Aust. J. Math. Anal. Appl.
Vol. **18** (2021), No. 1, Art. 5, 19 pp.
AJMAA

CONSTRUCTION OF A FRAME MULTIRESOLUTION ANALYSIS ON LOCALLY COMPACT ABELIAN GROUPS

RAJ KUMAR AND SATYAPRIYA

Received 24 May, 2020; accepted 9 December, 2020; published 12 February, 2021.

DEPARTMENT OF MATHEMATICS, KIRORI MAL COLLEGE, UNIVERSITY OF DELHI, DELHI, INDIA.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI, INDIA.
rajkmc@gmail.com
kmc.satyapriya@gmail.com

ABSTRACT. The frame multiresolution analysis (FMRA) on locally compact Abelian groups has been studied and the results concerning classical MRA have been worked upon to obtain new results. All the necessary conditions, which need to be imposed on the scaling function ϕ to construct a wavelet frame via FMRA, have been summed up. This process of construction of FMRA has aptly been illustrated by sufficient examples.

Key words and phrases: LCA groups; Frame multiresolution analysis; Wavelet frames; Refinable function; α-substantial.

2010 *Mathematics Subject Classification.* Primary 42C15, 42C40. Secondary 22B05.

ISSN (electronic): 1449-5910
© 2021 Austral Internet Publishing. All rights reserved.
This research work has been supported by the Junior Research Fellowship (JRF) of Human Resource Development Group, Council of Scientific and Industrial Research (HRDG-CSIR), India (Grant No: 09/045(1653)/2019-EMR-I)..

1. INTRODUCTION

Wavelet analysis, which may be treated as an alternative to the classical windowed Fourier analysis, has been studied extensively in recent decades. One of the principal frameworks for understanding a wavelet basis is the concept of multiresolution analysis (henceforth abbreviated as MRA). In recent years, the concept of MRA has become an important tool in Mathematics and in its applications. MRA not only helps us in understanding a wavelet basis but also enables us to construct a wavelet basis, i.e. a basis whose elements are scaled and translated version of finite number of functions. This theory, of MRA and wavelets, finds its main application in the field of signal and image processing and it is mainly concerned with decomposition of signals into subspaces of different resolutions.

Mathematically, a signal is an element of a Hilbert space. So, we can choose the space as per our requirements and then MRA is observed in that space only. The easiest spaces to work with are the Euclidean spaces. Naturally, MRA was developed for these spaces at first and then this concept was subsequently extended to other general spaces. In 1986 Mallet and Meyer developed the idea of MRA on $L^2(\mathbb{R})$. This was presented in detail in the paper [1]. Since then, MRA for Euclidean space $\mathbb{R}^n$ has also been studied extensively; see [1, 4].

We note that most of the frequently used spaces, like the compact spaces, the discrete spaces and the Euclidean spaces are all examples of a more general class of spaces, namely, the class of locally compact Abelian (henceforth abbreviated as LCA) groups. So, over the yeras, a unified theory was developed to study the generalized structure of LCA groups, which we shall briefly present in section two. In recent years, there has been a considerable interest in the study of topics like MRA and Gabor analysis in this generalized setting. Some notable works, in the setting of LCA groups, include: the study of Gabor analysis by K. Gröchenig in [17], the theory of shift invariant spaces by C. Cabrelli and V. Paternostro in [16] and the generalization of the definition of MRA by S. Dahlke in [5].

In his paper *Multiresolution Analysis and Wavelets on Locally Compact Abelian Groups* (see [5]), S. Dahlke constructed MRA and wavelets with the help of self-similar tiles and *B*-splines. All the MRA conditions were characterized in terms of the scaling and spectral functions by R.A. Kamyabi and R.R. Tousi in their paper *Some Equivalent Multiresolution Conditions on Locally Compact Abelian Group*, [6]. MRA conditions for a locally compact non Abelian group were given by Q. Yang and K. F. Taylor in their paper *Multiresolution Analysis and Harr-like Wavelet Bases on Locally Compact Groups*, [7]. These conditions were characterized without placing any decay conditions or regularity properties on the scaling function.

In this paper, these MRA conditions are modified to construct a wavelet frame for the space $L^2(G)$, G being an LCA group. Frames have an added advantage over bases due to the fact that, any element has multiple expressions as superpositions of frame elements as compared to the unique expression in case of a basis.

We have divided this paper in four sections. First section gives a general introduction to the theory of wavelets and MRA. We have also listed some of the classical works on MRA and wavelets in this section. In section two, a necessary background has been prepared which we shall require later to construct an FMRA. Our main work about the frame multiresolution analysis (henceforth abbreviated as FMRA) has been presented in section three. Some examples supporting our work have also been given in this section. Finally, we have concluded our work in section four.

2. PRELIMINARIES AND NOTATIONS

Some basic known results from the theory of LCA groups have been reviewed here. We refer [8, 9, 10], and the references therein, for a detailed study on LCA groups.

We call a group G, an LCA group if it is equipped with a Hausdorff topology, is metrizable and locally compact in this topology and can be written as countable union of compact sets. We denote group composition by $'+'$ and identity element by $'0'$.

The groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}_n$ are some of the frequently used LCA groups. These groups, along with their higher dimensional variants, are called *elementary LCA groups*.

A character γ on G is a homomorphism from G to the circle group $\mathbb{T}$. The set $\hat{G}$ of all continuous characters on G, called the dual group of G, also forms an LCA group (see [8]) under compact-open topology with the group operation:

$$(\gamma + \gamma')(x) = \gamma(x)\gamma'(x); \;\; \gamma, \gamma' \in \hat{G}, \; x \in G.$$

The *Pontryagin duality theorem* states that the dual group $\hat{\hat{G}}$ of $\hat{G}$ is topologically isomorphic to the group G; usually we can identify both the groups and we will simply write $\hat{\hat{G}} = G$. Thus $\gamma(x)$ can be interpreted as action of $\gamma \in \hat{G}$ on $x \in G$ or action of $x \in \hat{\hat{G}} = G$ on $\gamma \in \hat{G}$; for this reason, from now on, we will use the following notation:

$$\gamma(x) = (\gamma, x); \;\; \gamma \in \hat{G}, \; x \in G.$$

The group G is now equipped with Radon measure μ_G which is translation invariant, i.e.

$$\int_G f(x+y) d\mu_G(x) = \int_G f(x) d\mu_G(x); \;\; \forall \, y \in G$$

and for all continuous functions f on G with compact support. This measure is unique up to scalar multiplication and is called the *Haar measure*. For existence and uniqueness of Haar measure, see [9]. We will use a fixed Haar measure μ_G throughout this paper. Based on this Haar measure, we define the spaces $L^1(G)$, $L^2(G)$ and $L^\infty(G)$ in the usual way. Out of these spaces, only $L^2(G)$ is a Hilbert space. Moreover, due to our assumption of G, being a countable union of compact sets, $L^2(G)$ becomes a separable Hilbert space (see [15]). Throghout this paper, $\langle ., . \rangle$ will denote an inner product, and $||.||$ will denote a norm, in the space $L^2(G)$, unless stated otherwise.

The following theorem establishes an important relation between a group and its dual in the special case of either a compact group or a discrete group. For proof, we refer [8].

