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A NEW RELAXED *B*-METRIC TYPE AND FIXED POINT RESULTS

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ABSTRACT. The purpose of this paper is to introduce a new relaxed α, β b-metric type by relaxing the triangle inequality. We investigate the effect that this generalization has on fixed point theorems.

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1. INTRODUCTION

The concept of a b-metric was initiated from the contributions of Bourbaki [3] and Bakhtin [2]. Czerwik [4] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [5] discussed some kind of relaxation in the triangular inequality and called this new distance measure a non-linear elastic pattern matching. These applications led us to introduce the concept of a generalized b-metric type and that the results obtained for such spaces become viable in different fields.

Definition 1.1. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if there exists a real number $\alpha \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0 \iff x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq \alpha [d(x, z) + d(z, y)]$

The pair (X, d) is a called a b-metric space [9]. A b-metric with $\alpha = 1$ is exactly the usual metric.

Definition 1.2. Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}^+$ is a generalized α, β b-metric on X if there exists a real numbers $\alpha, \beta \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $\rho(x, y) = 0 \iff x = y$
(ii) $\rho(x, y) = \rho(y, x)$
(iii) $\rho(x, y) \leq \alpha\rho(x, z) + \beta\rho(z, y)$

We shall refer to (iii) as the α, β relaxed triangle inequality. The pair (X, ρ) is a called a generalized b-metric space. A generalized b-metric with $\alpha = \beta$ is exactly a b-metric. In the special case $\alpha = 1$ we obtain the a strong b-metric, [7, 9]. The following examples justify this generalization found in Definition 1.2.

Example 1.1. *Let $X = \{1, 2, 3\}$ be a discrete set and let $\rho : X \times X \to \mathbb{R}^+$ be a function defined by*

$$\rho(1,1) = \rho(2,2) = \rho(3,3) = 0$$
$$\rho(1,2) = \rho(2,1) = \frac{1}{3}$$
$$\rho(1,3) = \rho(3,1) = 3.$$

From the definition of the b-metric properties (i), (ii) are apparent. For all $x, y, z \in X$ it follows that

$$\rho(x, y) \leq 2\rho(x, z) + 3\rho(z, y).$$

Example 1.2. *Let $X = (1, 3)$ and let $\rho : X \times X \to \mathbb{R}^+$ be a function defined by*

$$\rho(x, y) = \begin{cases} e^{|x-y|}, & \text{if } x \neq y \\ 0, & \text{iff } x = y. \end{cases}$$

The first two properties of a generalized b-metric are inherent in the definition. We verify the α, β triangle inequality as follows:

For $x \neq y,\ z \in X$ *and* $\theta \in (0,1)$

$$\begin{aligned}\rho(x,y) &\leq e^{|x-z|+|z-y|}\\ &= e^{\theta|x-z|+(1-\theta)|z-y|}e^{(1-\theta)|x-z|+\theta|z-y|}\\ &\leq \sup_{x,y,z\in X} e^{\theta|x-z|+(1-\theta)|z-y|}\left((1-\theta)e^{|x-z|}+\theta e^{|z-y|}\right)\\ &\leq (1-\theta)e^2e^{|x-z|}+\theta e^2 e^{|z-y|}\\ &= (1-\theta)e^2\rho(x,z)+\theta e^2\rho(z,y).\end{aligned}$$

For $\theta = \frac{1}{3}$ *we have constants* $\alpha = \frac{2}{3}e^2$ *and* $\beta = \frac{1}{3}e^2$.

2. TOPOLOGICAL PROPERTIES OF THE GENERALIZED b-METRIC TYPE.

One introduces a topology on a generalized b-metric space (X,ρ) in the usual way. The open ball $B(x,r)$ with centre $x \in X$ and radius $r > 0$ is given by

$$B(x,r) = \{y \in X : \rho(x,y) < r\}$$

A subset A of X is open if for every $x \in A$ there is a number $r > 0$ such that $B(x,r) \subseteq A$. Denoting by τ_ρ the family of all open subsets of X it follows that τ_ρ satisfies the axioms of a topology.

Let (X,ρ) be a generalized b-metric space. Then the b-metric is continuous if
$\rho(x_n,x) \to 0$, $\rho(y_n,y) \to 0$ as $n \to \infty$ implies $\rho(x_n,y_n) \to 0$.

