &= x^{h(1)}\left\{\left[\left(\frac{x+1}{x}\right)^{h(1)} + 1\right]\ln\left(\frac{x+1}{x}\right) - \frac{1}{x}\left[\left(\frac{x+1}{x}\right)^{h(1)-1} + 1\right]\right\} \\ &= x^{h(1)}K_1(t), \end{aligned} \tag{4.56}$$

where:

$$t := \frac{x+1}{x} \in (1, \infty), \tag{4.57}$$

and

$$K_1(t) := \left(t^{h(1)}+1\right)\ln(t) - (t-1)\left(t^{h(1)-1}+1\right). \tag{4.58}$$

Step 2. Find the derivative of K_1 with respect to t.
For $n \in \mathbb{R}\setminus\{0, 1\}$, we have:

$$\begin{aligned} K_1'(t) &= [n+h(n)-1]\, t^{n+h(n)-2} + n(n-1)t^{n-2} - nh(n)t^{h(n)-1} \\ &\quad -n(n-1)t^{n-1} + (n-1)\,[h(n)-1]\, t^{h(n)-2} \\ &= t^{h(n)-2}K_2(t), \end{aligned} \tag{4.59}$$

where:

$$\begin{aligned} K_2(t) &:= [n+h(n)-1]\, t^{n} + n(n-1)t^{n-h(n)} - nh(n)t \\ &\quad -n(n-1)t^{n-h(n)+1} + (n-1)\,[h(n)-1]. \end{aligned} \tag{4.60}$$

For $n = 0$, we have:

$$\begin{aligned} K_1'(t) &= [h(0)+1]\, t^{h(0)} - h(0)t^{h(0)-1} + 1 \\ &\quad - \left[h(0)t^{h(0)-1}+1\right]\ln(t) - t^{h(0)-1} - 1 \\ &= t^{h(0)-1}\left\{[h(0)+1]\,(t-1) - \left[h(0)+t^{1-h(0)}\right]\ln(t)\right\} \\ &= t^{h(0)-1}K_2(t), \end{aligned} \tag{4.61}$$

where:

$$K_2(t) := [h(0)+1]\,(t-1) - \left[h(0)+t^{1-h(0)}\right]\ln(t). \tag{4.62}$$

For $n = 1$, we have:

$$\begin{aligned} K_1'(t) &:= h(1)t^{h(1)-1}\ln(t) + t^{h(1)-1} + t^{-1} - h(1)t^{h(1)-1} - 1 + [h(1)-1]\, t^{h(1)-2} \\ &= h(1)t^{h(1)-1}\ln(t) + [1-h(1)]\, t^{h(1)-1} + t^{-1} - 1 + [h(1)-1]\, t^{h(1)-2} \\ &=: K_2(t). \end{aligned} \tag{4.63}$$

Step 3. Find the derivative of K_2 with respect to t.
For $n \in \mathbb{R}\setminus\{0, 1\}$, we have:

$$\begin{aligned} K_2'(t) &= n\,[n+h(n)-1]\, t^{n-1} + n(n-1)\,[n-h(n)]\, t^{n-h(n)-1} - nh(n) \\ &\quad -n(n-1)\,[n-h(n)+1]\, t^{n-h(n)} \\ &= nK_3(t), \end{aligned} \tag{4.64}$$

where:

$$\begin{aligned} K_3(t) &:= [n+h(n)-1]\, t^{n-1} + (n-1)\,[n-h(n)]\, t^{n-h(n)-1} - h(n) \\ &\quad -(n-1)\,[n-h(n)+1]\, t^{n-h(n)}. \end{aligned} \tag{4.65}$$

For $n = 0$, we have:

$$\begin{aligned} K_2'(t) &= h(0)+1+[h(0)-1]\, t^{-h(0)}\ln(t) - h(0)t^{-1} - t^{-h(0)} \\ &=: K_3(t). \end{aligned} \tag{4.66}$$

For $n = 1$, we have:

$$\begin{aligned} K_2'(t) &= h(1)\,[h(1)-1]\, t^{h(1)-2}\ln(t) + h(1)t^{h(1)-2} - [h(1)-1]^2\, t^{h(1)-2} \\ &\quad -t^{-2} + [h(1)-1]\,[h(1)-2]\, t^{h(1)-3} \\ &= t^{h(1)-3}K_3(t), \end{aligned} \tag{4.67}$$

where:

$$\begin{aligned} K_3(t) &= h(1)\,[h(1)-1]\,t\ln(t) + \left[-h(1)^2+3h(1)-1\right]t - t^{1-h(1)} \\ &\quad + [h(1)-1]\,[h(1)-2]\,. \end{aligned} \tag{4.68}$$

Step 4. Find the derivative of K_3 with respect to t.
For all $n \in \mathbb{R}\setminus\{0,1\}$, we have:

$$\begin{aligned} K_3'(t) &= [n+h(n)-1]\,(n-1)t^{n-2} + (n-1)\,[n-h(n)]\,[n-h(n)-1]\,t^{n-h(n)-2} \\ &\quad -(n-1)\,[n-h(n)+1]\,[n-h(n)]\,t^{n-h(n)-1} \\ &= (n-1)t^{n-h(n)-2}K_4(t), \end{aligned} \tag{4.69}$$

where:

$$\begin{aligned} K_4(t) &:= [n+h(n)-1]\,t^{h(n)} + [n-h(n)]\,[n-h(n)-1] \\ &\quad - [n-h(n)+1]\,[n-h(n)]\,t. \end{aligned} \tag{4.70}$$

For $n=0$, we have:

$$\begin{aligned} K_3'(t) &= -h(0)\,[h(0)-1]\,t^{-h(0)-1}\ln(t) + [h(0)-1]\,t^{-h(0)-1} + h(0)t^{-2} \\ &\quad +h(0)t^{-h(0)-1} \\ &= -h(0)\,[h(0)-1]\,t^{-h(0)-1}\ln(t) + [2h(0)-1]\,t^{-h(0)-1} + h(0)t^{-2} \\ &= t^{-2}K_4(t), \end{aligned} \tag{4.71}$$

where

$$K_4(t) := h(0)\,[1-h(0)]\,t^{1-h(0)}\ln(t) + [2h(0)-1]\,t^{1-h(0)} + h(0). \tag{4.72}$$

For $n=1$, we have:

$$\begin{aligned} K_3'(t) &= h(1)\,[h(1)-1]\ln(t) + h(1)\,[h(1)-1] + \left[-h(1)^2+3h(1)-1\right] \\ &\quad + [h(1)-1]\,t^{-h(1)} \\ &= h(1)\,[h(1)-1]\ln(t) + [2h(1)-1] + [h(1)-1]\,t^{-h(1)} \\ &=: K_4(t). \end{aligned} \tag{4.73}$$

Step 5. Find the derivative of K_4 with respect to t.
For all $n \in \mathbb{R}\setminus\{0,1\}$, we have:

$$K_4'(t) = [n+h(n)-1]\,h(n)t^{h(n)-1} - [n-h(n)+1]\,[n-h(n)]\,. \tag{4.74}$$

Let us define the function:

$$K_5(t) := K_4'(t), \tag{4.75}$$

for all $t>1$.
For $n=0$, we have:

$$\begin{aligned} K_4'(t) &:= h(0)\,[1-h(0)]^2\,t^{-h(0)}\ln(t) + h(0)\,[1-h(0)]\,t^{-h(0)} \\ &\quad + [2h(0)-1]\,[1-h(0)]\,t^{-h(0)} \\ &= [1-h(0)]\,t^{-h(0)}K_5(t), \end{aligned} \tag{4.76}$$

where

$$K_5(t) := h(0)\,[1-h(0)]\ln(t) + [3h(0)-1]\,. \tag{4.77}$$

For $n = 1$, we have:

$$\begin{aligned} K_4'(t) &:= h(1)\,[h(1)-1]\,t^{-1} - h(1)\,[h(1)-1]\,t^{-h(1)-1} \\ &= h(1)\,[h(1)-1]\,t^{-h(1)-1} K_5(t), \end{aligned} \tag{4.78}$$

where

$$K_5(t) := t^{h(1)} - 1. \tag{4.79}$$

Step 6. Find the derivative of K_5 with respect to t.
For all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$K_5'(t) = [n + h(n) - 1]\,h(n)\,[h(n) - 1]\,t^{h(n)-2}, \tag{4.80}$$

for all $t > 1$.
For $n = 0$, we have:

$$K_5'(t) = h(0)\,[1 - h(0)]\,t^{-1}, \tag{4.81}$$

for all $t > 1$.
For $n = 1$, we have:

$$K_5'(t) = h(1)t^{h(1)-1}, \tag{4.82}$$

for all $t > 1$.

We can see that given any number n, the number $K_5'(t)$ has the same sign, for all $t > 1$.
Let us remember the definition of the signum function:

$$sgn(x) := \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}. \tag{4.83}$$

We extend the signum function by continuity at $-\infty$ and $+\infty$, by defining:

$$sgn(-\infty) := -1 \quad \text{and} \quad sgn(\infty) := 1. \tag{4.84}$$

We are going to apply these six steps in reverse order to the functions $h := f$ and $h := g$, from the text of our theorem.
Since, for any positive numbers a and b, we have:

$$S_{-\infty}(a, b) = H_{-\infty}(a, b), \tag{4.85}$$

$$S_{-1}(a, b) = H_0(a, b), \tag{4.86}$$

$$S_{1/2}(a, b) = H_{1/2}(a, b), \tag{4.87}$$

$$S_2(a, b) = H_1(a, b), \tag{4.88}$$

$$S_\infty(a, b) = H_\infty(a, b), \tag{4.89}$$

we must clearly have:

$$p(-\infty) = q(-\infty) = -\infty, \tag{4.90}$$

$$p(-1) = q(-1) = 0, \tag{4.91}$$

$$p\left(\frac{1}{2}\right) = q\left(\frac{1}{2}\right) = \frac{1}{2}, \tag{4.92}$$

$$p(2) = q(2) = 1, \tag{4.93}$$

$$p(\infty) = q(\infty) = \infty. \tag{4.94}$$

For this reason, we may assume that $n \in \mathbb{R} \setminus \{-1, 1/2, 2\}$.
Let us introduce the following notation:

$$\mathbb{R}_{-1,1/2,2} \;:=\; \mathbb{R} \setminus \{-1, 1/2, 2\}. \tag{4.95}$$

5. PROOF OF THE MAIN THEOREM

We have the following propositions and observations:

Proposition 5.1. *If $h := g$, where g is the function from the text of Theorem 4.4, and $n > 0$, then the signum function of $K_5'(t)$, for $t > 1$, depends on n is the following way:*

$$sgn\,(K_5'(t)) \;=\; \begin{cases} 1 & \text{if} \quad 0 < n < 1/2 \\ 0 & \text{if} \quad n = 1/2 \\ -1 & \text{if} \quad 1/2 < n < 1 \\ 1 & \text{if} \quad n = 1 \\ -1 & \text{if} \quad 1 < n < 2 \\ 0 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n > 2 \end{cases}, \tag{5.1}$$

for all $t > 1$. That implies, for all $t > 1$, and all $n \in (0, \infty) \setminus \{1\}$, we have:

$$sgn\,(K_5'(t)) \;=\; sgn\left(n - \frac{1}{2}\right) sgn(n-2). \tag{5.2}$$

Proof. Let us assume first that $n > 0$ and $n \neq 1$. We can see now that the factors $t^{g(n)-2}$ and $g(n) = (n-1)\ln(2)/\ln(n)$, from formula (4.80), which becomes now:

$$K_5'(t) \;=\; [n + g(n) - 1]\, g(n)\, [g(n) - 1]\, t^{g(n)-2}, \tag{5.3}$$

are both positive. Thus to find the signum of $K_5'(t)$, we need to find out the sign of $n + g(n) - 1$ and of $g(n) - 1$.
We have:

$$\begin{aligned} n + g(n) - 1 \;&=\; n - 1 + \frac{(n-1)\ln(2)}{\ln(n)} \\ &=\; \frac{n-1}{\ln(n)} \cdot \ln(2n). \end{aligned} \tag{5.4}$$