Theorem 2.1. *Let G be an LCA group and $\hat{G}$ be its dual. Then the following hold:*

(i) *If G is discrete, then $\hat{G}$ is compact.*
(ii) *If G is compact, then $\hat{G}$ is discrete.*

We now define the *Fourier transform* of a function $f \in L^1(G)$ by

$$\mathcal{F} : L^1(G) \to C_0(\hat{G}), \quad \mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int_G f(x)(\gamma, -x) d\mu_G(x).$$

Here, $C_0(\hat{G})$ is the space of all continuous functions on $\hat{G}$ vanishing at infinite.
The Haar measure, $\mu_{\hat{G}}$, on the dual group $\hat{G}$ can be normalized so that, for a specific class of functions, the following *inversion formula* holds (see [10]):

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)(\gamma, x) d\mu_{\hat{G}}(\gamma); \;\; \forall \, x \in G. \tag{2.1}$$

Throughout this paper, we shall appropriately normalize the measures μ_G and $\mu_{\hat{G}}$ so that inversion formula (2.1) always holds. Once this is done, the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\hat{G})$, the so called *Plancharel* *transformation*(see [9]), which is also denoted by $\mathcal{F}$ or $'\wedge'$. We now get a generalized version of *Parseval formula*:

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)}d\mu_G(x) = \int_{\hat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\mu_{\hat{G}}(\gamma) = \langle \hat{f}, \hat{g}\rangle. \tag{2.2}$$

To simplify the notations, from now onwards, we will denote $d\mu_G(x)$ by dx and $d\mu_{\hat{G}}(\gamma)$ by $d\gamma$; whenever the context is clear.

We now define an important class of subgroups of an LCA group, namely *the lattices* (or uniform lattices). A uniform lattice Λ is a countable closed subgroup of G such that the quotient group G/Λ is compact in the quotient topology. The *annihilator* $\Lambda^\perp$ of a lattice Λ, is a subgroup of $\hat{G}$, defined by:

$$\Lambda^\perp = \{\gamma \in \hat{G} : \gamma(\lambda) = 1,\ \ \forall \lambda \in \Lambda\}.$$

The topology of the group $\hat{G}$ implies that the annihilator $\Lambda^\perp$ of a lattice Λ is also a lattice in $\hat{G}$. Moreover, a lattice in G leads to a splitting of the group G, as well as the dual group $\hat{G}$, into disjoint cosets, as given in the lemma below. For the proof, we refer [11].

Lemma 2.2. *Let G be an LCA group and $\Lambda \subset G$ be a uniform lattice in G. Then the following hold:*

(i) *There exists a Borel measurable relatively compact set $\mathcal{Q} \subset G$ such that*

$$G = \bigcup_{\lambda\in\Lambda}(\lambda + \mathcal{Q}),\ \ (\lambda + \mathcal{Q}) \cap (\lambda' + \mathcal{Q}) = \emptyset \textit{ for } \lambda \neq \lambda';\ \ \lambda, \lambda' \in \Lambda. \tag{2.3}$$

(ii) *There exists a Borel measurable relatively compact set $\mathcal{S} \subseteq \hat{G}$ such that*

$$\hat{G} = \bigcup_{\omega\in\Lambda^\perp}(\omega + \mathcal{S}),\ \ (\omega + \mathcal{S}) \cap (\omega' + \mathcal{S}) = \emptyset \textit{ for } \omega \neq \omega';\ \ \omega, \omega' \in \Lambda^\perp.$$

Further note that the sets $\mathcal{Q}$ and $\mathcal{S}$ are in one-to-one correspondence with the quotient groups G/Λ and $\hat{G}/\Lambda^\perp$ respectively.

The set $\mathcal{Q}$ which appears in equation (2.3) is called a *fundamental domain* associated to the lattice Λ. For our convenience we will allow sets $\mathcal{Q}$ for which two conditions in (2.3) hold up to a set of measure zero. Throughout this paper, we will denote by $\mathcal{Q}$, a fundamental domain associated to the lattice $\Lambda \subset G$ and by $\mathcal{S}$, a fundamental domain associated to the lattice $\Lambda^\perp \subset \hat{G}$.

To avoid any confusion, we find it necessary to mention here, the two different uses of the symbol $\perp$. If H ia any closed subspace of G, then

- $z \in H^\perp$ will mean that z is an element of the annihilator of the subspace H; i.e.,
  - $z \in \hat{G}$.
  - $(z, h) = 1$ for all $h \in H$.
- $z \perp H$ will mean that z is in orthogonal complement of the subspace H; i.e.,
  - $z \in G$.
  - $\langle z, h\rangle = 0$ for all $h \in H$.

The main aim of this paper is to construct a frame for $L^2(G)$ via FMRA. For that, we need to define generalized versions of translation, modulation and dilation operators on $L^2(G)$ and $L^2(\hat{G})$. Corresponding to each $y \in G$:

- The operator $T_y : L^2(G) \to L^2(G)$ given by $T_y f(x) = f(x-y)$ represents a generalized translation operator on $L^2(G)$.
- The operator $\mathcal{E}_y : L^2(\hat{G}) \to L^2(\hat{G})$ given by $\mathcal{E}_y F(\gamma) = (\gamma, y)F(\gamma)$ represents a generalized modulation operator on $L^2(\hat{G})$.

Similarly, $\mathcal{T}_\gamma$ and E_γ define respectively, the generalized translation and modulation operators on $L^2(\hat{G})$ and $L^2(G)$.

Defining a generalized version of dilation operator is not that straightforward. To do this, we proceed via a method used by Dahlke in [5], for which we first need to define dilative automorphisms. An automorphism (algebraic automorphism and topological homeomorphism) $\alpha : G \to G$ is said to be dilative if for any compact set K in G and any open neighbourhood U of $0 \in G$, there exists $n_0 \in \mathbb{N}$ such that $K \subseteq \alpha^n(U),\ \ \forall n \geq n_0$. If $\alpha : G \to G$ is a dilative automorphism on G, then there exists a positive constant δ_α such that

$$\int_G f(x)dx = \delta_\alpha \int_G f(\alpha(x))dx.$$

This induces a unitary operator $D : L^2(G) \to L^2(G)$ given by $Df(x) = \delta_\alpha^{1/2} f(\alpha(x))$. This D is called the dilation operator on $L^2(G)$.

The pair $(\Lambda$,$\alpha)$, where Λ is a uniform lattice in G and α is a dilative automorphism on G such that $\alpha(\Lambda) \subseteq \Lambda$, is called a *scaling system on G*. The following lemma gives us some essential properties of this scaling system. Proof of this lemma may be found in [7].

Lemma 2.3. *Let (Λ, α) be a scaling system. Then the following conditions hold:*

(i) *Λ is not an open subgroup of G.*
(ii) $\mu_G(\Lambda) = 0$.
(iii) *For any $j_0 \in \mathbb{Z}$, $\underset{j \geq j_0}{\cup} \alpha^{-j}(\Lambda)$ is dense in G.*

Throughout this paper, we shall assume that (Λ, α) is a scaling system on G. Note that, more often than not, we will work on the dual group $\hat{G}$ instead of G; so it becomes imperative for us to know the corresponding scaling system on $\hat{G}$. Based on some of the existing information given in [5, 6, 7], we construct the following lemma which gives us a dilation operator on $L^2(\hat{G})$ and thus a scaling system on $\hat{G}$. The straightforward proof has been skipped.

Lemma 2.4. *Let G be an LCA group and $\hat{G}$ be its dual group. Suppose $\alpha : G \to G$ is a dilative automorphism on G. Then the following hold:*

(i) *The map, $\hat{\alpha} : \hat{G} \to \hat{G}$ given by*

$$(\hat{\alpha}(\gamma), x) = (\gamma, \alpha(x));\ \ x \in G,$$

is a dilative automorphism on $\hat{G}$.

(ii) *For any appropriately defined function F on $\hat{G}$, we have*

$$\int_{\hat{G}} F(\gamma)d\gamma = \delta_\alpha \int_{\hat{G}} F(\hat{\alpha}(\gamma))d\gamma.$$

for any appropriately defined function F on $\hat{G}$.