Furthermore, the b-metric is separately continuous if for every $x \in X$,
$\rho(y_n,y) \to 0$ as $n \to \infty$ implies $\rho(x,y_n) \to \rho(x,y)$.

The topology τ_ρ generated by a generalized b-metric ρ has a peculiar property in that a ball $B(x,r)$ need not be τ_ρ- open as illustrated by the following example, [8].

Example 2.1. *Let* $X = \mathbb{Z}^+ \cup \{0\}$, $\epsilon > 0$ *and define* $\rho : X \times X \to \mathbb{R}^+$ *by*

$$\begin{aligned}\rho(0,1) &= 1\\ \rho(1,m) &= \frac{1}{m}\\ \rho(0,m) &= 1+\epsilon \ \text{ for } \ m \geq 2\\ \rho(n,m) &= \frac{1}{n}+\frac{1}{m} \ \text{ for } \ n \geq 2\\ \rho(n,n) &= 0.\end{aligned}$$

Then

$$\rho(m,n) \leq \alpha\rho(m,k)+\beta\rho(k,n)$$

for all $m,n,k \in X$. *The ball* $B(0,1+\frac{\epsilon}{2}) = \{0,1\}$ *and the ball* $B(1,r)$ *contains an infinite number of terms for every* $r > 0$. *Now since* $1 \in B(0,1+\frac{\epsilon}{2})$ *it follows that* $B(1,r) \not\subset B(0,1+\frac{\epsilon}{2})$ *for every* $r > 0$ *illustrating that the ball* $B(0,1+\frac{\epsilon}{2})$ *is not* τ_ρ *open.*

Let X be a non-empty set and $B : X \times (0,\infty) \to \mathbb{P}(X)$ satisfying

(i) $\bigcap_{r>0} B(x,r) = \{x\}$

(ii) $\bigcup_{r>0} B(x,r) = X$
(iii) $0 < r_1 \leq r_2 \implies B(x,r_1) \subset B(x,r_2)$
(iv) there exists $c \geq 1$ such that $y \in B(x,r) \implies B(x,r) \subset B(y,cr)$ and $B(y,r) \subset B(x,cr)$ for all $x \in X$ and $r > 0$.

A family of subsets satisfying properties (i)-(iv) generates a b-metric on X, [1].

Condition (i)-(iii) are verified easily for an α, β b-metric. We shall show property (iv). If $y \in B(x,r)$ and $z \in B(x,r)$ then $\rho(y,z) \leq \alpha\rho(y,x) + \beta\rho(x,z) < (\alpha+\beta)r$ where $\alpha+\beta \geq 2$ thus $z \in B(y,(\alpha+\beta)r)$ and in a similar manner, if $y \in B(x,r)$ and $z \in B(y,r)$ then $\rho(x,z) \leq \alpha\rho(x,y) + \beta\rho(y,z) < (\alpha+\beta)r$ thus $z \in B(x,(\alpha+\beta)r)$. Thus this family of subsets also generates an α, β b-metric.

3. COMPLETENESS

Definition 3.1. Let (X,ρ) be a generalized b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to x, if $\lim_{n\to\infty} \rho(x_n,x) = 0$.
(ii) The sequence $\{x_n\}$ is a Cauchy in (X,ρ) if $\lim_{n,m\to\infty} \rho(x_n,x_m) = 0$.
(iii) The space (X,ρ) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\lim_{n,m\to\infty} \rho(x_n,x_m) = \lim_{n\to\infty} \rho(x_n,x) = 0$.

Definition 3.2.

(i) If (X_1,ρ_1) and (X_2,ρ_2) are generalized b-metric spaces then a mapping $i : X_1 \to X_2$ is an isometric embeddings if

$$\rho_2(i(x), i(y)) = \rho_1(x,y)$$

for all $x, y \in X_1$.

(ii) A completion of a generalized b-metric space (X,ρ) is a complete b-metric space (Y,ρ) such that there exists an isometric embedding $i : X \to Y$ with $i(X) \subset Y$ dense in Y.