Since $(n-1)/\ln(n) > 0$, for all $n > 0$, we conclude that the signum of $n + g(n) - 1$ is the same as the signum of $\ln(2n)$, that means:

$$sgn\,(n + g(n) - 1) \;:=\; \begin{cases} -1 & \text{if} \quad 0 < n < 1/2 \\ 0 & \text{if} \quad n = 1/2 \\ 1 & \text{if} \quad n > 1/2 \end{cases}. \tag{5.5}$$

On the other hand, since we proved that for $n > 0$, the function $g(n) = (n-1)\ln(2)/\ln(n)$ is increasing and $g(2) = 1$, we conclude that for all $0 < n < 2$, $g(n) < 1$, while for all $n > 2$, $g(n) > 1$. Thus, we have:

$$sgn\,(g(n) - 1) \;:=\; \begin{cases} -1 & \text{if} \quad 0 < n < 2 \\ 0 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n > 2 \end{cases}. \tag{5.6}$$

Therefore, for $n > 0$, $n \neq 1$, we have:

$$\begin{aligned} sgn\,(K_5'(t)) &= sgn\,(n + g(n) - 1) \cdot sgn\,(g(n) - 1) \\ &= \begin{cases} 1 & \text{if} \quad 0 < n < 1/2 \\ 0 & \text{if} \quad n = 1/2 \\ -1 & \text{if} \quad 1/2 < n < 2 \\ 0 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n > 1/2 \end{cases}, \end{aligned} \tag{5.7}$$

for all $t > 1$.
For $n = 1$, we can see from formula (4.82) that:

$$\begin{aligned} sgn\,(K_5'(t)) &= sgn\,(h(1)) \\ &= sgn\,(g(1)) \\ &= 1, \end{aligned} \tag{5.8}$$

since $g(1)$ is defined as:

$$\begin{aligned} g(1) &= \lim_{n \to 1} \frac{(n-1)\ln(2)}{\ln(n)} \\ &= \ln(2). \end{aligned} \tag{5.9}$$

∎

Proposition 5.2. *If $h := g$, where g is the function from the text of Theorem 4.4, then for any $n \in (0, \infty) \setminus \{1/2, 2\}$, we have:*

$$sgn\,(K_4(1)) = \begin{cases} -1 & \text{if} \quad 0 < n < 1/2 \\ 1 & \text{if} \quad 1/2 < n < 2 \\ -1 & \text{if} \quad n > 2 \end{cases}, \tag{5.10}$$

that means:

$$sgn\,(K_4(1)) = (-1) sgn(n+1) sgn\left(n - \frac{1}{2}\right) sgn(n-2), \tag{5.11}$$

and

$$sgn\,(K_4(\infty)) = \begin{cases} 1 & \text{if} \quad 0 < n < 1/2 \\ -1 & \text{if} \quad 1/2 < n < 2 \\ 1 & \text{if} \quad n > 2 \end{cases}, \tag{5.12}$$

that means:

$$sgn\,(K_4(\infty)) = sgn(n+1) sgn\left(n - \frac{1}{2}\right) sgn(n-2), \tag{5.13}$$

where the factor $sgn(n+1)$ may be omitted, since $n + 1 > 0$, for all $n > 0$.
Hence, for all $n \in (0, \infty) \setminus \{1/2, 2\}$, we have:

$$sgn\,(K_4(1)) \cdot sgn\,(K_4(\infty)) = -1. \tag{5.14}$$

Proof. We can see from formula (4.70), that, for all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$\begin{aligned} K_4(1) &= [n + h(n) - 1] + [n - h(n)]\,[n - h(n) - 1] \\ &\quad - [n - h(n) + 1]\,[n - h(n)] \\ &= 3h(n) - n - 1. \end{aligned} \tag{5.15}$$

We analyze four cases:
Case 1. If $0 < n < 1/2$, then we have:

$$\begin{aligned} K_4(1) &= 3h(n) - n - 1 \\ &= 3\left[h(n) - \frac{n+1}{3}\right] \\ &= 3\left[\frac{(n-1)\ln(2)}{\ln(n)} - \frac{n+1}{3}\right] \\ &< 0, \end{aligned} \tag{5.16}$$

due to the result from Corollary 4.3. Since, we have $g = \ln(2) \cdot L$, on $(0, \infty)$, it follows from Proposition 4.2, that g is increasing on $(0, \infty)$, and thus for all $0 < n < 2$, we have:

$$\begin{aligned} h(n) &= g(n) \\ &< g(2) \\ &= 1. \end{aligned} \tag{5.17}$$

Thus, we have:

$$\begin{aligned} K_4(\infty) &= \lim_{t\to\infty} \left\{[n + h(n) - 1]\, t^{h(n)} + [n - h(n)]\,[n - h(n) - 1] \right. \\ &\qquad \left. - [n - h(n) + 1]\,[n - h(n)]\, t\right\} \\ &= -\lim_{t\to\infty} \{[n - h(n) + 1]\,[n - h(n)]\, t\} \\ &= -sgn\,([n - h(n) + 1]\,[n - h(n)])\,\infty. \end{aligned} \tag{5.18}$$

We can see that:

$$\begin{aligned} n - h(n) + 1 &> -h(n) + 1 \\ &> 0. \end{aligned} \tag{5.19}$$

On the other hand, since $h = g$ is concave on $[0,\, 1/2]$, and $g(0) = 0$ and $g(1/2) = 1/2$, due to the fact that the portion of the graph of g, in between the points $(0,\, g(0))$ and $(1/2,\, g(1/2))$, is above the chord (line segment) joining these points, we conclude that for $0 < n < 1/2$, we have $n < h(n)$. Thus, we obtain:

$$K_4(\infty) = \infty. \tag{5.20}$$

Case 2. If $1/2 < n < 2$, $n \neq 1$, then it follows from Corollary 4.3 that:

$$\begin{aligned} K_4(1) &= 3\left[\frac{(n-1)\ln(2)}{\ln(n)} - \frac{n+1}{3}\right] \\ &> 0. \end{aligned} \tag{5.21}$$

Since g is increasing, we have as before $g(n) < g(2) = 1$. Thus:

$$\begin{aligned} K_4(\infty) &= \lim_{t\to\infty} \left\{[n + h(n) - 1]\, t^{h(n)} + [n - h(n)]\,[n - h(n) - 1] \right. \\ &\qquad \left. - [n - h(n) + 1]\,[n - h(n)]\, t\right\} \\ &= -\lim_{t\to\infty} \{[n - h(n) + 1]\,[n - h(n)]\, t\} \\ &= -sgn\,([n - h(n) + 1]\,[n - h(n)])\,\infty. \end{aligned} \tag{5.22}$$

As before, we have $n - h(n) + 1 > -h(n) + 1 > 0$, while the concavity of the function g, on the interval $[0,\, \infty)$ implies that the portion of the graph of g to the right of the point $(1/2,\, g(1/2))$

is below the line segment joining the points $(0, g(0))$ and $(1/2, g(1/2))$. Hence, $n - h(n) > 0$, and thus:

$$K_4(\infty) = -\infty. \tag{5.23}$$

Case 3. If $n = 1$, then $h(1) = g(1) = \ln(2)$, and formula (4.73) implies:

$$\begin{aligned} K_4(1) &= 3h(1) - 2 \\ &= 3\left[h(1) - \frac{2}{3}\right] \\ &= 3\left[g(1) - f(1)\right] \\ &> 0. \end{aligned} \tag{5.24}$$

On the other hand, we have:

$$\begin{aligned} K_4(\infty) &= \lim_{t\to\infty} K_4(t) \\ &= \lim_{t\to\infty} \left\{h(1)\left[h(1)-1\right]\ln(t) + 2h(1) - 1 + \left[h(1)-1\right]t^{-h(1)}\right\} \\ &= \lim_{t\to\infty} \left\{h(1)\left[h(1)-1\right]\ln(t)\right\} \\ &= \lim_{t\to\infty} \left\{\ln(2)\left[\ln(2)-1\right]\ln(t)\right\} \\ &= -\infty. \end{aligned} \tag{5.25}$$

Case 4. If $n > 2$, then we have:

$$\begin{aligned} K_4(1) &= 3\left[\frac{(n-1)\ln(2)}{\ln(n)} - \frac{n+1}{3}\right] \\ &< 0, \end{aligned} \tag{5.26}$$

due to the result of Corollary 4.3.
In this case we have $h(n) > h(2) = 1$. Thus:

$$\begin{aligned} K_4(\infty) &= \lim_{t\to\infty} \left\{\left[n + h(n) - 1\right]t^{h(n)} + \left[n - h(n)\right]\left[n - h(n) - 1\right] \right.\\ &\qquad \left. - \left[n - h(n) + 1\right]\left[n - h(n)\right]t\right\} \\ &= \lim_{t\to\infty} \left\{\left[n + h(n) - 1\right]t^{h(n)}\right\} \\ &= \infty, \end{aligned} \tag{5.27}$$

since $n + h(n) - 1 > n - 1 > 0$. ∎

Observation 1. *For all functions h, we have:*

$$K_1(1) = 0, \tag{5.28}$$

$$K_2(1) = 0, \tag{5.29}$$

$$K_3(1) = 0. \tag{5.30}$$

Proof. We can see from formula (4.52) that, for all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$\begin{aligned} K_1(1) &= 1^{n+h(n)-1} - 1 + n1^{n-1} - n1^{h(n)} - (n-1)1^n + (n-1)1^{h(n)-1}. \\ &= 0. \end{aligned}$$

For $n = 0$, formula (4.55) implies:

$$K_1(1) = 0.$$

For $n = 1$, formula (4.58) implies:

$$K_1(1) = 0.$$

We can also see from formula (4.60) that, for all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$\begin{aligned} K_2(1) &= [n + h(n) - 1]\, 1^n + n(n-1)1^{n-h(n)} - nh(n)1 \\ &\quad -n(n-1)1^{n-h(n)+1} + (n-1)\,[h(n) - 1] \\ &= 0. \end{aligned}$$

For $n = 0$, formula (4.62) implies:

$$K_2(1) = 0.$$

For $n = 1$, formula (4.63) implies:

$$K_2(1) = 0.$$

Finally, we can see from formula (4.65) that, for all $n \in \mathbb{R} \setminus \{0, 1\}$, we have:

$$\begin{aligned} K_3(1) &= [n + h(n) - 1]\, 1^{n-1} + (n-1)\,[n - h(n)]\, 1^{n-h(n)-1} - h(n) \\ &\quad -(n-1)\,[n - h(n) + 1]\, 1^{n-h(n)} \\ &= 0. \end{aligned}$$

For $n = 0$, formula (4.66) implies:

$$K_3(1) = 0.$$

For $n = 1$, formula (4.68) implies:

$$K_3(1) = 0.$$

∎

Proposition 5.3. *Let $h := g$. Then, for all $n \in (0, \infty) \setminus \{1/2, 2\}$, the sign of $K_3(\infty)$ depends on n in the following way:*

$$sgn\,(K_3(\infty)) = \begin{cases} -1 & \text{if} \quad 0 < n < 1/2 \\ 1 & \text{if} \quad 1/2 < n < 1 \\ -1 & \text{if} \quad 1 \le n < 2 \\ 1 & \text{if} \quad n > 2 \end{cases}, \tag{5.31}$$

that means, for all $n \in (0, \infty) \setminus \{1/2, 2\}$, we have:

$$sgn\,(K_3(\infty)) = sgn\left(n - \frac{1}{2}\right) sgn^+(n-1) sgn(n-2), \tag{5.32}$$

where $sgn^+(u) := \lim_{v \to u^+} sgn(v)$, for all $v \in \mathbb{R}$.