(iii) *The operator $\mathcal{D} : L^2(\hat{G}) \to L^2(\hat{G})$ given by $\mathcal{D}F(\gamma) = \delta_\alpha^{1/2} F(\hat{\alpha}(\gamma))$ is a unitary operator on $L^2(\hat{G})$. This operator $\mathcal{D}$ works as dilation operator on $L^2(\hat{G})$.*

(iv) *$(\Lambda^\perp, \hat{\alpha})$ is a scaling system on $\hat{G}$.*

We note that it is not always easy to find a dilative automorphism on G. In fact, it may be possible that no automorphism on G is dilative:

Example 2.1. *Consider the LCA group $G = Z_n$. Take $U = \{0\}$, a neighbourhood of $0 \in G$; and $K = \{1\}$, a compact set in G. Let $\alpha : G \to G$ be any automorphism of G. Then it is easy to see that, for any $n \in \mathbb{N}$, $\alpha^n(U) = U$. Thus, for any $n \in \mathbb{N}$, K can't be contained in $\alpha^n(U)$ and hence α fails to be dilative.*

More generally, let G be any discrete group with identity e. Then any automorphism α of G fails to be dilative as the condition fails for the neighbourhood $U = \{e\}$ of $e \in G$ and the compact set $K = \{x\}$ $(x \neq e)$ in G.

Example 2.1 along with Theorem 2.1 and Lemma 2.4 suggests us that the existence of dilative automorphism also fails for all the compact groups. But discrete groups and compact groups are not the only ones where this happens. Given below is an example of a non discrete and non compact group on which we are unable to define a dilative automorphism.

Example 2.2. *Let* $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}\,,\, ab > 0 \right\}$*. Then G forms an LCA group under the operation of matrix multiplication and the topology induced by the Eucledian space* $\mathbb{R}^2$*. The matrix* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is the identity element of G. Note that* $G = G_1 \cup G_2$*, where*

$$G_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a, b > 0 \right\};$$

and

$$G_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a, b < 0 \right\}.$$

Let U be a neighbourhood of I contained entirely in G_1, and let $K \subseteq G$ be a compact set contained entirely in G_2. Then $K \nsubseteq \alpha^n(U)$, for any automorphism α of G and $n \in \mathbb{N}$. Hence, α can't be dilative.
We use a similar approach to show that the existence of dilative automorphism also fails for all those LCA groups which are disconnected but have a connected neighbourhood of identity.

To construct an FMRA on G via the methods given in this paper, first we must ensure that there exists a dilative automorphism on G. Once we find a dilative automorphism on G and define the subsequent dilation operator on $L^2(G)$, then translation, modulation and dilation operators follow the same commutative relations amongst them, and behave similarly under Fourier transform, as in the case of $G = \mathbb{R}$.

We will now list some more notations corresponding to the group G, which shall also hold analogously for the group $\hat{G}$.

- If $H \subset G$, then the function $\mathcal{X}_H$ given by:

$$\mathcal{X}_H(x) = \begin{cases} 1 & , x \in H \\ 0 & , x \notin H \end{cases}$$

  is called the *indicator function of* H or *the characteristic function of* H.
- For any $H \subset G$, we say that a function $f : G \to \mathbb{C}$ is H*-periodic*, if

$$f(x + h) = f(x); \;\; \forall\, x \in G \text{ and } \forall\, h \in H.$$

- For any $f, g \in L^2(G)$, the tem fg will denote the pointwise product of f and g, and the term $f * g$ will give us the *convolution* of f and g; i.e., for any $x \in G$, we have

$$fg(x) = f(x)g(x) \quad \text{and} \quad f * g(x) = \int_G f(y)g(x - y)dy.$$

  Moreover, both fg and $f * g$ are members of $L^1(G)$.

Now, since, the quotient G/Λ is in one to one correspondence with the fundamental domain $\mathcal{Q} \subset G$ associated to the lattice $\Lambda \subset G$, therefore we are here tempted to assert a relation between the spaces $L^p(G/\Lambda)$ and $L^p(\mathcal{Q})$ (p=1 or 2 or ∞). But before that, we define the spaces $L^p(\mathcal{Q})$:

$$L^p(\mathcal{Q}) = \{f \in L^p(G) : f = 0 \text{ a.e. } G/\mathcal{Q}\}; \quad \text{p=1, 2 or } \infty.$$

Analogously, we define the space $L^p(\mathcal{S})$ (p=1,2 or ∞), $\mathcal{S}$ being the fundamental domain associated to the lattice $\Lambda^\perp \subset \hat{G}$. The following remark provides us an orthonormal family in $L^2(\mathcal{S})$ which is also its basis. For more details, we refer [16].

Remark 2.1. Let, for each $\lambda \in \Lambda$, $\eta_\lambda : \hat{G} \to \mathbb{C}$ be defined by $\eta_\lambda(\gamma) = (\gamma, \lambda)\mathcal{X}_\mathcal{S}(\gamma)$. Then the family,

$$\left\{ \frac{1}{\sqrt{\mu_{\hat{G}}(\mathcal{S})}} \eta_\lambda \right\}_{\lambda \in \Lambda}$$

forms an orthonormal basis for $L^2(\mathcal{S})$.

Using the above given notation of the periodic functions, we note that there is a one to one correspondence between $L^2(G/\Lambda)$ and the set of all $\Lambda-$periodic functions f such that $f\mathcal{X}_\mathcal{Q} \in L^2(\mathcal{Q})$. So, with a slight abuse of notation, we write $f \in L^2(G/\Lambda)$ whenever f is a $\Lambda-$periodic function on G satisfying $f\mathcal{X}_\mathcal{Q} \in L^2(\mathcal{Q})$. Analogously, we give the definition for the space $L^2(\hat{G}/\Lambda^\perp)$.

Now, Remark 2.1 and the notations of the above paragraph help us in constructing the following lemma which gives us an explicit representation of the elements of the space $L^2(\hat{G}/\Lambda^\perp)$.

Lemma 2.5. *If, for each $\lambda \in \Lambda$, the functions η_λ are defined as in Remark 2.1, then the following are equivalent :*

(i) $F \in L^2(\hat{G}/\Lambda^\perp)$.

(ii) *There exists a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that*

$$F = \sum_{\lambda\in\Lambda} c_\lambda \varepsilon_\lambda;$$

where $\varepsilon_\lambda : \hat{G} \to \mathbb{C}$ is given by, $\varepsilon_\lambda(\gamma) = (\gamma, \lambda)$.

Proof. First suppose that $F \in L^2(\hat{G}/\Lambda^\perp)$. This means that F is $\Lambda^\perp$-periodic and that $F\mathcal{X}_\mathcal{S} \in L^2(\mathcal{S})$. Since $\{\eta_\lambda\}_{\lambda\in\Lambda}$ is orthogonal basis for $L^2(\mathcal{S})$, therefore there exists a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that

$$F\mathcal{X}_\mathcal{S} = \sum_{\lambda\in\Lambda} c_\lambda \eta_\lambda;$$

i.e, for any $\gamma \in \hat{G}$,

$$F\mathcal{X}_\mathcal{S}(\gamma) = \sum_{\lambda\in\Lambda} c_\lambda (\gamma, \lambda)\mathcal{X}_\mathcal{S}(\gamma).$$

Using $\Lambda^\perp$-periodicity of the function F we get that, for any $\gamma \in \hat{G}$,

$$F(\gamma) = \sum_{\lambda\in\Lambda} c_\lambda(\gamma, \lambda) \quad \text{i.e.} \quad F = \sum_{\lambda\in\Lambda} c_\lambda \varepsilon_\lambda.$$

The fact that $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ can be proved using orthogonality of the family $\{\eta_\lambda\}_{\lambda\in\Lambda}$ and the definition of the space $L^2(\mathcal{S})$.