4. FIXED POINT THEOREM FOR GENERALIZED b-METRIC SPACES

As a consequence of (iii) of Definition 1.2, the α, β triangle inequality, we get for $n, m \in \mathbb{N}$ with $m > n$

$$\begin{aligned} &\rho(x_n,x_m) \\ &\leq \alpha\rho(x_n,x_{n+1}) + \beta\rho(x_{n+1},x_m) \\ &\leq \alpha\rho(x_n,x_{n+1}) + \beta[\alpha\rho(x_{n+1},x_{n+2}) + \beta\rho(x_{n+2},x_m)] \end{aligned} \tag{4.1}$$

Successively applying the α, β triangle inequality, we obtain

$$\rho(x_n,x_m) \leq \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(x_{n+i},x_{n+i+1}) + \beta^{m-n-1}\rho(x_{m-1},x_m). \tag{4.2}$$

In line with the α, β triangle inequality, one may consider the α, β relaxed polygonal inequality given by

$$\rho(x_n,x_m) \leq \frac{(\alpha+\beta)}{2} \sum_{i=0}^{m-n-1} \rho(x_{n+i},x_{n+i+1}) \tag{4.3}$$

where $x_i \in X$ for all i.

Definition 4.1. Let (X,ρ) be a generalized b-metric space then a mapping $T : X \to X$ is a contraction on X if there is a real number $0 < \lambda < 1$ such that for all $x, y \in X$

$$\rho(Tx, Ty) \le \lambda \rho(x, y). \tag{4.4}$$

The Banach fixed point theorem gives a constructive procedure for obtaining approximations to the fixed point called iterations. By the definition, in this method we choose an arbitrary x_0 and calculate recursively a sequence from the relation

$$x_n = T(x_{n-1}) = T^n(x_0). \tag{4.5}$$

By a repeated use of (4.4) and (4.5) we get

$$\begin{aligned}\rho(x_{n+i}, x_{n+i+1}) &= \rho(T^{n+i}(x_0), T^{n+i}(x_1)) \\ &\le \lambda\rho(T^{n+i-1}(x_0), T^{n+i-1}(x_1)) \\ &\vdots \\ &\le \lambda^{n+i}\rho(x_0, x_1). \end{aligned} \tag{4.6}$$

Theorem 4.1. *Let (X,ρ) be a complete generalized b-metric space, where ρ satisfies the α, β triangle inequality and $T : X \to X$ a contraction mapping such that $0 < \lambda < \frac{1}{\beta}$. Then T has a unique fixed point $x^* \in X$.*

Proof. We begin by proving that $\{x_n\}$ is a Cauchy sequence. Using (4.2) and (4.6) we get

$$\begin{aligned}\rho(x_n, x_{n+k+1}) &\le \alpha \sum_{i=0}^{k-1} \beta^i \rho(x_{n+i}, x_{n+i+1}) + \beta^k \rho(x_{n+k}, x_{n+k+1}) \\ &\le \alpha \sum_{i=0}^{k-1} \beta^i \lambda^{n+i} \rho(x_0, x_1) + \beta^k \lambda^{n+k} \rho(x_0, x_1) \\ &= \lambda^n \left[\alpha \sum_{i=0}^{k-1} \beta^i \lambda^i + \beta^k \lambda^k\right] \rho(x_0, x_1) \\ &= \lambda^n \left[\alpha \frac{1-\beta^k\lambda^k}{1-\beta\lambda} + \beta^k\lambda^k\right] \rho(x_0, x_1) \\ &= \frac{\lambda^n}{(1-\beta\lambda)} \left[\alpha - \beta^k\lambda^k(\alpha + \beta\lambda - 1)\right] \rho(x_0, x_1) \\ &< \lambda^n \frac{\alpha}{(1-\beta\lambda)} \rho(x_0, x_1). \end{aligned} \tag{4.7}$$

Since $\lim_{n\to\infty} \lambda^n = 0$, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of (X,ρ) it follows that there exists $x^* \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^*) = 0$. Furthermore, using the α, β triangle inequality, we have

$$\begin{aligned}\rho(x^*, Tx^*) &\le \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, Tx^*) \\ &\le \alpha\rho(x^*, x_{n+1}) + \beta\rho(x_{n+1}, x^*) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.8}$$