Proof. For $n > 0$ and $n \neq 1$, using formula (4.65), we have:

$$\begin{aligned} K_3(t) &:= [n + h(n) - 1]\, t^{n-1} + (n-1)\,[n - h(n)]\, t^{n-h(n)-1} - h(n)t^0 \\ &\quad -(n-1)\,[n - h(n) + 1]\, t^{n-h(n)}. \end{aligned}$$

This means that the behavior of K_3 at ∞ is given by the term that contains t to the highest power in the above formula. We call this term the leading term.

- If $0 < n < 1/2$, then according to inequality (4.17), we have $h(n) = g(n) > n$. Thus all the powers of t in formula (4.65), $n - 1$, $n - h(n) - 1$, and $n - h(n)$, are negative. Hence, we conclude that:

$$\begin{aligned} \lim_{t \to \infty} K_3(t) &= -h(n) \\ &< 0. \end{aligned} \tag{5.33}$$

- If $1/2 < n < 1$, then since $h = g$ is increasing, we have $h(n) < h(2) = 1$. Thus the leading term of $K_3(t)$ is $-(n-1)[n-h(n)+1]t^{n-h(n)}$. The exponent of t in this term $n - h(n) = n - g(n)$ is positive due to inequality (4.18). The coefficient of this term $-(n-1)[n-h(n)+1]$ is positive, since $n - 1 < 0$ and $n + [1 - h(n)] > 0 + 0 = 0$. Thus we have:

$$\lim_{t\to\infty} K_3(t) \quad > \quad 0. \tag{5.34}$$

- If $1 < n < 2$, then $h(n) < h(2) = 1$ and, according to inequality (4.18), $n - h(n) > 0$. Thus the leading term of $K_3(t)$ is again $-(n-1)[n-h(n)+1]t^{n-h(n)}$. This time though, the leading coefficient $-(n-1)[n-h(n)+1]$ is negative. Hence, we have:

$$\lim_{t\to\infty} K_3(t) \quad < \quad 0. \tag{5.35}$$

- If $n > 2$, then $h(n) > h(2) = 1$. Thus the leading term of $K_3(t)$ is $[n + h(n) - 1]t^{n-1}$, which clearly has a positive coefficient, $n + h(n) - 1$. Therefore, $sgn(K_3(\infty)) = 1$.

For $n = 1$, in formula (4.68), the dominant term is $h(1)[h(1) - 1]t\ln(t)$. Since we have $h(1)[h(1) - 1]t\ln(t) = \ln(2)[\ln(2) - 1]t\ln(t) < 0$, we conclude that:

$$\begin{aligned} K_3(\infty) &= \lim_{t\to\infty} K_3(t) \\ &= -\infty. \end{aligned} \tag{5.36}$$

$\blacksquare$

Proposition 5.4. *Let $h := g$. Then, for all $n \in (0, \infty) \setminus \{1/2, 2\}$, the sign of $K_2(\infty)$ depends on n in the following way:*

$$sgn\left(K_2(\infty)\right) \quad = \quad \begin{cases} -1 & \text{if} \quad 0 < n < 1/2 \\ 1 & \text{if} \quad 1/2 < n < 1 \\ -1 & \text{if} \quad 1 \le n < 2 \\ 1 & \text{if} \quad n > 2 \end{cases}. \tag{5.37}$$

That means, for all for all $n \in (0, \infty) \setminus \{1/2, 2\}$, we have:

$$K_2(\infty) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn^{+}(n-1)sgn(n-2) >} \quad 0, \tag{5.38}$$

where $sgn^{+}(u) := \lim_{v\to u^{+}} sgn(v)$, for all $u \in \mathbb{R}$.

Proof. For $n > 0$ and $n \neq 1$, using formula (4.60), we have:

$$\begin{aligned} K_2(t) \quad := \quad & [n + h(n) - 1]\, t^{n} + n(n-1)t^{n-h(n)} - nh(n)t \\ & -n(n-1)t^{n-h(n)+1} + (n-1)\,[h(n) - 1]\, t^{0}. \end{aligned}$$

We see from here that we have the following:

- If $0 < n < 1/2$, then according to inequality (4.17), $h(n) = g(n) > n$. This implies $n - h(n) + 1 < 1$. Therefore, the leading term of $K_2(t)$ is $-nh(n)t$, which has a negative coefficient $-nh(n) = -n(n-1)\ln(2)/\ln(n)$. Thus, in this case, we have:

$$\lim_{t\to\infty} K_2(t) \quad < \quad 0. \tag{5.39}$$

- If $1/2 < n < 1$, then we have $n - h(n) + 1 > n$ and $n - h(n) + 1 > 1$. Thus, the leading term in $K_2(t)$ is $-n(n-1)t^{n-h(n)+1}$, which has a positive coefficient $-n(n-1)$. Therefore,

$$\begin{aligned} \lim_{t\to\infty} K_2(t) &= +\infty \\ &> 0. \end{aligned} \tag{5.40}$$

- If $1 < n < 2$, then we have $n-h(n)+1 > n$ and $n-h(n)+1 > 1$. Thus, the leading term in $K_2(t)$ is $-n(n-1)t^{n-h(n)+1}$, which has a negative coefficient $-n(n-1)$. Therefore,

$$\begin{aligned} \lim_{t\to\infty} K_2(t) &= -\infty \\ &< 0. \end{aligned} \tag{5.41}$$

- If $n > 2$, then the leading term in $K_2(t)$ is $[n + h(n) - 1]t^n$, which has a positive coefficient $n + h(n) - 1$. Therefore, we have:

$$\begin{aligned} \lim_{t\to\infty} K_2(t) &= +\infty \\ &> 0. \end{aligned} \tag{5.42}$$

For $n = 1$, we can see that, in formula (4.63), the leading term is -1. Hence,

$$\begin{aligned} K_2(\infty) &= \lim_{t\to\infty} K_2(t) \\ &= -1 \\ &< 0. \end{aligned} \tag{5.43}$$

∎

Proposition 5.5. *Let $h := g$. Then, for all $n \in (0, \infty) \setminus \{1/2, 2\}$, the sign of $K_1(\infty)$ depends on n in the following way:*

$$sgn\,(K_1(\infty)) = \begin{cases} -1 & \text{if} \quad 0 < n < 1/2 \\ 1 & \text{if} \quad 1/2 < n < 1 \\ -1 & \text{if} \quad 1 \le n < 2 \\ 1 & \text{if} \quad n > 2 \end{cases}. \tag{5.44}$$

Thus, for all $n > 0$ and $n \neq 1$, we have:

$$sgn\,(K_1(\infty)) = sgn(n-1/2)sgn(n-1)sgn(n-2). \tag{5.45}$$

Proof. For $n > 0$ and $n \neq 1$, using formula (4.52), we have:

$$K_1(t) = t^{n+h(n)-1} - 1 + nt^{n-1} - nt^{h(n)} - (n-1)t^n + (n-1)t^{h(n)-1}.$$

We have the following cases:

- If $0 < n < 1/2$, then the leading term of $K_1(t)$ is $-nt^{h(n)}$ since, according to inequality (4.17), $h(n) > n$. Thus, we have:

$$\begin{aligned} K_1(\infty) &= \lim_{t\to\infty} K_1(t) \\ &= sgn(-n) \lim_{n\to\infty} t^{h(n)} \\ &= -\infty \\ &< 0. \end{aligned} \tag{5.46}$$

- If $1/2 < n < 1$, then the leading term of $K_1(t)$ is $-(n-1)t^n$, which has a positive coefficient $-(n-1)$. Thus, we have:

$$\begin{aligned} K_1(\infty) &= +\infty \\ &> 0. \end{aligned} \tag{5.47}$$

- If $1 < n < 2$, then the leading term of $K_1(t)$ is $-(n-1)t^n$, which has a negative coefficient $-(n-1)$. Thus, we have:

$$\begin{aligned} K_1(\infty) &= -\infty \\ &< 0. \end{aligned} \tag{5.48}$$

- If $n > 2$, then the leading term of $K_1(t)$ is $t^{n+h(n)-1}$. Thus, we have:

$$\begin{aligned} K_1(\infty) &= +\infty \\ &> 0. \end{aligned} \tag{5.49}$$

If $n = 1$, then we can see that in formula (4.58), the leading term is $-t$. Hence,

$$K_1(\infty) = -\infty. \tag{5.50}$$

∎

Observation 2. *For all $n \in \mathbb{R}$ and all real values of $h(n)$, we have:*

$$\lim_{x\to\infty} F_1(x) = 0. \tag{5.51}$$

Proof. We analyze the following cases:
Case 1. If $n \notin \{0, 1\}$ and $h(n) \neq 0$, then for all $x > 0$, applying Lagrange Mean Value Theorem on the interval $[x, x+1]$, to $\varphi(t) := t^n$, there exists $c_x \in (x, x+1)$, such that:

$$(x+1)^n - x^n = nc_x^{n-1}. \tag{5.52}$$

Since, for all $x > 0$, we also have:

$$2x^{h(n)} \boxed{sgn\,(h(n)) <} x^{h(n)} + (x+1)^{h(n)} \boxed{sgn\,(h(n)) <} 2(x+1)^{h(n)},$$

applying the Intermediate Value Property (Darboux Property) to the continuous function $\phi(t) = 2t^n$, there exists $d_x \in (x, x+1)$, such that:

$$x^{h(n)} + (x+1)^{h(n)} = 2d_x^{h(n)}. \tag{5.53}$$

Thus, we obtain:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty} \left[\frac{1}{n-1}\ln\left(\frac{(x+1)^n - x^n}{n}\right) - \frac{1}{h(n)}\ln\left(\frac{x^{h(n)} + (x+1)^{h(n)}}{2}\right)\right] \\ &= \lim_{x\to\infty} \left[\frac{1}{n-1}\ln\left(\frac{nc_x^{n-1}}{n}\right) - \frac{1}{h(n)}\ln\left(\frac{2d_x^{h(n)}}{2}\right)\right] \\ &= \lim_{x\to\infty} \ln\left(\frac{c_x}{d_x}\right) \\ &= \ln(1) \\ &= 0, \end{aligned} \tag{5.54}$$

since we have:

$$\frac{x}{x+1} < \frac{c_x}{d_x} < \frac{x+1}{x}, \tag{5.55}$$

due to the fact that for all $x > 0$, $x < c_x < x+1$ and $x < d_x < x+1$, and both $x/(x+1)$ and $(x+1)/x$ converge to 1, as $x \to \infty$.