For the converse, suppose $F = \sum_{\lambda\in\Lambda} c_\lambda \varepsilon_\lambda$ for some sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\lambda)$. Then clearly F is $\Lambda^\perp$-periodic. Further, note that

$$\int_\mathcal{S} |F\mathcal{X}_\mathcal{S}(\gamma)|^2 d\gamma = \sum_{\lambda\in\Lambda} |c_\lambda|^2.$$

Therefore, $\displaystyle\int_{\mathcal{S}} |F\mathcal{X}_{\mathcal{S}}(\gamma)|^2 d\gamma \ < \ \infty$ and thus $F\mathcal{X}_{\mathcal{S}} \in L^2(\mathcal{S})$. So, by our definition of the space $L^2(\hat{G}/\Lambda^{\perp})$, we conclude that $F \in L^2(\hat{G}/\Lambda^{\perp})$. This completes the proof. ∎

We finish this section with a brief discussion on frames in an arbitrary separable Hilbert space. For a detailed study on frames and their properties, we refer [12].

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. Then, a sequence of elements $\{f_\beta\}_{\beta\in\mathbb{I}}$ is called frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{\beta\in\mathbb{I}} |\langle f, f_\beta\rangle|^2 \leq B||f||^2, \quad \forall \ f \in \mathcal{H}.$$

The numbers A and B, appearing in Definition 2.1, are called *frame bounds*. More precisely, A is the *lower bound* and B is the *upper bound*. The frame is *exact*, if it ceases to be a frame whenever any single element is deleted. The frame is *tight* if $A = B$; moreover, when $A = B = 1$, the frame is called *Parseval frame*.

The following lemma gives us one of the main characterizations of the frames in a separable Hilbert space. It does not involve any knowledge of the frame bounds. Proof of this lemma may be found in [12].

Lemma 2.6. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. Then, a sequence $\{f_\beta\}_{\beta\in\mathbb{I}}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if and only if the map $T : l^2(\mathbb{I}) \to \mathcal{H}$, given by*

$$T(\{c_\beta\}) = \sum_{\beta\in\mathbb{I}} c_\beta f_\beta,$$

is well defined and onto.

3. Frame Multiresolution Analysis

The definition of frame multiresolution analysis, for the special case $G = \mathbb{R}$, was first given by J. J. Benedetto and S. Li in the paper [13]. The following definition, of FMRA on LCA groups, may be treated as a generalized version of the analogous definition by Benedetto and Li.

Definition 3.1. A frame multiresolution analysis for $L^2(G)$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(G)$ and a function $\phi \in V_0$ such that

(i) $... \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq ....$
(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$.
(iii) $V_j = D^j V_0$.
(iv) $f \in V_0 \implies T_\lambda f \in V_0, \forall \ \lambda \in \Lambda$.
(v) $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is a frame for V_0.

The function ϕ, which appears in Definition 3.1, is called *the scaling function* and the subspaces V_j are called *multiresolution subspaces* or *approximation spaces*. If the conditions in Definition 3.1 are satisfied, it follows that

$$V_j = D^j(\overline{span}\{T_\lambda\phi\}_{\lambda\in\Lambda}) = \overline{span}\{D^j T_\lambda \phi\}_{\lambda\in\Lambda}, \quad j \in \mathbb{Z} \tag{3.1}$$

Remark 3.1. Equation (3.1), Definition 3.1 and the fact that D^j is a unitary operator together implies that for $j \in \mathbb{Z}$, $\{D^j T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame for V_j with same frame bounds as that of the frame $\{T_\lambda\phi\}_{\lambda\in\Lambda}$. See [12, Lemma 5.3.3] for more details.

To start the construction of a FMRA, the foremost requirement is chosing a function ϕ in $L^2(G)$ such that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame for V_0. Assuming that the subspaces V_j are defined as in (3.1), R A Kamyabi, in his paper [6], proved the triviality of the intersection for the classical case i.e. the case where $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ forms a orthonormal basis for V_0. We modify that result to get triviality of intersection in the case of frames.

Theorem 3.1 (Triviality of the intersection). *If $\phi \in L^2(G)$ is such that the sequence $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence and $\{V_j\}_{j\in\mathbb{Z}}$ is a sequence of closed subspaces of $L^2(G)$ defined as in* (3.1), *then $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$.*

Proof. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$ and $\epsilon > 0$ be arbitrary. Then there exists a compactly supported continuous function $\tilde{f} \in L^2(G)$ such that $||f - \tilde{f}|| < \epsilon$. Let $\mathcal{K}$ denote the support of $\tilde{f}$.
If, for $j \in \mathbb{Z}$, $P_j : L^2(G) \to V_j$ denotes the orthogonal projection onto the subspace V_j, then we have

$$||f - P_j(\tilde{f})|| = ||P_j(f) - P_j(\tilde{f})|| \leq ||f - \tilde{f}|| < \epsilon.$$

This implies that

$$||f|| \leq ||P_j(\tilde{f})|| + \epsilon. \tag{3.2}$$

Our aim is to find a bound for $||P_j(\tilde{f})||$ which can be made arbitrarily small.
Now, since $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence and the subspaces are defined as in (3.1), therefore, it is a frame for V_0. Let A and B denote the frame bounds for this frame. Further Remark 3.1 tells us that $\{D^jT_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame for V_j with bounds A and B. All this information can be clubbed together to write

$$||P_j(\tilde{f})|| \leq A^{-1/2}\left(\sum_{\lambda\in\Lambda}|\langle\tilde{f}, D^jT_\lambda\phi\rangle|^2\right). \tag{3.3}$$

Note that, for any $j \in \mathbb{Z}$ and $\lambda \in \Lambda$, we have;

$$|\langle\tilde{f}, D^jT_\lambda\phi\rangle|^2 = \left|\int_G \tilde{f}(x)(D^jT_\lambda)\phi(x)dx\right|^2.$$

An appropriate manipulation now yields

$$\sum_{\lambda\in\Lambda}|\langle\tilde{f}, D^jT_\lambda\phi\rangle|^2 \leq \mu_G(\mathcal{K})||\tilde{f}||_\infty^2 \int_{\mathcal{K}_j}|\phi(x)|^2dx;$$

where $\mathcal{K}_j = \bigcup_{\lambda\in\Lambda}(\alpha^j(\mathcal{K}) - \lambda)$. We rewrite above inequality as follows:

$$\sum_{\lambda\in\Lambda}|\langle\tilde{f}, D^jT_\lambda\phi\rangle|^2 \leq \mu_G(\mathcal{K})||\tilde{f}||_\infty^2 \int_G \mathcal{X}(\mathcal{K}_j)|\phi(x)|^2dx.$$

It is easy to see that $\mathcal{X}(\mathcal{K}_j) \to 0$ as $j \to -\infty$.
It then follows from Lebesgue dominated convergence theorem that

$$\int_G \mathcal{X}(\mathcal{K}_j)|\phi(x)|^2dx \to 0 \text{ as } j \to -\infty.$$

When this is substitued in (3.3) and then in (3.2), we get that $||f|| < \epsilon$. This proves our result. ∎

Theorem 3.1 tells us that the condition of triviality of the intersection is a redundant condition whenever we are given that the subspaces are defined as in (3.1) and $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence. Keeping this in mind, it is convenient to work with a shorter definition of frame multiresolution analysis in which all the redundancies are removed.

Definition 3.2. A function $\phi \in L^2(G)$ generates an FMRA if $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is a frame sequence and the spaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by (3.1) satisfy the conditions

(i) $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$.

(ii) $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G)$.

So it turns out that nested property of the subspaces and density of the union, is all that we need to prove to ensure that the function ϕ generates an FMRA.

Before investigating these conditions, we wish to introduce another notation, namely Φ, a complex valued function on $\hat{G}$, which is defined by:

$$\Phi(\gamma) = \sum_{\omega\in\Lambda^\perp} |\hat{\phi}(\gamma+\omega)|^2.$$

It is easy to see that Φ is $\Lambda^\perp$-periodic and $\Phi\mathcal{X}_{\mathcal{S}} \in L^1(\mathcal{S})$; thus, by the definition of the space $L^1(\hat{G}/\Lambda^\perp)$, we can say that $\Phi \in L^1(\hat{G}/\Lambda^\perp)$. We now state a lemma which gives us bounds for the function Φ. This lemma, which can be treated as a generalized version of a result given by Benedetto and Li (see [13, Theorem 3.4]), shows that the frame properties of $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ can be completely described in terms of the function Φ.