Hence $\rho(x^*, Tx^*) = 0$ thus $Tx^* = x^*$. Suppose there exists $x^{**}, x^* \in X$ such that $Tx^* = x^*$ and $Tx^{**} = x^{**}$. Then

$$\rho(x^{**}, x^*) = \rho(Tx^{**}, Tx^*) \le \lambda\rho(x^{**}, x^*), \tag{4.9}$$

which implies that $\rho(x^{**}, x^*) = 0$, i.e., $x^{**} = x^*$. ∎

Remark 4.1. Using (4.7) we get

$$\begin{aligned} \rho(x_n, x^*) &\leq \alpha\rho(x_n, x_{n+k+1}) + \beta\rho(x_{n+k+1}, x^*) \\ &\leq \lambda^n \frac{\alpha^2}{(1-\beta\lambda)}\rho(x_0, x_1) + \beta\rho(x_{n+k+1}, x^*). \end{aligned} \tag{4.10}$$

Letting $k \to \infty$ we get the order of convergence:

$$\rho(x_n, x^*) \leq \lambda^n \frac{\alpha^2}{(1-\beta\lambda)}\rho(x_0, x_1) \tag{4.11}$$

which implies at least linear convergence.

Theorem 4.2. *Let (X, ρ) be a complete generalized b-metric space, where ρ satisfies the α, β relaxed polygonal inequality and $T : X \to X$ a contraction mapping such that $0 < \lambda < 1$. Then T has a unique fixed point $x^* \in X$.*

Proof. The proof follows in line with the Theorem 4.1 ∎

Remark 4.2. Using (4.3) and (4.6) results in

$$\begin{aligned} \rho(x_n, x_{n+k}) &\leq \frac{(\alpha+\beta)}{2}\sum_{i=0}^{k-1}\rho(x_{n+i}, x_{n+i+1}) \\ &\leq \frac{(\alpha+\beta)}{2}\left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+k}\right)\rho(x_0, x_1) \\ &= \frac{(\alpha+\beta)}{2}\lambda^n \frac{1-\lambda^{k+1}}{1-\lambda}\rho(x_0, x_1). \end{aligned} \tag{4.12}$$

Hence we obtain the priori estimate

$$\rho(x_n, x^*) \leq \frac{(\alpha+\beta)}{2(1-\lambda)}\lambda^n\rho(x_0, x_1). \tag{4.13}$$

Definition 4.2. Let (X, ρ) be a complete b-metric space such that a mapping $T : X \to X$ is a Kannan contraction [6] if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$\rho(Tx, Ty) \leq \lambda\left[\rho(x, Tx) + \rho(y, Ty)\right] \tag{4.14}$$

for all $x, y \in X$.

Let $x_0 \in X$ be fixed, then for $n \in \mathbb{N}$

$$\begin{aligned} \rho(T^n x_0, T^{n+1}x_0) &= \rho(TT^{n-1}x_0, TT^n x_0) \\ &\leq \lambda\left[\rho(T^n x_0, T^{n-1}x_0) + \rho(T^{n+1}x_0, T^n x_0)\right], \end{aligned} \tag{4.15}$$

then it follows that

$$\rho(T^n x_0, T^{n+1}x_0) - \lambda\rho(T^{n+1}x_0, T^n x_0) \leq \lambda\rho(T^n x_0, T^{n-1}x_0) \tag{4.16}$$

which implies that

$$\rho(T^n x_0, T^{n+1}x_0) \leq \frac{\lambda}{1-\lambda}\rho(T^n x_0, T^{n-1}x_0). \tag{4.17}$$

Successively using (4.17) we obtain that

$$\rho(T^n x_0, T^{n+1}x_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^n \rho(x_0, Tx_0). \tag{4.18}$$

Theorem 4.3. *Let (X, ρ) be a complete generalized b-metric space and let $T : X \to X$ be a mapping for which there exists $\lambda \in \left[0, \frac{1}{\beta+1}\right)$ such that*

$$\rho(Tx, Ty) \leq \lambda \left[\rho(x, Tx) + \rho(y, Ty)\right] \tag{4.19}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. We begin by showing that for $x_0 \in X$ fixed, and for $n \in \mathbb{N}$
$\{T^n x_0\}$ is a Cauchy sequence in (X, ρ): for $m, n \in X$ with $m > n$, and using inequality (4.2) and (4.18), we get