Case 2. If $n \notin \{0, 1\}$ and $h(n) = 0$, then we have:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty}\left[\frac{1}{n-1}\ln\left(\frac{(x+1)^n - x^n}{n}\right) - \ln\left(\sqrt{x(x+1)}\right)\right] \\ &= \lim_{x\to\infty}\left[\frac{1}{n-1}\ln\left(\frac{nc_x^{n-1}}{n}\right) - \ln\left(\sqrt{x(x+1)}\right)\right] \\ &= \lim_{x\to\infty}\ln\left(\frac{c_x}{\sqrt{x(x+1)}}\right) \\ &= \ln(1) \\ &= 0, \end{aligned} \tag{5.56}$$

since we have:

$$\sqrt{\frac{x}{x+1}} < \frac{c_x}{\sqrt{x(x+1)}} < \sqrt{\frac{x+1}{x}}, \tag{5.57}$$

due to the fact that for all $x > 0$, $x < c_x < x + 1$.

Case 3. If $n = 0$ and $h(n) \neq 0$, then:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty}\left[\ln\left(\frac{x+1-x}{\ln(x+1)-\ln(x)}\right) - \frac{1}{h(n)}\ln\left(\frac{x^{h(n)} + (x+1)^{h(n)}}{2}\right)\right] \\ &= \lim_{x\to\infty}\left[-\ln\left(\ln\left(\frac{x+1}{x}\right)\right) - \frac{1}{h(n)}\ln\left(\frac{2d_x^{h(n)}}{2}\right)\right] \\ &= -\lim_{x\to\infty}\ln\left(\ln\left(\left[\frac{x+1}{x}\right]^{d_x}\right)\right) \\ &= -\ln\left(\ln\left(\lim_{x\to\infty}\left[1+\frac{1}{x}\right]^{d_x}\right)\right) \\ &= -\ln\left(\ln(e)\right) \\ &= 0, \end{aligned} \tag{5.58}$$

since, for all $x > 0$, we have:

$$\left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{x}\right)^{d_x} < \left(1+\frac{1}{x}\right)^{x+1} \tag{5.59}$$

and both $(1 + 1/x)^x$ and $(1 + 1/x)^{x+1}$ converge to e, as $x \to \infty$.

Case 4. If $n = 0$ and $h(n) = 0$, then:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty}\left[\ln\left(\frac{1}{\ln(x+1)-\ln(x)}\right) - \ln\left(\sqrt{x(x+1)}\right)\right] \\ &= -\lim_{x\to\infty}\ln\left(\ln\left(\left[\frac{x+1}{x}\right]^{\sqrt{x(x+1)}}\right)\right) \\ &= -\ln\left(\ln\left(\lim_{x\to\infty}\left[1+\frac{1}{x}\right]^{\sqrt{x(x+1)}}\right)\right) \\ &= -\ln\left(\ln(e)\right) \\ &= 0, \end{aligned} \tag{5.60}$$

since, for all $x > 0$, we have:

$$\left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{x}\right)^{\sqrt{x(x+1)}} < \left(1+\frac{1}{x}\right)^{x+1}. \tag{5.61}$$

Case 5. If $n = 1$ and $h(n) \neq 0$, then:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty}\left[\ln\left(\frac{1}{e}\frac{(x+1)^{x+1}}{x^x}\right) - \frac{1}{h(n)}\ln\left(\frac{x^{h(n)}+(x+1)^{h(n)}}{2}\right)\right] \\ &= -1+\lim_{x\to\infty}\left[\ln\left(\frac{(x+1)^{x+1}}{x^x}\right) - \frac{1}{h(n)}\ln\left(\frac{2d_x^{h(n)}}{2}\right)\right] \\ &= -1+\lim_{x\to\infty}\ln\left(\frac{(x+1)^{x+1}}{x^x d_x}\right) \\ &= -1+\ln\left(\lim_{x\to\infty}\frac{(x+1)^{x+1}}{x^x d_x}\right) \\ &= -1+\ln(e) \\ &= 0, \end{aligned} \tag{5.62}$$

since for all $x > 0$, we have:

$$\frac{(x+1)^x}{x^x} < \frac{(x+1)^{x+1}}{x^x d_x} < \frac{(x+1)^{x+1}}{x^{x+1}}. \tag{5.63}$$

Case 6. If $n = 1$ and $h(n) = 0$, then:

$$\begin{aligned} F_1(\infty) &= \lim_{x\to\infty}\left[\ln\left(\frac{1}{e}\frac{(x+1)^{x+1}}{x^x}\right) - \ln\left(\sqrt{x(x+1)}\right)\right] \\ &= -1+\lim_{x\to\infty}\left[\ln\left(\frac{(x+1)^{x+1}}{x^x}\right) - \ln\left(\sqrt{x(x+1)}\right)\right] \\ &= -1+\lim_{x\to\infty}\ln\left(\frac{(x+1)^{x+(1/2)}}{x^{x+(1/2)}}\right) \\ &= -1+\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+(1/2)}\right) \\ &= -1+\ln(e) \\ &= 0. \end{aligned} \tag{5.64}$$

∎

Proposition 5.6. *For all $n \in \mathbb{R}\setminus\{-1, 1/2, 2\}$, the $((-1)sgn(n+1)sgn(n-1/2)sgn(n-2))$est $q = q(n)$, such that for all a and b positive numbers, we have:*

$$S_n(a,b) \ \boxed{(sgn(n+1)sgn(n-1/2)sgn(n-2)) \leq} \ H_q(a,b), \tag{5.65}$$

is $q = f(n)$, where $f(n) = (n+1)/3$.

Proof. We show first that, for all a and b positive numbers, we have:

$$S_n(a,b) \ \boxed{(sgn(n+1)sgn(n-1/2)sgn(n-2)) \leq} \ H_{f(n)}(a,b). \tag{5.66}$$

Using formula (4.70), for $h(n) := f(n) = (n+1)/3$, where $n \in \mathbb{R}_{-1,1/2,2} \setminus \{0, 1\}$, we have:

$$\begin{aligned}
K_4(t) &:= [n + h(n) - 1]\, t^{h(n)} + [n - h(n)]\, [n - h(n) - 1] \\
&\quad - [n - h(n) + 1]\, [n - h(n)]\, t \\
&= \left(n + \frac{n+1}{3} - 1\right) t^{(n+1)/3} + \left(n - \frac{n+1}{3}\right) \left(n - \frac{n+1}{3} - 1\right) t^0 \\
&\quad - \left(n - \frac{n+1}{3} + 1\right) \left(n - \frac{n+1}{3}\right) t^1 \\
&= 2 \cdot \frac{2n-1}{3} t^{(n+1)/3} + \frac{2n-1}{3} \cdot 2 \cdot \frac{n-2}{3} t^0 - 2 \cdot \frac{n+1}{3} \cdot \frac{2n-1}{3} t^1 \\
&= \frac{2(2n-1)}{9} \left[3t^{(n+1)/3} + (n-2)t^0 - (n+1)t^1\right] \\
&= \frac{2(2n-1)}{9} \cdot (n+1) \left[\frac{3}{n+1} t^{(n+1)/3} + \frac{n-2}{n+1} t^0 - t^1\right]. \qquad (5.67)
\end{aligned}$$

Since for any $t > 1$, the function $x \mapsto t^x$ is strictly convex, and $3/(n+1)$ and $(n-2)/(n+1)$ are non-zero weights that add up to 1, and are both positive if and only if $n > 2$ (this is the case when the number $3/(n+1) \cdot (n+1)/3 + (n-2)/(n+1) \cdot 0$ is in between the numbers $(n+1)/3$ and 0), Jensen inequality, written in the language of our action of the monoid $M_{-,0,+}$ on the set $\{<, =, >\}$, multiplied by $2(2n-1)/9 \cdot (n+1)$, implies that:

$$\begin{aligned}
& K_4(t) \\
= \;& \frac{2(2n-1)}{9} \cdot (n+1) \\
& \times \left[\frac{3}{n+1} t^{(n+1)/3} + \frac{n-2}{n+1} t^0 - t^1\right] \\
\boxed{sgn\left((2n-1)(n+1)\right) \boxed{sgn(n-2) >}} \;& \frac{2(2n-1)}{9} \cdot (n+1) \\
& \times \left[t^{3/(n+1) \cdot (n+1)/3 + (n-2)/(n+1) \cdot 0} - t^1\right] \\
= \;& \frac{2(2n-1)}{9} \cdot (n+1)(t - t) \\
= \;& 0. \qquad (5.68)
\end{aligned}$$

Therefore, $K_4(t) \boxed{sgn(n+1) sgn(n-1/2) sgn(n-2) >} 0$, for all $t > 0$.
If $n = 0$, then $h(0) = f(0) = (0+1)/3 = 1/3$, and formula (4.72) becomes:

$$\begin{aligned}
K_4(t) &= h(0)\, [1 - h(0)]\, t^{1-h(0)} \ln(t) + [2h(0) - 1]\, t^{1-h(0)} + h(0) \\
&= \frac{2}{9} t^{2/3} \ln(t) - \frac{1}{3} t^{2/3} + \frac{1}{3} \\
&= \frac{1}{3} t^{2/3} \left[-\ln\left(t^{-2/3}\right) - 1 + t^{-2/3}\right] \\
&= \frac{1}{3} t^{2/3} \left[s - 1 - \ln(s)\right] \\
&> 0, \qquad (5.69)
\end{aligned}$$

where $s := t^{-2/3} > 0$, due to the well-known inequality $\ln(s) \leq s - 1$, for all $s > 0$.
Let us observe that for $n = 0$, $(n+1)(n-1/2)(n-2) = 1 > 0$.
Therefore, $K_4(t) \boxed{sgn(n+1) sgn(n-1/2) sgn(n-2) >} 0$, for all $t > 1$.