Lemma 3.2. *Let $\phi \in L^2(G)$ be given and the subspace V_j be defined by* (3.1)*. Then $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame for V_0 with bounds A and B if and only if $A \leq \Phi(\gamma) \leq B$; for all $\gamma \in \mathcal{S}/\mathcal{N}$, where $\mathcal{N}$ is the null set of Φ, given by*

$$\mathcal{N} = \{\gamma \in \mathcal{S} : \Phi(\gamma) = 0\}.$$

We now return back to our discussion of investigating the two properties listed in Definition 3.2. Both these properties can be obtained by modifying the corresponding results in the case of classical MRA. See [2, 3] for detailed discussion on classical MRA.

We now list the equivalent conditions for the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ to be nested.

Theorem 3.3 (**Nested Property**). *Let G be an LCA group with the dual group $\hat{G}$ and let (Λ, α) be a scaling system defined on G. Assume that $\phi \in L^2(G)$ and that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence. If the subspaces V_j are defined by* (3.1)*, then, the following conditions are equivalent:*

(i) *$V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$.*

(ii) *There exists a function $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ such that*

$$\hat{\phi}(\gamma) = H_0(\hat{\alpha}^{-1}(\gamma))\hat{\phi}(\hat{\alpha}^{-1}(\gamma)). \tag{3.4}$$

Proof. First assume that the subspaces V_j are nested. Since $\phi \in V_0 \subseteq V_1 = DV_0$, therefore $D^{-1}\phi \in V_0$.

A direct implication of Lemma 2.6 gives us existence of a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ so that for any $x \in G$, we can write

$$(D^{-1}\phi)(x) = \sum_{\lambda\in\Lambda} c_\lambda T_\lambda \phi(x).$$

So, for any $\gamma \in \hat{G}$, we have

$$\hat{\phi}(\hat{\alpha}(\gamma)) = \delta_\alpha^{-1/2} F(\gamma)\hat{\phi}(\gamma); \quad \text{where } F(\gamma) = \sum_{\lambda\in\Lambda} c_{-\lambda}(\gamma, \lambda).$$

Clearly, $F \in L^2(\hat{G}/\Lambda^\perp)$. Further, if we choose $H_0(\gamma) = \delta_\alpha^{-1/2}F(\gamma)$, then it is evident that $H_0 \in L^2(\hat{G}/\Lambda^\perp)$. Now, it only remains to show that H_0 is bounded. To see this, first note that

$$\Phi(\hat{\alpha}(\gamma)) \geq |H_0(\gamma)|^2\Phi(\gamma).$$

If A, B denote the frame bounds for the frame $\{T_\lambda \phi\}_{\lambda\in\Lambda}$, then using Lemma 3.2, we can write

$$A \leq \Phi(\gamma) \leq B;$$

whenever $\Phi(\gamma) \neq 0$. Therefore, for such γ, we have

$$|H_0(\gamma)|^2 \leq \frac{B}{A}.$$

Choosing $H_0(\gamma) = 0$ whenever $\Phi(\gamma) = 0$, gives us that H_0 is bounded. This means that $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ and thus (ii) holds.
For the converse implication, suppose there exists a function $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ such that (3.4) holds.
Let $f \in V_j$. Then Lemma 2.6 gives us existence of a sequence $\{d_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that for any $x \in G$, we have

$$f(x) = \sum_{\lambda\in\Lambda} d_\lambda D^j T_\lambda \phi(x).$$

This means that, for any $\gamma \in \hat{G}$, we can write

$$\hat{f}(\hat{\alpha}^j(\gamma)) = F(\gamma)\hat{\phi}(\gamma); \quad \text{where } F(\gamma) = \delta_\alpha^{j/2} \sum_{\lambda\in\Lambda} d_{-\lambda}(\gamma, \lambda).$$

Clearly, $F \in L^2(\hat{G}/\Lambda^\perp)$. Further

$$\hat{f}(\hat{\alpha}^{j+1}(\gamma)) = F(\hat{\alpha}(\gamma))H_0(\gamma)\hat{\phi}(\gamma).$$

If we write $G(\gamma) = F(\hat{\alpha}(\gamma))H_0(\gamma)$, then $G \in L^2(\hat{G}/\Lambda^\perp)$ and the expression

$$\hat{f}(\hat{\alpha}^{j+1}(\gamma)) = G(\gamma)\hat{\phi}(\gamma)$$

holds. Moreover, Lemma 2.5 gives us existence of a sequence $\{g_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that, for any $\gamma \in \hat{G}$, we have

$$G(\gamma) = \sum_{\lambda\in\Lambda} g_\lambda(\gamma, \lambda).$$

Using this notation, we can write

$$\hat{f}(\hat{\alpha}^{j+1}(\gamma)) = \sum_{\lambda\in\Lambda} g_\lambda(\gamma, \lambda)\hat{\phi}(\gamma); \ \ \gamma \in G.$$

This implies that, for any $x \in G$, we have

$$f(x) = \sum_{\Lambda\in\Lambda} g_\lambda^1 D^{j+1} T_\lambda \phi(x); \quad \text{where } \{g_\lambda^1\}_{\lambda\in\Lambda} = \{\delta_\alpha^{j+1/2} g_{-\lambda}\}_{\lambda\in\Lambda} \in l^2(\Lambda).$$

Using a straightforward application of Lemma 2.6, we get that $f \in V_{j+1}$. Thus (i) holds. This completes the proof. ∎

An equation of the form (3.4) is called **refinement equation** and such ϕ is called **refinable**.

Remark 3.2. In the classical MRA, where $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is an orthonormal basis for V_0, the function H_0 is always unique. But this might not be the case for FMRA. It may happen that the the set $\{\gamma \in \mathcal{S} : \Phi(\gamma) = 0\}$ has a positive measure, in which case, the function H_0 has more than one choice to take. Trivially, we can choose $H_0 = 0$ on this set, as in Theorem 3.3. The function H_0 obtained in this way is called *two scale symbol* or *refinement mask* for the FMRA.

Now it only remains to show that the union of the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ is dense in $L^2(G)$. This can be achieved in more ways than one. R A Kamyabi, [6], used the theory of spectral functions and spectral radius to prove the density; I Daubechies, [2], proved density of the union by working with the Fourier transform of the scaling function; Q.Yang and K. F. Taylor, [7], did not use either of the above, but he proved this using translation invariant subspaces and the concept of zero divisors.

We will follow the works of Q. Yang and K. F. Taylor to prove the density of the union. For that, we need a few definitions:

- A subspace X of $L^2(G)$, is called a *translation invariant subspace*, if $T_x f \in X$, for all $x \in G$ and $f \in X$.
- We call a family $\mathcal{F} \subseteq L^2(G)$, a *zero divisor* in $L^2(G)$, if there exists a nonzero $g \in L^2(G)$ such that the convolution $f * g = 0$, for all $f \in \mathcal{F}$.
- Given an automorphism α of G and subsequent dilation operator D defined on $L^2(G)$, we say that a function $f \in L^2(G)$ is *α-substantial* if and only if the family $\{D^j f : j \in \mathbb{Z}\}$ is not a zero divisor in $L^2(G)$.

The following lemma lists all the results which are required to prove the density of the union of the subspaces V_j.

Lemma 3.4. *Let G be an LCA group and let (Λ, α) be a scaling system defined on G. If the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ are defined by* (3.1) *and ϕ is a refinable function in $L^2(G)$, then the following hold:*

(i) *The map $f \mapsto \tilde{f}$, where*

$$\tilde{f}(x) = \overline{f(-x)}; \;\; x \in G,$$

is a norm preserving conjugate linear bijection on $L^2(G)$.