$$\begin{aligned}
&\rho(T^n x_0, T^m x_0)\\
&\leq \alpha \sum_{i=0}^{m-n-2} \beta^i \rho(T^{n+i} x_0, T^{n+i+1} x_0) + \beta^{m-n-1} \rho(T^{m-1} x_0, T^m x_0)\\
&\leq \alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{n+i} \rho(x_0, Tx_0) + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-1} \rho(x_0, Tx_0)\\
&= \left[\alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{n+i} + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-1}\right] \rho(x_0, Tx_0)\\
&= \left(\frac{\lambda}{1-\lambda}\right)^{n} \left[\alpha \sum_{i=0}^{m-n-2} \beta^i \left(\frac{\lambda}{1-\lambda}\right)^{i} + \beta^{m-n-1} \left(\frac{\lambda}{1-\lambda}\right)^{m-n-1}\right] \rho(x_0, Tx_0)\\
&= \left(\frac{\lambda}{1-\lambda}\right)^{n} \left[\alpha \frac{1 - \left(\frac{\beta\lambda}{1-\lambda}\right)^{m-n-1}}{1 - \left(\frac{\beta\lambda}{1-\lambda}\right)} + \left(\frac{\beta\lambda}{1-\lambda}\right)^{m-n-1}\right] \rho(x_0, Tx_0). \qquad (4.20)
\end{aligned}$$

It follows that as $n \to \infty$ that the sequence $\{T^n x_0\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete there exists a $z_0 \in X$ such that

$$\lim_{n\to\infty} \rho(T^n x_0, z_0) = 0.$$

By the contraction (4.19), we obtain that

$$\rho(T^{n+1} x_0, Tz_0) \leq \lambda \rho(T^{n+1} x_0, T^n x_0) + \lambda \rho(z_0, Tz_0) \tag{4.21}$$

In the limit as $n \to \infty$, we get

$$\rho(z_0, Tz_0) \leq \lambda \rho(z_0, Tz_0). \tag{4.22}$$

Since $\lambda < 1$ we deduce that $\rho(z_0, Tz_0) = 0$. If we assume that $z' \in X$ is any fixed point then we obtain

$$\begin{aligned}
\rho(z_0, z') &= \rho(Tz_0, Tz')\\
&\leq \lambda \left[\rho(z_0, Tz_0) + \rho(z', Tz')\right]\\
&= \lambda \left[\rho(z_0, z_0) + \rho(z', z')\right]\\
&= 0 \qquad (4.23)
\end{aligned}$$

which implies that $z' = z_0$. ∎

Remark 4.3. From (4.20) we obtain the priori estimate

$$\rho(T^n(x_0), z_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^n \left(\frac{\alpha}{1 - \frac{\beta\lambda}{1-\lambda}}\right) \rho(x_0, Tx_0). \tag{4.24}$$

5. CONCLUSION

In this paper we have presented a relaxed α, β b-metric and proved some fixed point results for this new class of generalized metric.

REFERENCES

[1] H. AIMAR, B. IAFFEI and L. NITTI, On the Macías-Segovia metrization of quasi-metric spaces, *Rev. Union Math. Argent.*, **41**, (1998), no. 2, pp. 67–75.

[2] I. A. BAKHTIN, The contraction mapping principle in almost metric spaces, *Funct. Anal.*,**30**, (1989), pp. 26–37.

[3] N. BOURBAKI, Topologie Generale, Herman: Paris, France, (1974).

[4] S. CZERWIK, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostra.*,**1**, (1993), pp. 5–11.

[5] R. FAGIN and L. STOCKMEYER, Relaxing the triangle inequality in pattern matching, *Int. J. Comput. Vis.*, **30**, (1998), pp. 219–231.

[6] R. KANNAN, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60**, (1968), pp. 71–76.

[7] W. KIRK and N. SHAHZAD, Fixed point theory in distance spaces, Cham: Springer, (2014).

[8] M. PALUSZY'NSKI and K. STEMPAK, On quasi-metric and metric spaces, *Proc. Amer. Math. Soc.*, **137**, (2009), no. 12, pp. 4307–4312.

[9] T. VAN AN and N. VAN DUNG, Answers to Kirk-Shahzad's questions on strong b-metric spaces, *Taiwanese J. Math.*, **20**, (2016), no. 5, pp. 1175–1184.