If $n = 1$, then $h(1) = f(1) = (1+1)/3 = 2/3$, and formula (4.73) becomes now:

$$\begin{aligned} K_4(t) &= h(1)\,[h(1)-1]\ln(t) + 2h(1) - 1 + [h(1)-1]\,t^{-h(1)} \\ &= -\frac{2}{9}\ln(t) + \frac{1}{3} - \frac{1}{3}t^{-2/3} \\ &= \frac{1}{3}\left[\ln\left(t^{-2/3}\right) + 1 - t^{-2/3}\right] \\ &= \frac{1}{3}\left[\ln(s) + 1 - s\right] \\ &< 0, \end{aligned} \tag{5.70}$$

where $s := t^{-2/3} > 0$.
Therefore, $K_4(t) < 0$, for all $t > 1$.
Let us observe that for $n = 1$, $(n+1)(n-1/2)(n-2) = -1 < 0$.
Therefore, $K_4(t)$ $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >}$ 0, for all $t > 1$.
For $n \in \mathbb{R}_{-1,1/2,2} \setminus \{0, 1\}$, formula (4.69):

$$K_3'(t) = (n-1)t^{n-h(n)-2}K_4(t)$$

implies now:

$$K_3'(t) \quad \boxed{sgn(n-1)\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >}} \quad 0, \tag{5.71}$$

for all $t > 0$. The last inequality is equivalent to:

$$K_3'(t) \quad \boxed{sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0, \tag{5.72}$$

for all $t > 1$. Since the function K_3 is continuous on $[1, \infty)$, we conclude that K_3 is strictly $sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2)creasing$ on $[1, \infty)$. Because we know from Observation 1 that $K_3(1) = 0$, we conclude that for all $t > 1$, we have:

$$\begin{aligned} K_3(t) \quad &\boxed{sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad K_3(1) \\ &= \quad 0. \end{aligned} \tag{5.73}$$

For $n = 0$, since for all $t > 1$, according to formula (4.71),

$$K_3'(t) = t^{-2}K_4(t),$$

we conclude that $K_3'(t) > 0$. Thus, K_3 is strictly increasing on $[1, \infty)$, and since according to Observation 1, $K_3(1) = 0$, we conclude that for all $t > 1$, we have:

$$\begin{aligned} K_3(t) &> K_3(1) \\ &= 0. \end{aligned} \tag{5.74}$$

For $n = 1$, since for all $t > 1$, according to formula (4.73),

$$K_3'(t) = K_4(t),$$

we conclude that $K_3'(t) < 0$. Thus, K_3 is strictly decreasing on $[1, \infty)$, and since according to Observation 1, $K_3(1) = 0$, we conclude that for all $t > 1$, we have:

$$\begin{aligned} K_3(t) &< K_3(1) \\ &= 0. \end{aligned} \tag{5.75}$$

Therefore, if $n \in \{0, 1\}$, then for all $t > 1$, we have:

$$K_3(t) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0. \tag{5.76}$$

For $n \in \mathbb{R}_{-1,1/2,2} \setminus \{0, 1\}$, formula (4.64):

$$K_2'(t) \;=\; nK_3(t)$$

implies now that, for all $t > 1$, we have:

$$(5.77) \qquad K_2'(t) \;\boxed{sgn(n)\boxed{sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >}}\; 0.$$

Therefore, for all $t > 1$, we have:

$$(5.78) \qquad K_2'(t) \;\boxed{sgn(n)sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0.$$

For $n = 0$, formula (4.66):

$$(5.79) \qquad K_2'(t) \;=\; K_3(t),$$

and for $n = 1$, formula (4.67):

$$(5.80) \qquad K_2'(t) \;=\; t^{h(1)-3}K_3(t)$$

imply that for all $n \in \{0, 1\}$, we have:

$$(5.81) \qquad K_2'(t) \;\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0,$$

for all $t > 1$.
We conclude from here that, for any $n \in \mathbb{R} \setminus \{-1, 0, 1/2, 1, 2\}$, the function K_2 is $sgn(n)sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2)creasing$ on the interval $[1, \infty)$; while for $n \in \{0, 1\}$, K_2 is $sgn(n+1)sgn(n-1/2)sgn(n-2)creasing$ on $[1, \infty)$. Since $K_2(1) = 0$, it follows that for all $n \notin \{0, 1\}$, we have:

$$(5.82) \qquad K_2(t) \;\boxed{sgn(n)sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0,$$

while for all $n \in \{0, 1\}$, we have:

$$(5.83) \qquad K_2(t) \;\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0,$$

for all $t > 1$.
Formula (4.59):

$$(5.84) \qquad K_1'(t) \;=\; t^{h(n)-2}K_2(t),$$

for $n \notin \{0, 1\}$, formula (4.61):

$$(5.85) \qquad K_1'(t) \;=\; t^{h(0)-1}K_2(t),$$

for $n = 0$, and formula (4.63):

$$(5.86) \qquad K_1'(t) \;=\; K_2(t),$$

for $n = 1$, and the fact that $K_1(1) = 0$, imply now that, for all $n \notin \{0, 1\}$, we have:

$$(5.87) \qquad K_1(t) \;\boxed{sgn(n)sgn(n-1)sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0,$$

while for all $n \in \{0, 1\}$, we have:

$$(5.88) \qquad K_1(t) \;\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >}\; 0,$$

for all $t > 1$.
Formulas (4.50), (4.53), and (4.56) tell us that the sign of $G_1(x)$ is the same as the sign of $F_1(t)$, where $t = (x+1)/x$, for all $x > 0$, in all three cases $n \notin \{0, 1\}$, $n = 0$, and $n = 1$.
For $n \notin \{0, 1\}$, formula (4.45):

$$G_1(x) \;:=\; (n-1)\left[(x+1)^n - x^n\right]\left[(x+1)^{h(n)} + x^{h(n)}\right]F_1'(x),$$

implies that:

$$sgn\left(F_1'(x)\right) = sgn(n-1)sgn\left((x+1)^n - x^n\right)sgn\left(G_1(x)\right). \tag{5.89}$$

Since, for all $x > 0$, $x+1 > x$, we have:

$$sgn\left((x+1)^n - x^n\right) = sgn(n). \tag{5.90}$$

Thus, since $G_1(x)$ has the same sign as $K_1(t)$, for $t = (x+1)/x$, formulas (5.87), (5.89), and (5.90) imply:

$$F_1'(x) \quad \boxed{sgn\left((n-1)n\right)\boxed{sgn\left((n-1)n(n+1)(n-1/2)(n-2)\right)} >} \quad 0.$$

This is equivalent to:

$$F_1'(x) \quad \boxed{sgn\left((n-1)^2\right)sgn\left(n^2\right)sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0.$$

Since $n^2 > 0$ and $(n-1)^2 > 0$, for $n \notin \{0, 1\}$, we conclude that, for all $x > 0$, we have:

$$F_1'(x) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0.$$

For $n \in \{0, 1\}$, formulas (4.47) and (4.49) imply that $F_1'(x)$ and $G_1(x)$ have the same sign, for all $x > 0$. Since the sign of $G_1(x)$ is the same as the sign of $K_1(t)$, for $t = (x+1)/x$, formula (5.88) implies:

$$F_1'(x) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0,$$

for all $x > 0$.
Therefore, in both cases $n \notin \{0, 1\}$ and $n \in \{0, 1\}$, for all $x > 1$, we have:

$$F_1'(x) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) >} \quad 0. \tag{5.91}$$

Thus, for all $n \in \mathbb{R}\backslash\{-1, 1/2, 2\}$, the function F_1 is $sgn(n+1)sgn(n-1/2)sgn(n-2)creasing$ on $(0, \infty)$. Since, according to Observation 2, $F_1(\infty) = 0$, for all $x > 1$, we have:

$$\begin{aligned} F_1(x) \quad & \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) <} \quad F(\infty) \\ & = \quad 0. \end{aligned} \tag{5.92}$$

Because $F_1(x) = S_n(x, x+1) - H_{h(n)}(x, x+1)$, we conclude that
$S_n(x,\, x+1)$ $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) <}$ $H_{h(n)}(x,\, x+1)$, for all $x > 0$. This proves that for all a and b positive numbers, $a \neq b$, we have
$S_n(a, b)$ $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) <}$ $H_{h(n)}(a, b)$.
Therefore, the $(-1)sgn(n+1)sgn(n-1/2)sgn(n-2)est\ q = q(n)$ such that for all a and b positive numbers, we have:

$$S_n(a,b) \leq H_q(a,b)$$

satisfies the inequality:

$$q(n) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \leq} \quad f(n) = \frac{n+1}{3}. \tag{5.93}$$

We will show now the opposite inequality:

$$q(n) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \geq} \quad f(n) = \frac{n+1}{3}. \tag{5.94}$$

Indeed, suppose that q is a real number, such that, for all a and b positive numbers, we have:

$$S_n(a,b) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \leq} \quad H_q(a,b).$$

Case 1. If $n \neq 1$, then choosing $a = 1$ and $b = x$ arbitrarily positive, and raising both sides of the above inequality to the power $n-1$, we obtain:

$$\frac{x^n - 1}{n(x-1)} \quad \boxed{sgn(n-1)\boxed{sgn\left((n+1)(n-1/2)(n-2)\right) \leq}} \quad \left(\frac{x^q+1}{2}\right)^{(n-1)/q}. \tag{5.95}$$

We define the functions:

$$u(x) \;:=\; \frac{x^n - 1}{n(x-1)} \tag{5.96}$$

and

$$v(x) \;:=\; \left(\frac{x^q+1}{2}\right)^{(n-1)/q}. \tag{5.97}$$

These functions are defined on $(0, \infty)$, since even though the formula of $u(x)$ seems to not make sense for $x = 1$, we can see that the function $z \mapsto (z^n - 1)/(n(z-1))$ has an analytic extension at $z = 1$. Moreover, for $n = 0$, $u(x) := \ln(x)/(x-1)$, extended by continuity at $x = 1$, as $u(1) = 1$.
For all $x \in (0, \infty)$, we have:

$$u(x) \quad \boxed{sgn\left((n-1)(n+1)(n-1/2)(n-2)\right) \leq} \quad v(x). \tag{5.98}$$

Moreover, we have:

$$\begin{aligned}
u(1) &= \lim_{x\to 1} \frac{x^n - 1}{n(x-1)} \\
&= \frac{1}{n}\frac{d}{dx}_{|x=1}(x^n) \\
&= \frac{1}{n} \cdot n x^{n-1}_{|x=1} \\
&= 1 && (5.99)\\
&= v(1). && (5.100)
\end{aligned}$$

Since 1 is in the interior of $(0, \infty)$, applying Lemma 4.1, we conclude that:

$$u''(1) \quad \boxed{sgn\left((n-1)(n+1)(n-1/2)(n-2)\right) \leq} \quad v''(1). \tag{5.101}$$

Writing $x = 1 + t$, for $t \in (-1, 1)$, and using the binomial expansion, we have:

$$\begin{aligned}
u(x) &= u(1+t) \\
&= \frac{(1+t)^n - 1}{nt} && (5.102)\\
&= \frac{1}{nt}\left[1 + nt + \frac{n(n-1)}{2}t^2 + \frac{n(n-1)(n-2)}{3!}t^3 + \cdots - 1\right] \\
&= 1 + \frac{n-1}{2}t + \frac{(n-1)(n-2)}{6}t^2 + \cdots \\
&= 1 + \frac{n-1}{2}(x-1) + \frac{(n-1)(n-2)}{6}(x-1)^2 + \cdots. && (5.103)
\end{aligned}$$

We can see from here that:

$$u''(1) \;=\; \frac{(n-1)(n-2)}{3}. \tag{5.104}$$

On the other hand, for all $x > 0$, we have:

$$\begin{aligned} v'(x) &= \frac{(n-1)}{q}\left(\frac{x^q+1}{2}\right)^{(n-1-q)/q} \cdot \frac{q}{2}x^{q-1} \\ &= \frac{(n-1)}{2^{(n-1)/q}}x^{q-1}\left(x^q+1\right)^{(n-1-q)/q}. \end{aligned} \tag{5.105}$$

Differentiating one more time, for all $x > 0$, we have:

$$\begin{aligned} v''(x) &= \frac{(n-1)}{2^{(n-1)/q}}\Big[(q-1)x^{q-2}\left(x^q+1\right)^{(n-1-q)/q} \\ &\qquad +(n-1-q)x^{2(q-1)}(x^q+1)^{(n-1-2q)/q}\Big]. \end{aligned} \tag{5.106}$$