(ii) *For any $f, g \in L^2(G)$ and any $x \in G$, we have:*

$$f * g(x) = \langle T_{-x} f, \tilde{g} \rangle.$$

(iii) *If $W = \overline{\underset{j\in\mathbb{Z}}{\cup} V_j}$, then W is a translation invariant subspace of $L^2(G)$. Thus we can write:*

$$W = \overline{span}\{T_x D^j \phi : x \in G, j \in \mathbb{Z}\}.$$

We finally give the proof of the density theorem in terms of α-substantiality of the scaling function ϕ.

Theorem 3.5 (Density of the Union). *Let ϕ be a refinable function in $L^2(G)$ and $\{V_j\}_{j\in\mathbb{Z}}$ be defined by* (3.1)*. Then the following conditions are equivalent:*

(i) $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G)$.

(ii) *ϕ is α-substantial.*

Proof. First assume that $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G)$. Suppose that, for some $g \in L^2(G)$, we have:

$$D^j \phi * g = 0; \;\; \forall\, j \in \mathbb{Z}.$$

Then using Lemma 3.4, we get that:

$$\langle T_{-x} D^j \phi, \tilde{g} \rangle = 0; \;\; \forall\, x \in G \text{ and } \forall\, j \in \mathbb{Z}.$$

This gives us that

$$\tilde{g} \perp \left(\overline{span} \left\{ T_x D^j \phi : x \in G, j \in \mathbb{Z} \right\} \right).$$

So by our assumption we get that $\tilde{g} = 0$ and hence $g = 0$. This implies that ϕ is α-substantial.

Conversely, suppose that ϕ is α-substantial. Let $g \perp \left(\overline{\underset{j\in\mathbb{Z}}{\cup} V_j}\right)$. Then Lemma 3.4-$(iii)$ gives us

$$\langle T_{-x}D^j\phi, g\rangle = 0;\ \ \forall x \in G \text{ and } \forall j \in \mathbb{Z}.$$

This implies that

$$D^j\phi * \tilde{g}(x) = 0;\ \ \forall x \in G \text{ and } \forall j \in \mathbb{Z}.$$

i.e.

$$D^j\phi * \tilde{g} = 0;\ \ \forall j \in \mathbb{Z}.$$

Since ϕ is α-substantial, therefore we must have $\tilde{g} = 0$; and hence $g = 0$. This implies that

$$\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G).$$

This proves the result. ∎

We now club all the conditions which need to be imposed on a function ϕ to get a frame multireoluton analysis.

Theorem 3.6. *A function $\phi \in L^2(G)$ generates an FMRA if it satisfies the following conditions:*

(i) *$\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence.*

(ii) *$\{V_j\}_{j\in\mathbb{Z}}$ are defined as in equation* (3.1).

(iii) *There exists a function $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ satisfying the relation:*

$$\hat{\phi}(\hat{\alpha}(\gamma)) = H_0(\gamma)\hat{\phi}(\gamma).$$

(iv) *ϕ is α-substantial.*

We now present some examples where a function ϕ satisfies all the properties listed in Theorem 3.6 and, thus, generates a frame multiresolution analysis:

Example 3.1. *Let $G = \mathbb{R}_+$ denote the multiplicative group of positive real numbers. This group together with the topology induced by the Euclidean group $\mathbb{R}$ forms an LCA group. For any Borel set $\mathcal{B}$ in G, a Haar measure μ_G on G is given by:*

$$\mu_G(\mathcal{B}) = \int_S d\mu_G(t);\ \textit{where}\ d\mu_G(t) = \frac{(\log 2)^{-1}}{t}dt.$$

The set $\Lambda = \{2^n : n \in \mathbb{Z}\}$ works as a uniform lattice in G, and the map $\alpha : x \mapsto x^2$ works as a dilative automorphism of G. For any $x, \xi \in G$, the map $x \mapsto x^{i\log\xi}$ is a continuous homomorphism from G to $\mathbb{T}$. Defining the characters of G in this way, i.e. by writing $(\xi, x) = \phi_\xi(x)$, we get that the Pontryagin dual group of $\mathbb{R}_+$ is $\mathbb{R}_+$ i.e., $\hat{G} = G$. The measure $\mu_{\hat{G}}$ is normalized appropriately so that the inversion formula and the Parseval formula hold. Further, the annihilator $\Lambda^\perp$ of Λ and the automorphism $\hat{\alpha}$ of $\hat{G}$(corresponding to the automorphism α of G) can be derived accordingly. We choose the set

$$\mathcal{Q} = \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right)$$

as a fundamental domain associated to Λ in G, and the set

$$\mathcal{S} = \left[e^{\frac{-\pi}{\log 2}}, e^{\frac{\pi}{\log 2}}\right)$$

as a fundamental domain associated to $\Lambda^\perp$ in $\hat{G}$. With the chosen measures, it is evident that

$$\mu_G(\mathcal{Q}) = 1 = \mu_{\hat{G}}(\mathcal{S}).$$

We now define a function $\phi \in L^2(G)$ via its Fourier transform as:

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_1}(\gamma); \text{ where } A_1 = \left[e^{\frac{-\pi}{\log 4}}, e^{\frac{\pi}{\log 4}}\right).$$

Then for $\gamma \in \left[e^{-\pi/\log 2}, e^{\pi/\log 2}\right)$, we have that

$$\sum_{m\in\mathbb{Z}} |\hat{\phi}(\gamma e^{\frac{2\pi m}{\log 2}})|^2 = \mathcal{X}_{A_1}(\gamma).$$

Thus, $\Phi(\gamma) = 1$ outside its null set and hence by Lemma 3.2, we get that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence.
Further note that, for $\gamma \in \mathcal{S}$, $\hat{\phi}(\hat{\alpha}(\gamma))$, i.e. $\hat{\phi}(\gamma^2)$ is given by:

$$\hat{\phi}(\gamma^2) = \mathcal{X}_{B_1}(\gamma); \text{ where } B_1 = \left[e^{\frac{-\pi}{\log 16}}, e^{\frac{\pi}{\log 16}}\right).$$

Define a function H on $\mathcal{S}$ by $H(\gamma) = \mathcal{X}_{B_1}(\gamma)$. Let H_0 denote the $\Lambda^\perp$-periodic extension of the function H. Then as per the definition of the space $L^\infty(\hat{G}/\Lambda^\perp)$, $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ and it satisfies the refinement equation

$$\hat{\phi}(\hat{\alpha}(\gamma)) = H_0(\gamma)\hat{\phi}(\gamma).$$

This means that the function ϕ is refinable.
We finally show that ϕ is α-substantial. For that, suppose there is a $g \in L^2(G)$ such that $D^j\phi * g = 0$ for all $j \in \mathbb{Z}$. Then, using the convolution theorem , we get that $(D^j\hat{\phi})(\hat{g}) = 0$ for all $j \in \mathbb{Z}$, i.e. $\hat{\phi}(\gamma^{2^j})\hat{g}(\gamma) = 0\, \forall j \in \mathbb{Z}$ and $\gamma \in \hat{G}$. Note that $\hat{\phi}(\gamma^{2^j}) \neq 0$ only when $\gamma \in \left[\left(e^{\frac{-\pi}{\log 4}}\right)^{2^{-j}}, \left(e^{\frac{\pi}{\log 4}}\right)^{2^{-j}}\right)$. Thus $\hat{g}$ has to be zero on the set $\bigcup_{j\in\mathbb{Z}} \left[\left(e^{\frac{-\pi}{\log 4}}\right)^{2^{-j}}, \left(e^{\frac{\pi}{\log 4}}\right)^{2^{-j}}\right)$.
This implies that $\hat{g} = 0$ on $\mathbb{R}_+$ and hence $g = 0$ on $\mathbb{R}_+$. So ϕ is α-substantial.
Defining the subspaces via (3.1)*, we conclude that the function ϕ generates an FMRA for $L^2(G)$.*

In most of the practical applications, the scaling function ϕ is defined via its Fourier transform, as in Example 3.1. Many authors, working with MRA and FMRA, prefer to give the density property in terms of behaviour of the function $\hat{\phi}$ around the identity $0 \in \hat{G}$ (see [2]). Keeping this in mind, we give the following lemma which connects α-substantiality of ϕ and the behaviour of $\hat{\phi}$ around $0 \in \hat{G}$.

Lemma 3.7. *If $\phi \in L^2(G)$ is such that $|\hat{\phi}| > 0$ on a neighbourhood of $0 \in \hat{G}$, then ϕ is α-substantial.*

Proof. Let U be a neighbourhood of $0 \in \hat{G}$ such that

$$\hat{\phi}(\gamma) \neq 0;\ \ \forall\, \gamma \in U.$$

Now if $D^j\phi * g = 0$ for some $g \in L^2(G)$, then, this means that

$$\hat{\phi}(\hat{\alpha}^j(\gamma))\hat{g}(\gamma) = 0;\ \ \forall\, \gamma \in \hat{G}.$$

For any $j \in \mathbb{Z}$, $\hat{\phi}(\hat{\alpha}^j(\gamma)) \neq 0$ whenever $\gamma \in \hat{\alpha}^{-j}(U)$. This means that $\hat{g} = 0$ on the set $\bigcup_{j\in\mathbb{Z}} \hat{\alpha}^{-j}(U)$. Since $\hat{\alpha}$ is a dilative automorphism on $\hat{G}$, therefore $\bigcup_{j\in\mathbb{Z}} \hat{\alpha}^{-j}(U) = \hat{G}$ and hence $\hat{g} = 0$ on $\hat{G}$. This implies that $g = 0$ on G and hence ϕ is α-substantial. ∎

We now give some more examples in the Euclidean space $\mathbb{R}^2$, where a function ϕ generates an FMRA.

Example 3.2. *Consider the LCA group $G = \mathbb{R}^2$ with the standard Haar measure μ given by*

$$\mu(\mathcal{B}) = \iint_{\mathcal{B}} dx_1 dx_2 \ ; \quad \mathcal{B} \subset \mathbb{R}^2 \text{ is a Borel set};$$

and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents a general point of the space G. It is a well known fact that the dual group, $\hat{G}$, of G is G; with exactly the same structure and same measure μ. Thus, $\hat{G}$ can be identified with G in this case. Further, the action of a character $\xi \in \mathbb{R}^2$ on an element $x \in \mathbb{R}^2$ is given by $(\xi, x) = e^{2\pi i \xi . x}$, where $\xi.x$ represents the usual dot product in $\mathbb{R}^2$. The set $\Lambda = \mathbb{Z} \times \mathbb{Z}$ works as a uniform lattice and the map $\alpha : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$ works as a dilative automorphism of G. Further, $\Lambda^{\perp} = \Lambda$ gives the annihilator of the lattice Λ and the coorosponding dilative automorphism $\hat{\alpha}$ on $\hat{G}$ may be identified with α. The set

$$\mathcal{S} = \left[\frac{-1}{2}, \frac{1}{2}\right) \times \left[\frac{-1}{2}, \frac{1}{2}\right)$$

is a fundamental domain associated to Λ as well as $\Lambda^{\perp}$ in $\mathbb{R}^2$. Moreover, with the chosen measure, we have

$$\mu(\mathcal{S}) = 1.$$

A function ϕ is now defined in $L^2(G)$ via its Fourier transform by:

$$\hat{\phi}(\gamma) = 4\mathcal{X}_{A_2}(\gamma) + 3\mathcal{X}_{A_3}(\gamma) - 7\mathcal{X}_{A_4}(\gamma);$$

where

$$\begin{aligned} A_2 &= \left[\frac{-1}{4}, \frac{-1}{4}\right) \times \left[\frac{-1}{4}, \frac{-1}{4}\right), \\ A_3 &= \left[\frac{-1}{3}, \frac{1}{3}\right) \times \left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right)\right), \\ A_4 &= \left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right)\right) \times \left[\frac{-1}{3}, \frac{1}{3}\right). \end{aligned}$$

Now if $\gamma \in \mathcal{S}$, then

$$\sum_{k \in \Lambda} |\hat{\phi}(\gamma + k)|^2 = 16\mathcal{X}_{B_2}(\gamma) + 9\mathcal{X}_{B_3}(\gamma) + 49\mathcal{X}_{B_4}(\gamma);$$

where

$$\begin{aligned} B_2 =& A_2 \bigcup \left(\left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right) \right) \times \left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right) \right) \right), \\ B_3 =& \left[\frac{-1}{4}, \frac{1}{4}\right) \times \left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right)\right), \\ B_4 =& \left(\left[\frac{-1}{3}, \frac{-1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{3}\right)\right) \times \left[\frac{-1}{4}, \frac{1}{4}\right). \end{aligned}$$

This implies that $9 \leq \Phi(\gamma) \leq 49$ outside its null set and hence $\{T_{\lambda}\phi\}_{\lambda \in \Lambda}$ is a frame sequence with lower bound 9 and upper bound 49.

Next we define a function H on $\mathcal{S}$ such that $H(\gamma) = \hat{\phi}(2\gamma)$. We then extend this function $\Lambda^{\perp}$-periodically to whole of $\mathbb{R}^2$ to get another function H_0. Clearly $H_0 \in L^{\infty}(\hat{G}/\Lambda^{\perp})$ and satisfies the refinement equation as stated below:

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma).$$

This means that the function ϕ is refinable.
Finally note that $|\hat{\phi}| > 0$ on any neighbourhood of $0 \in \hat{G}$ which is contained entirely contained in A_2. So we can conclude α-substantiality of ϕ using Lemma 3.7. Further, defining the subspaces V_j as in (3.1), we get that the function ϕ generates an FMRA on $L^2(G)$.

Example 3.3. *On the LCA group $G = \mathbb{R}^2$ (defined in Example 3.2) we define a function ϕ by:*

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_5}(\gamma);$$

where

$$A_5 = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| - \frac{1}{4}\sqrt{1 - 16x_1^2} \leq x_2 \leq |x_1| + \frac{1}{4}\sqrt{1 - 16x_1^2}, |x_1| \leq \frac{1}{4}\right\}.$$

Now if $\gamma \in \mathcal{S}$, then

$$\sum_{k \in \Lambda} |\hat{\phi}(\gamma + k)|^2 = \mathcal{X}_{A_5}(\gamma).$$

We infer from here that $\Phi(\gamma) = 1$ outside its null set, and this further implies that the sequence $\{T_k\phi\}_{k\in\Lambda}$ generates a tight frame sequence with frame bound equal to 1.
Let

$$B_5 = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| - \frac{1}{8}\sqrt{1 - 64x_1^2} \leq x_2 \leq |x_1| + \frac{1}{8}\sqrt{1 - 64x_1^2}, |x_1| \leq \frac{1}{8}\right\}.$$

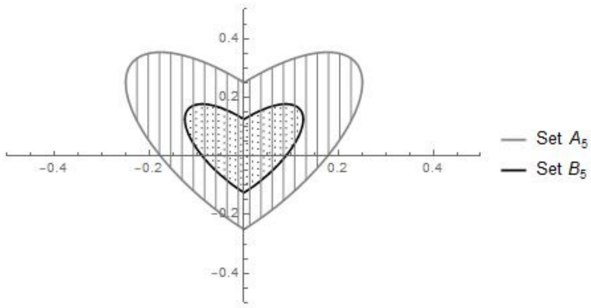


FIGURE 1. A relative diagram comparing the sets A_5 and B_5 in $\mathbb{R}^2$

We now define a function H on $\mathcal{S}$ by:

$$H(\gamma) = \mathcal{X}_{B_5}(\gamma).$$

Extending this function $\Lambda^\perp$-periodically, we obtain another function H_0. Clearly, H_0 is in $L^\infty(\hat{G}/\Lambda^\perp)$ and satisfies:

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma).$$

This means that ϕ is refinable.
Since $|\hat{\phi}|$ is continuous at $0 \in \hat{G}$ and $\hat{\phi}(0) \neq 0$, therefore α-substantiality of ϕ can be deduced by using Lemma 3.7. Thus, by defining the subspaves V_j as in equation (3.1), we conclude that the function ϕ generates an FMRA on $L^2(G)$.