Setting $x = 1$, we obtain:

$$\begin{aligned} v''(1) &= \frac{(n-1)}{2^{(n-1)/q}}\left((q-1)2^{(n-1-q)/q} + (n-1-q)2^{(n-1-2q)/q}\right) \\ &= \frac{(n-1)}{2^{(n-1)/q}} \cdot 2^{(n-1-2q)/q}\left[2(q-1)+n-1-q\right] \\ &= \frac{n-1}{4}(n+q-3). \end{aligned} \tag{5.107}$$

Inequality $u''(1)\ \boxed{sgn\left((n-1)(n+1)(n-1/2)(n-2)\right) \leq}\ v''(1)$ becomes now:

$$\frac{(n-1)(n-2)}{3} \quad \boxed{sgn\left((n-1)(n+1)(n-1/2)(n-2)\right) \leq} \quad \frac{(n-1)(n+q-3)}{4}.$$

Multiplying both sides of this inequality by the number $12/(n-1)$, we obtain:

$$4(n-2) \quad \boxed{sgn(n-1)\boxed{sgn\left((n-1)(n+1)(n-1/2)(n-2)\right) \leq}} \quad 3(n+q-3).$$

That means,

$$4(n-2) \quad \boxed{sgn\left((n+1)(n-1/2)(n-2)\right) \leq} \quad 3(n+q-3). \tag{5.108}$$

Solving this inequality for q, we obtain:

$$q \quad \boxed{sgn\left((n+1)(n-1/2)(n-2)\right) \geq} \quad \frac{n+1}{3}. \tag{5.109}$$

Therefore, we have proved that the $\boxed{(-1)sgn((n+1)(n-1/2)(n-2))}est\ q = q(n)$ is:

$$q(n) = \frac{n+1}{3}. \tag{5.110}$$

Case 2. If $n = 1$, then $sgn(n+1)sgn(n-1/2)sgn(n-2) = -1$, and we know that for all a and b positive numbers, we have:

$$S_1(a,b) \geq H_q(a,b). \tag{5.111}$$

Choosing $a = 1$ and $b = x$ arbitrarily positive, the above inequality becomes:

$$\frac{1}{e}\left(\frac{x^x}{1^1}\right)^{1/(x-1)} \geq \left(\frac{1^q+x^q}{2}\right)^{1/q}. \tag{5.112}$$

Applying $\ln$ to both sides, we conclude that for all $x > 0$, we have:

$$-1 + \frac{x}{x-1} \cdot \ln(x) \geq \frac{1}{q}\ln\left(x^q+1\right) - \frac{\ln(2)}{q}. \tag{5.113}$$

Writing $x/(x-1) = 1 + [1/(x-1)]$, we obtain:

$$-1 + \ln(x) + \frac{\ln(x)}{x-1} \;\geq\; \frac{1}{q}\ln\left(x^q + 1\right) - \frac{\ln(2)}{q}, \tag{5.114}$$

for all $x > 0$, where the value of $\ln(x)/(x-1)$ at $x = 1$, is understood as:

$$\lim_{x\to 1}\frac{\ln(x)}{x-1} \;=\; 1. \tag{5.115}$$

In fact, using the Taylor expansion of the function $x \mapsto \ln(x)$ around the point $x = 1$, we have:

$$\frac{\ln(x)}{x-1} \;=\; 1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \cdots, \tag{5.116}$$

for all $x \in (1, 2)$, from which we can see that:

$$\frac{d^2}{dx^2}_{|x=1}\left[\frac{\ln(x)}{x-1}\right] \;=\; \frac{2}{3}. \tag{5.117}$$

Defining the functions $u, v : (0, \infty) \to \mathbb{R}$

$$u(x) \;:=\; -1 + \ln(x) + \frac{\ln(x)}{x-1} \tag{5.118}$$

and

$$v(x) \;:=\; \frac{1}{q}\ln\left(x^q + 1\right) - \frac{\ln(2)}{q}, \tag{5.119}$$

we have that, for all $x > 0$, $u(x) \geq v(x)$. Since $u(1) = v(1) = 0$, Lemma 4.1 implies:

$$u''(1) \;\geq\; v''(1). \tag{5.120}$$

We have:

$$\begin{aligned} u''(1) &= \frac{d^2}{dx^2}_{|x=1}(\ln(x)) + \frac{d^2}{dx^2}_{|x=1}\left(\frac{\ln(x)}{x-1}\right) \\ &= -\frac{1}{x^2}_{|x=1} + \frac{2}{3} \\ &= -1 + \frac{2}{3} \\ &= -\frac{1}{3}. \end{aligned} \tag{5.121}$$

On the other hand, we have:

$$\begin{aligned} v''(1) &= \frac{1}{q}\frac{d^2}{dx^2}_{|x=1}\left(\ln\left(x^q + 1\right)\right) \\ &= \frac{1}{q}\frac{q(q-1)x^{q-2}(x^q+1) - q^2x^{2q-1}}{(x^q+1)^2}_{|x=1} \\ &= \frac{q-2}{4}. \end{aligned} \tag{5.122}$$

The inequality $u''(1) \geq v''(1)$ becomes now:

$$-\frac{1}{3} \;\geq\; \frac{q-2}{4}. \tag{5.123}$$

Solving this inequality, we obtain:

$$\begin{aligned} q &\leq \frac{2}{3} \\ &= \frac{1+1}{3} \\ &= f(1). \end{aligned} \tag{5.124}$$

Therefore, we obtain:

$$q(1) \leq f(1), \tag{5.125}$$

which together with the previous inequality $q(1) \geq f(1)$, imply the fact that the optimal (the greatest) $q = q(1)$, for which:

$$S_1(a,b) \geq H_q(a,b), \tag{5.126}$$

for all positive numbers a and b, is

$$q(1) = f(1) = \frac{2}{3}. \tag{5.127}$$

The proof of our proposition is now complete. Moreover, we have proven that for every $n \in \mathbb{R} \setminus \{-1, 1/2, 2\}$, the equality:

$$S_n(a,b) = H_{f(n)}(a,b) \tag{5.128}$$

occurs if and only if:

$$a = b. \tag{5.129}$$

∎

Proposition 5.7. *For all $n \in \mathbb{R} \setminus \{-1, 1/2, 2\}$, the $(sgn(n+1)sgn(n-1/2)sgn(n-2))est$ $p = p(n)$, such that for all a and b positive numbers, we have:*

$$S_n(a,b) \boxed{(sgn(n+1)sgn(n-1/2)sgn(n-2)) \geq} H_p(a,b), \tag{5.130}$$

is $p = g(n)$, where:

$$g(n) = \begin{cases} \frac{(n-1)\ln(2)}{\ln(n)} & \text{if} \quad n > 0, n \neq 1 \\ \ln(2) & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \leq 0 \end{cases}.$$

Proof. Let $h(n) := g(n) = (n-1)\ln(2)/\ln(n)$ for $n > 0$, $n \neq 1$, extended by continuity at $n = 1$ and defined to be identically zero for $n \leq 0$.

We prove first that $p(n) \boxed{(sgn(n+1)sgn(n-1/2)sgn(n-2)) \geq} g(n)$.

Case 1. If $n > 0$ and $n \neq 1$, then it follows from Proposition 5.1 and formula (4.80), that for all $t > 1$, we have:

$$\begin{aligned} K_5'(t) &= [n + h(n) - 1]\, h(n)\, [h(n) - 1]\, t^{h(n)-2} \\ &\boxed{sgn(n-1/2)sgn(n-2) >} 0. \end{aligned} \tag{5.131}$$

If $n = 1$, we have $K_5'(t) = \ln(2)t^{\ln(2)-1} > 0$, for all $t > 1$.

Since $K_5' = K_4''$, we conclude that:

If $n > 0$ and $n \neq 1$, then K_4 is strictly $\boxed{sgn(n-1/2)sgn(n-1)}vex$ on $[1, \infty)$.

If $n = 1$, then K_4 is strictly convex on $[1, \infty)$.

If $n > 0$ and $n \neq 1$, then since, according to Proposition 5.2, we have

$K_4(1)\ \boxed{sgn(n-1/2)sgn(n-2) <}\ 0$ and $K_4(\infty)\ \boxed{sgn(n-1/2)sgn(n-2) >}\ 0$, and K_4 is continuous on $[1, \infty)$, by the Intermediate Value Theorem, there exists $t_4 \in (1, \infty)$, such that:

$$K_4(t_4) = 0. \tag{5.132}$$

Moreover, since K_4 is strictly $\boxed{sgn(n-1/2)sgn(n-1)}vex$ on $[1, \infty)$, the point t_4 is unique with the above property. Indeed, assuming that there exists $t_4' \neq t_4$ in $(1, \infty)$, such that:

$$K_4(t_4') = 0,$$

then the graph of K_4 will be either completely below or completely above the line $y = 0$ joining the points $(t_4, 0)$ and $(t_4', 0)$, for t in between t_4 and t_4', and the other way around the same line outside of the interval (t_4, t_4') or (t_4', t_4) (it depends on which of t_4 and t_4' is greater). Since 1 and ∞ (here by ∞, we mean a sufficiently large t) are for sure outside of this interval, this will imply that $K_4(1)K_4(\infty) > 0$, which is false.

The uniqueness of t_4, and the fact that $K_4(1)\ \boxed{sgn(n-1/2)sgn(n-2) <}\ 0$ and $K_4(\infty)\ \boxed{sgn(n-1/2)sgn(n-2) >}\ 0$, imply that:

- For all $t \in (1, t_4)$, $K_4(t)\ \boxed{sgn(n-1/2)sgn(n-2) <}\ 0$.
- For all $t \in (t_4, \infty)$, $K_4(t)\ \boxed{sgn(n-1/2)sgn(n-2) >}\ 0$.

If $n = 1$, then since, according to Proposition 5.2, we have $K_4(1) > 0$ and $K_4(\infty) < 0$, there exists a point $t_4 \in (1, \infty)$, such that $K_4(t_4) = 0$. Since K_4 is strictly convex on $[1, \infty)$, t_4 is unique and we have;

- For all $t \in (1, t_4)$, $K_4(t) > 0$.
- For all $t \in (t_4, \infty)$, $K_4(t) < 0$.

For $n > 0$ and $n \neq 1$, formula (4.69),

$$K_3'(t) = (n-1)t^{n-h(n)-2}K_4(t),$$

implies now that $K_3'(t)$ has the same sign as $(n-1)K_4(t)$, for all $t > 1$. Hence,

- For all $t \in (1, t_4)$, $K_3'(t)\ \boxed{sgn(n-1)\boxed{sgn(n-1/2)sgn(n-2) <}}\ 0$.
- For all $t \in (t_4, \infty)$, $K_3'(t)\ \boxed{sgn(n-1)\boxed{sgn(n-1/2)sgn(n-2) >}}\ 0$.

Thus, the function K_3 is $(-1)sgn(n-1)sgn(n-1/2)sgn(n-2)creasing$ on the interval $[1, t_4]$ and $sgn(n-1)sgn(n-1/2)sgn(n-2)creasing$ on $[t_4, \infty)$.
Since we know from Observation 1, that $K_3(1) = 0$, we conclude that $K_3(t)\ \boxed{(-1)sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ K_3(1) = 0$, for all $t \in (1, t_4]$. That means, $K_3(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$, for all $t \in (1, t_4]$.
We also know from Proposition 5.3 that $K_3(\infty)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$. Thus $K_3(t_4)$ and $K_3(\infty)$ have opposite signs. This combined with the fact that K_3 is continuous and strictly monotone on $[t_4, \infty)$ imply that there exists a unique $t_3 \in (t_4, \infty)$, such that $K_3(t_3) = 0$. Moreover, we have:

- For all $t \in [t_4, t_3)$, $K_3(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$.
- For all $t \in (t_3, \infty)$, $K_3(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$.

If $n = 1$, then since, according to formula (4.73), $K_3'(t) = K_4(t)$, we conclude that $K_3'(t)$ has the same sign as $K_4(t)$, for all $t > 1$. Thus K_3 is strictly increasing on $[1, t_4]$ and strictly decreasing on $[t_4, \infty)$. Since $K_3(1) = 0$ we have $K_3(t_4) > K_3(1) = 0$. Since $sgn(K_3(\infty)) = -1$, there exists a unique $t_3 \in (t_4, \infty)$, such that $K_3(t_3) = 0$. Moreover, we have:

- For all $t \in (1, t_3)$, $K_3(t) > 0$.
- For all $t \in (t_3, \infty)$, $K_3(t) < 0$.

If $n > 0$ and $n \neq 1$, then since formula (4.64) says that $K_2'(t) = nK_3(t)$, we conclude that for all $t > 1$, $K_2'(t)$ has the same sign as $K_3(t)$. Thus K_2 is strictly $\boxed{(-1)sgn(n-1)sgn(n-1/2)sgn(n-2)}creasing$ on $[1, t_3]$, and strictly $\boxed{sgn(n-1)sgn(n-1/2)sgn(2)}creasing$ on $[t_3, \infty)$. Since, according to Observation 1, $K_2(1) = 0$, we conclude that $K_2(t_3)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$. Now, due to Proposition 5.4, we have $K_2(\infty)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$. Since $K_2(t_3)$ and $K_2(\infty)$ have opposite signs and K_2 is continuous and strictly monotone on $[t_3, \infty)$, there exists a unique $t_2 \in (t_3, \infty)$, such that $K_2(t_2) = 0$. Moreover, we have:

- For all $t \in (1, t_2)$, $K_3(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$.
- For all $t \in (t_2, \infty)$, $K_3(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$.

If $n = 1$, then according to formula(4.67), $K_2'(t)$ and $K_3(t)$ have the same sign for all $t > 1$. Thus, K_2 is strictly increasing on $[1, t_3]$ and strictly decreasing on $[t_3, \infty)$. Since, according to Observation 1, $K_2(1) = 0$, we conclude that $K_2(t_3) > K_2(1) = 0$. Proposition 5.4 tells us that $K_2(\infty) < 0$. Since K_2 is continuous and strictly monotone on $[t_3, \infty)$, the Darboux Property implies that there exists a unique $t_2 \in (t_3, \infty)$, such that $K_2(t_2) = 0$. Moreover, we have:

- For all $t \in (1, t_2)$, $K_2(t) > 0$.
- For all $t \in (t_2, \infty)$, $K_2(t) < 0$.

If $n > 0$ and $n \neq 1$, then according to formula (4.59), $K_1'(t)$ has the same sign as $K_2(t)$. Thus, K_1 is strictly $\boxed{(-1)sgn(n-1)sgn(n-1/2)sgn(n-2)}creasing$ on $[1, t_2]$, and strictly $\boxed{sgn(n-1)sgn(n-1/2)sgn(n-2)}creasing$ on $[t_2, \infty)$. Since, according to Observation 1, $K_1(1) = 0$, we conclude that: $K_1(t_2)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$. However, we know from Proposition 5.5, that: $K_1(\infty)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$. The continuity of K_1 and its strict monotonicity on $[t_3, \infty)$ imply that there exists a unique $t_1 \in (t_2, \infty)$, such that $K_1(t_1) = 0$. Moreover, we have:

- For all $t \in (1, t_1)$, $K_1(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}\ 0$.
- For all $t \in (t_2, \infty)$, $K_1(t)\ \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}\ 0$.

If $n = 1$, formula (4.63) implies that for all $t > 1$, $K_1'(t)$ has the same sign as $K_2(t)$. Thus K_1 is strictly increasing on $[1, t_2]$, and strictly decreasing on $[t_2, \infty)$. Since, according to Observation 1, $K_2(1) = 0$, we conclude that $K_2(t_2) > K_2(1) = 0$. On the other hand, Proposition 5.5 tells us that $K_1(\infty) < 0$. The continuity and strict monotonicity of K_1 on $[t_2, \infty)$ imply that there exists a unique $t_1 \in (t_2, \infty)$, such that $K_1(t_1) = 0$. Moreover, we have:

- For all $t \in (1, t_1)$, $K_1(t) > 0$.
- For all $t \in (t_1, \infty)$, $K_1(t) < 0$.

If $n > 0$ and $n \neq 1$, according to formula (4.50), for all $x > 0$, $G_1(x)$ has the same sign as $K_1(t)$, where $t := (x + 1)/x$, or equivalently $x = 1/(t - 1)$. Since for $t \in (1, t_1)$, we have $x \in (1/(t_1 - 1), \infty)$, while for $t \in (t_1, \infty)$, we have $x \in (0, 1/(t_1 - 1))$, defining $x_1 := 1/(t_1 - 1)$, we conclude that:

- For all $x \in (0, x_1)$, $G_1(x) \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >} 0$.
- For all $x \in (x_1, \infty)$, $G_1(x) \boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <} 0$.

If $n = 1$, then according to formula (4.56), $G_1(x)$ has the same sign as $K_1(t)$, where $t := (x+1)/x > 1$. Doing the same reasoning as before, we conclude that:

- For all $x \in (0, x_1)$, $G_1(x) < 0$.
- For all $x \in (x_1, \infty)$, $G_1(x) > 0$.

If $n > 0$ and $n \neq 1$, the according to formula (4.45), for all $x > 0$, we have:

$$G_1(x) \quad = \quad (n-1)\left[(x+1)^n - x^n\right]\left[(x+1)^{h(n)} + x^{h(n)}\right] F_1'(x).$$

Since $x + 1 > x > 0$ and $n > 0$, we conclude that $(x+1)^n - x^n > 0$, for all $x > 0$. Thus, for all $x > 0$, $F_1'(x)$ has the same sign as $(n-1)G_1(x)$. Therefore, we have:

- For all $x \in (0, x_1)$,
  $G_1(x) \boxed{sgn(n-1)\boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) >}} 0$.
- For all $x \in (x_1, \infty)$,
  $G_1(x) \boxed{sgn(n-1)\boxed{sgn(n-1)sgn(n-1/2)sgn(n-2) <}} 0$.

Since, $sgn^2(n-1) = +1$, we conclude that:

- For all $x \in (0, x_1)$, $G_1(x) \boxed{sgn(n-1/2)sgn(n-2) >} 0$.
- For all $x \in (x_1, \infty)$, $G_1(x) \boxed{sgn(n-1/2)sgn(n-2) <} 0$.

Thus, F_1 is strictly $\boxed{sgn(n-1/2)sgn(n-2)}creasing$ on $(0, x_1)$, and strictly $\boxed{(-1)sgn(n-1/2)sgn(n-2)}creasing$ on (x_1, ∞).
Since $n > 0$, we have $n + 1 > 0$, so $sgn(n+1) = +1$. Thus we can include $sgn(n+1)$ in our inequalities without affecting the inequality symbols.
This implies that for all $x \in (0, \infty)$, we have:

$$\begin{aligned} & \boxed{(-1)sgn(n+1)sgn(n-1/2)sgn(n-2)}mum\{F_1(x) \mid x > 0\} \\ = & \left\{\lim_{x\to 0^+} F_1(x), \lim_{x\to\infty} F_1(x)\right\} \\ = & \boxed{(-1)sgn(n+1)sgn(n-1/2)sgn(n-2)}mum\left\{\lim_{x\to 0^+} F_1(x), 0\right\}, \end{aligned} \tag{5.133}$$

since we saw in Observation 2 that, for every function h, we have $\lim_{x\to\infty} F_1(x) = 0$.
Therefore, the statement:

$$F_1(x) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \geq} \quad 0, \tag{5.134}$$

for all $x > 0$, holds true if and only if:

$$\begin{aligned} F_1(0) & := \lim_{x\to 0^+} F_1(x) \\ & \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \geq} \quad 0. \end{aligned} \tag{5.135}$$

Since for $n > 0$, $n \neq 1$, $h(n) > 0$, we have:

$$\begin{aligned} F_1(0) &= \lim_{x\to 0^+} \left\{ \frac{1}{n-1} \ln\left((x+1)^n - x^n\right) - \frac{\ln(n)}{n-1} \right\} \\ &\quad - \lim_{x\to 0^+} \left\{ \frac{1}{h(n)} \ln\left((x+1)^{h(n)} + x^{h(n)}\right) - \frac{\ln(2)}{h(n)} \right\} \\ &= 0 - \frac{\ln(n)}{n-1} - 0 + \frac{\ln(2)}{h(n)}. \end{aligned} \tag{5.136}$$

Because for all $n > 0$, $n \neq 1$, we have $(n-1)/\ln(n) > 0$, we can see from here that $F_1(0)$ $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \geq}$ 0 if and only if:

$$\begin{aligned} h(n) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2) \leq} \quad & \frac{(n-1)\ln(2)}{\ln(n)} \\ = \quad & g(n). \end{aligned} \tag{5.137}$$

If $n = 1$, then according to formula (4.49), we have:

$$G_1(x) = \left[(x+1)^{h(1)} + x^{h(1)}\right] F_1'(x).$$

This implies that for all $x > 0$, $F_1'(x)$ has the same sign as $G_1(x)$. Thus F_1 is strictly decreasing on $[1, x_1]$, and strictly increasing on $[x_1, \infty)$. Therefore, we have:

$$\begin{aligned} \sup\{F_1(x) \mid x > 0\} &= \max\{F_1(0), F_1(\infty)\} \\ &= \max\{F_1(0), 0\}. \end{aligned} \tag{5.138}$$

It follows from here that the statement:

$$F_1(x) \leq 0, \tag{5.139}$$

for all $x > 0$, holds true if and only if:

$$\begin{aligned} F_1(0) &:= \lim_{x\to 0^+} F_1(x) \\ &\leq 0. \end{aligned} \tag{5.140}$$

Since, if $h(n) > 0$, then

$$\begin{aligned} F_1(0) &= \lim_{x\to 0^+} \left\{ \ln\left(\frac{1}{e} \cdot \frac{(x+1)^{x+1}}{x^x} \right) \right\} \\ &\quad - \frac{1}{h(n)} \lim_{x\to 0^+} \left\{ \ln\left(\frac{(x+1)^{h(n)} + x^{h(n)}}{2} \right) \right\} \\ &= -1 + \lim_{x\to 0^+} \left[(x+1)\ln(x+1)\right] - \lim_{x\to 0^+} \left[x \ln(x)\right] + \frac{\ln(2)}{h(n)} \\ &= -1 + \frac{\ln(2)}{h(n)}, \end{aligned} \tag{5.141}$$

while if $h(n) \leq 0$, we have:

$$F_1(0) = \infty, \tag{5.142}$$

we conclude that $F_1(0) \leq 0$ if and only if

$$\begin{aligned} h(n) &\geq \ln(2) \\ &= g(1). \end{aligned} \tag{5.143}$$

Thus, for all $n > 0$, the $\boxed{sgn(n+1)sgn(n-1/2)sgn(n-2))}est\ p = p(n)$, such that for all a and b positive, we have:

$$S_n(a,b) \quad \boxed{sgn(n+1)sgn(n-1/2)sgn(n-2)) \geq} \quad H_p(a,b) \tag{5.144}$$

is $p(n) = g(n) = (n-1)\ln(2)/\ln(n)$. The equality in the above inequality holds if and only if $a = b$.