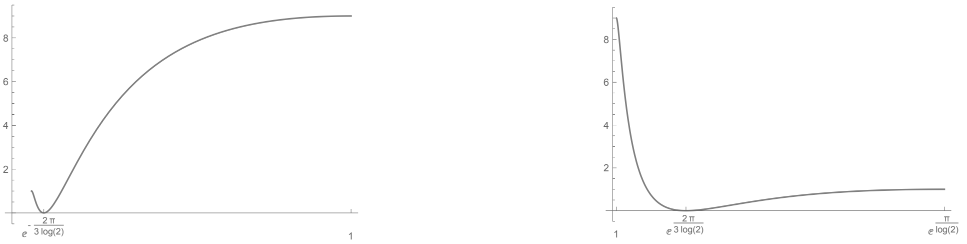
A function ϕ does not generate an FMRA whenever it fails to satisfy one or more conditions listed in Theorem 3.6. In the following example we present various scenarios where we are not able to generate an FMRA.

Example 3.4. *Consider the LCA group $G = \mathbb{R}_+$ as defined in Example 3.1.*

(a) *We define a function ϕ on G by:*

$$\phi(x) = \mathcal{X}_{A_1}(x); \quad A_1 = \left[\frac{1}{2}, 4\right).$$

It is easy to note that the funtion Φ is continuous and it has isolated zeroes at the points $e^{\frac{-2\pi}{3\log 2}}$ and $e^{\frac{2\pi}{3\log 2}}$ in $\mathcal{S}$ (see Figure 2).



(A) Φ on the interval $\left[e^{\frac{-\pi}{\log 2}}, 1\right)$ (B) Φ on the interval $\left[1, e^{\frac{\pi}{\log 2}}\right)$

FIGURE 2. Isolated zeroes of the functon Φ in $\left[e^{\frac{-\pi}{\log 2}}, e^{\frac{\pi}{\log 2}}\right)$

Lemma 3.2 implies that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ can't generate a frame sequence and thus ϕ can't generate an FMRA.

(b) *If a function ϕ on G is defined by*

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_2}(\gamma); \quad A_2 = \left[e^{\frac{\pi}{\log 2}}, e^{\frac{2\pi}{\log 2}}\right),$$

then it is evident that for no $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$, the refinement equation (3.4) *can be satisfied. Hence, ϕ doesnot generate an FMRA.*

(c) *A function ϕ on G is defined by:*

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_3}(\gamma); \quad A_3 = \left[1, e^{\frac{\pi}{\log 2}}\right).$$

It is easy to see that ϕ is refinable and that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence. But ϕ can't be α-substantial, as suggested by Lemma 3.7. Therefore ϕ can't generate an FMRA.

4. CONCLUSION

We combined the definition of MRA on LCA groups, given in [5], and the definition of FMRA on the Euclidean group $\mathbb{R}$, given in [13], to give the definition of FMRA on LCA groups (see Definition 3.1). The foremost steps in the construction of FMRA are:

- Choosing a function ϕ such that $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a frame sequence; so that the condition (v) of Definition 3.1 holds true.
- Defining the subspaces V_j by (3.1); due to which the conditions (iii) and (iv) of Definition 3.1 always hold true.

With these choices of ϕ and V_j, we see that the triviality of intersection of the subspaces V_j, mentioned at (ii) in above definition, becomes a redundant property, i.e., no extra assumption is required to prove this property. After that, we shifted our entire focus in establishing the remaining two properties, i.e., the density of the union and nested property of the subspaces V_j. With substant changes to the result of classical MRA, we proved an equivalent condition for the subspaces V_j to be nested. This equivalent condition is given in terms of the Fourier transform of the scaling function ϕ. To prove the density of the union, we used the concept of α-substantiality of the scaling function ϕ. Finally, in Theorem 3.6, we gathered all the conditions which need to be imposed on a function $\phi \in L^2(G)$ to generate an FMRA.

The next step is the construction of a frame for $L^2(G)$ via the given FMRA. For the case $G = \mathbb{R}$, the construction of a frame via a given FMRA has been studied by J.J. Benedetto and O. M. Treiber in [14]. They proved that, unlike the case of classical MRA, it may happen that we may not be able to find a function $\psi \in L^2(\mathbb{R})$ such that $\{D^j T_\lambda \psi\}_{\lambda \in \Lambda}$ is a frame for $L^2(\mathbb{R})$, inspite of an FMRA given to us. Establishing an analogous result for an arbitrary LCA group needs some additional study.

We emphasize that this paper deals explicitly with the construction of FMRA on LCA groups. As we mentioned earlier, there are several works which give us the theory of MRA on LCA groups, but none of them mentions FMRA on these groups. Hence, to best of our knowledge, the work done in this paper is a new work.

REFERENCES

[1] S. G. MALLAT, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$, *Trans. Amer. Math. Soc.*, **315** (1989), No. 1, pp. 69–87.

[2] I. DAUBECHIES, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1992).

[3] E. HERNÁNDEZ and G. WEISS, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, (1996).

[4] L. W. BAGGETT, H. A. MEDINA and K. D. MERRILL, Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.*, **5** (1999), No. 6, pp. 563–573.

[5] S. DAHLKE, Multiresolution analysis and wavelets on locally compact Abelian groups, in *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*,pp. 141–156, A K Peters, Wellesley, MA.

[6] R. A. KAMYABI-GOL and R. RAISI TOUSI, Some equivalent multiresolution conditions on locally compact Abelian groups, *Proc. Indian Acad. Sci. Math. Sci.*, **120** (2010), No. 3, pp. 317–331.

[7] Q. YANG and K. F. TAYLOR, Multiresolution analysis and Harr-like wavelet bases on locally compact groups, *J. Appl. Funct. Anal.*, **7** (2012), No. 4, pp. 413–439.

[8] A. DEITMAR and S. ECHTERHOFF, *Principles of harmonic analysis*, second edition, Universitext, Springer, Cham, (2014).

[9] G. B. FOLLAND, *A course in abstract harmonic analysis*, second edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, (2016).

[10] W. RUDIN, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York, (1962).

[11] E. KANIUTH and G. KUTYNIOK, Zeros of the Zak transform on locally compact Abelian groups, *Proc. Amer. Math. Soc.*, **126** (1998), No. 12, pp. 3561–3569.

[12] O. CHRISTENSEN, *An introduction to frames and Riesz bases*, second edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, (2016).

[13] J. J. BENEDETTO and S. LI, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.*, **5** (1998), No. 4, pp. 389–427.

[14] J. J. BENEDETTO and O. M. TREIBER, Wavelet frames: multiresolution analysis and extension principles, in Wavelet transforms and time-frequency signal analysis, pp. 3–36, *Appl. Numer. Harmon. Anal*, Birkhäuser Boston, Boston, MA.

[15] E. HEWITT and K. ROSS: *Abstract Harmonic Analysis, vols. 1 and 2*. Springer, Berlin (1963).

[16] C. CABRELLI and V. PATERNOSTRO, Shift-invariant spaces on LCA groups, *J. Funct. Anal.*, **258** (2010), No. 6, pp. 2034–2059.

[17] K. GRÖCHENIG, Aspects of Gabor analysis on locally compact abelian groups, in Gabor analysis and algorithms, pp. 211–231, *Appl. Numer. Harmon. Anal*, Birkhäuser Boston, Boston, MA.