If $-1 < n \leq 0$, then for all a and b positive, we have:

$$\begin{aligned} S_n(a,b) &\geq S_{-1}(a,b) \\ &= H_0(a,b). \end{aligned} \tag{5.145}$$

The equality happens if and only if $a = b$. Thus $p(n) \geq 0$.

To show that $p(n) \leq 0$, we show that if $p > 0$, then the inequality:

$$S_n(a,b) \geq H_p(a,b) \tag{5.146}$$

cannot hold for all a and b positive. Indeed, assuming that it holds for all a and b positive, then let $a = 1$ and $b = x > 0$. Letting $x \to 0^+$, we obtain:

$$\lim_{x\to 0^+} S_n(1,x) \geq \lim_{x\to 0^+} H_p(1,x). \tag{5.147}$$

For $n = 0$, the above inequality becomes:

$$\lim_{x\to 0^+} \frac{1-x}{\ln(1)-\ln(x)} \geq \lim_{x\to 0^+} \left(\frac{1+x^p}{2}\right)^{1/p}. \tag{5.148}$$

That means:

$$0 \geq \frac{1}{2^{1/p}}, \tag{5.149}$$

which is clearly a contradiction.

For $-1 < n < 0$, the inequality (5.147) becomes:

$$\lim_{x\to 0^+} \left(\frac{1-x^n}{n(1-x)}\right)^{1/(n-1)} \geq \lim_{x\to 0^+} \left(\frac{1+x^p}{2}\right)^{1/p}. \tag{5.150}$$

Since $n < 0$, $x^n \to \infty$, as $x \to 0^+$, and since $n - 1 < 0$, the limit from the left is 0, while the limit from the right is a strictly positive number. Thus, the above inequality is impossible.

If $n < -1$, the for all a and b positive, we have:

$$\begin{aligned} S_n(a,b) &\leq S_{-1}(a,b) \\ &= H_0(a,b). \end{aligned} \tag{5.151}$$

The equality happens if and only if $a = b$. Thus $p(n) \leq 0$.

To show that $p(n) \geq 0$, we show that if $p < 0$, then the inequality:

$$S_n(a,b) \leq H_p(a,b) \tag{5.152}$$

cannot hold for all a and b positive. Indeed, assuming that it holds for all a and b positive, then let $a = 1$ and $b = x > 0$. Letting $x \to \infty$, we obtain:

$$\lim_{x\to\infty} S_n(1,x) \leq \lim_{x\to\infty} H_p(1,x). \tag{5.153}$$

This inequality becomes:

$$\lim_{x\to\infty} \left(\frac{x^n-1}{n(x-1)}\right)^{1/(n-1)} \leq \lim_{x\to\infty} \left(\frac{x^p+1}{2}\right)^{1/p}. \tag{5.154}$$

That means $\infty \leq 2^{-1/p}$, which is a contradiction. The proof is now complete. ∎

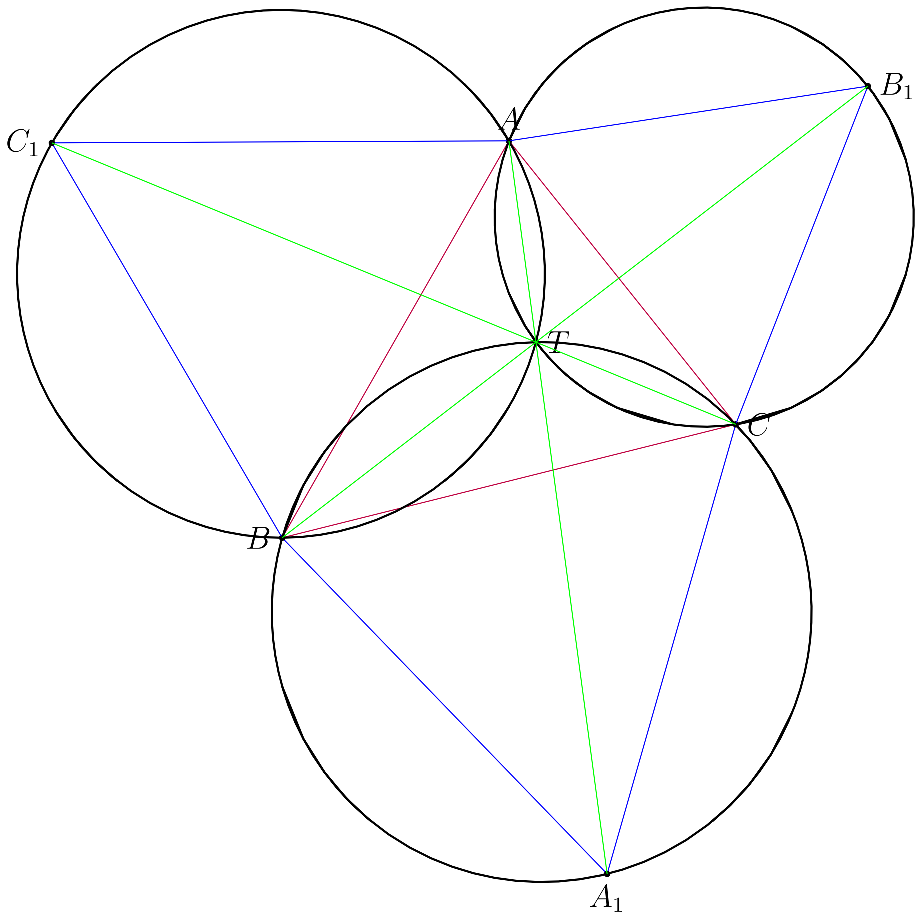

Figure 1: M1

6. APPLICATION TO THE FERMAT-TORRICELLI POINT OF A TRIANGLE

We end the paper with an application.
Let us consider a triangle ABC in which the measure of each interior angle is less than or equal to $120°$. Then the point T, for which the sum of the distances $|TA| + |TB| + |TC|$ is minimum possible, is the Fermat-Torricelli point. This point is denoted by $X(13)$ in [7] and is constructed in the following way:
Step 1. Construct equilateral triangles A_1BC, B_1CA, and C_1AB on the sides of the triangle ABC, in the exterior of the triangle.
Step 2. The circles circumscribed to the triangles A_1BC, B_1CA, and C_1AB are concurrent at a point T, which is the Fermat-Torricelli point.
Alternatively, the lines AA_1, BB_1, and CC_1 are concurrent at one point T, which is the Fermat-Torricelli point. See Figure 1 above.
Let us observe that since the quadrilateral TBA_1C is cyclic, we have:

$$\begin{aligned} m(\sphericalangle BTC) &= 180° - 60° \\ &= 120°. \end{aligned} \tag{6.1}$$

Denoting the length of the sides $|BC|$, $|CA|$, and $|AB|$, of the triangle ABC, by a, b, and c, respectively, and applying the Law of Cosines in the triangle TBC, we obtain:

$$\begin{aligned} a^2 &= |BC|^2 \\ &= |TB|^2 + |TC|^2 - 2|TB||TC|\cos(\sphericalangle BTC) \\ &= |TB|^2 + |TC|^2 - 2|TB||TC|\cos(120^\circ) \\ &= |TB|^2 + |TC|^2 + |TB||TC| \\ &= \frac{|TC|^3 - |TB^3|}{|TC| - |TB|} \\ &= 3\left[S_3(|TB|, |TC|)\right]^2. \end{aligned} \tag{6.2}$$

Thus we obtain,

$$a = S_3(|TB|, |TC|)\sqrt{3}. \tag{6.3}$$

Applying the inequality that we just proved, for $n = 3$, we obtain:

$$H_{2\ln(2)/\ln(3)}(|TB|, |TC|)\sqrt{3} \le a \le H_{4/3}(|TB|, |TC|)\sqrt{3}. \tag{6.4}$$

Raising both sides of the inequality:

$$H_{\log_3(4)}(|TB|, |TC|) \le 3^{-1/2}a \tag{6.5}$$

to the positive power $\log_3(4)$, and both sides of

$$3^{-1/2}a \le H_{4/3}(|TB|, |TC|) \tag{6.6}$$

to the positive power $4/3$, we obtain:

$$\frac{|TB|^{\log_3(4)} + |TC|^{\log_3(4)}}{2} \le \frac{1}{2}a^{\log_3(4)}, \tag{6.7}$$

and

$$\frac{|TB|^{4/3} + |TC|^{4/3}}{2} \ge \frac{\sqrt[3]{3}}{3}a^{4/3}, \tag{6.8}$$

with the equality if and only if $|TB| = |TC|$, which is equivalent to $b = c$. The last statement follows from the fact that since $|A_1B| = |A_1C|$ (as sides of the equilateral triangle A_1BC), A_1 belongs to the perpendicular bisector d of the segment $|BC|$. Therefore, we have $|TB| = |TC|$, if and only if the perpendicular bisector of $|BC|$ is the line TA_1, which is the same as the line AA_1 (since A, T, and A_1 are collinear). This is equivalent to the fact that the vertex A of the triangle ABC belongs to the perpendicular bisector of $|BC|$. That means $|AC| = |AB|$. Similarly, we obtain the inequalities:

$$\frac{|TC|^{\log_3(4)} + |TA|^{\log_3(4)}}{2} \le \frac{1}{2}b^{\log_3(4)},, \tag{6.9}$$

$$\frac{|TC|^{4/3} + |TA|^{4/3}}{2} \ge \frac{\sqrt[3]{3}}{3}b^{4/3}, \tag{6.10}$$

$$\frac{|TA|^{\log_3(4)} + |TB|^{\log_3(4)}}{2} \le \frac{1}{2}c^{\log_3(4)}, \tag{6.11}$$

and

$$\frac{|TA|^{4/3} + |TB|^{4/3}}{2} \ge \frac{\sqrt[3]{3}}{3}c^{4/3}. \tag{6.12}$$

Summing up inequalities (6.7), (6.9), and (6.11), and inequalities (6.8), (6.10), and (6.12), we obtain the following:

Lemma 6.1. *In any triangle ABC, in which each interior angle has a measure less than or equal to $120°$, the following two inequalities hold:*

$$|TA|^{\log_3(4)} + |TB|^{\log_3(4)} + |TC|^{\log_3(4)} \leq \frac{1}{2}\left(a^{\log_3(4)} + b^{\log_3(4)} + c^{\log_3(4)}\right)$$

and

$$|TA|^{4/3} + |TB|^{4/3} + |TC|^{4/3} \geq \frac{\sqrt[3]{3}}{3}\left(a^{4/3} + b^{4/3} + c^{4/3}\right).$$

The equality happens in any one of these two inequalities if and only if $a = b = c$.
Here T is the Fermat-Torricelli point, and a, b, and c are the length of the sides of the triangle ABC.

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