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SWEEPING SURFACES WITH DARBOUX FRAME IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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ABSTRACT. The curve on a regular surface has a moving frame and it is called Darboux frame. We introduce sweeping surfaces along the curve relating to the this frame and investigate their geometrical properties. Moreover, we obtain the necessary and sufficient conditions for these surfaces to be developable ruled surfaces. Finally, an example to illustrate the application of the results is introduced.

Key words and phrases: Darboux frame; Sweeping surface; Singularity.

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1. INTRODUCTION

Sweeping surface is a surface generated by the motion of the plane curve (the profile curve or generatrix) while this movement of the plane in the space is in the same direction of the normal to the plane. In geometrical modeling, sweeping is an essential and useful tool and it has some applications in specially in geometric design. The idea depends on choosing a geometrical object, that is called generator, and sweeping it along a spine curve, which is called trajectory ([1]-[9]). In recent years, the properties of sweeping surfaces and their offsets surfaces have been examined in Euclidean and non-Euclidean spaces (See for instance Refs. ([6]-[13])). In view of the mentioned references , tubular surface, pipe surface, string, and canal surface are considered as different names for the sweeping surfaces ([11]-[13]). So far as we know, there is no previous studies in regard to curves lying in surfaces as the initial objects with the consideration of singularities and convexity of sweeping surfaces. In order to extend the work in [12], this study focuses on the geometrical properties of sweeping surfaces whose center curves in surfaces in Euclidean 3-space $\mathbb{E}^3$. Furthermore, in kinematics, the sweeping surfaces, the ruled surfaces, are introduced as one-dimensional line manifolds created by oriented moving line in the space, playing an important role of the line trajectory. As a consequence, considering the sweeping surfaces as a special ruled surfaces is important in both kinematics and differential line geometry theory.

In this work , the differential geometry of the sweeping surface with Darboux frame is developed. We also show that the parametric curves on this surface are lines of curvature. Then we study local singularities and convexity of a sweeping surface. In terms of this, we derived the necessary and sufficient condition for a sweeping surface to become the developable ruled surface. Additionally, an example of application is introduced and explained in detail.

2. PRELIMINARIES

The general references are used ([14],[15]) in this work. Let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to \mathbb{E}^3$ is the unit speed curve; we will use $\kappa(s)$ and $\tau(s)$ to define curvature and torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, in the same order. Let $\boldsymbol{\alpha}^{''}(s) \neq 0$ for all $s \in [0, L]$, which gives a straight line. At this research, $\boldsymbol{\alpha}^{'}(s)$ defines the derivative of $\boldsymbol{\alpha}$ respecting to s the arc length parameter. At every point of $\boldsymbol{\alpha}(s)$, the set $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is named Serret–Frenet frame through $\boldsymbol{\alpha}(s)$, such that $\mathbf{t}(s) = \boldsymbol{\alpha}^{'}(s)$ defines a unit tangent, $\mathbf{n}(s) = \frac{\boldsymbol{\alpha}^{''}(s)}{\left\|\boldsymbol{\alpha}^{''}(s)\right\|}$ being the unit principal normal, also $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ defines a unit binormal vector. The derivative of the Serret–Frenet frame respecting to the arc length is given as:

$$\begin{pmatrix} \mathbf{t}^{'} \\ \mathbf{n}^{'} \\ \mathbf{b}^{'} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{2.1}$$

Let F be the regular surface, and $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to F$ is the unit speed curve on F. At this surface, the Darboux frame is $\{\boldsymbol{\alpha}(s);\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$; $\mathbf{e}_1(s)$ is a unit tangent vector to $\boldsymbol{\alpha}(s)$, $\mathbf{e}_3 = \mathbf{e}_3(s)$ is a unit normal to the surface restricted to $\boldsymbol{\alpha}$, and $\mathbf{e}_2(s)= \mathbf{e}_3 \times \mathbf{e}_1$ is the unit tangent to the surface F. Then, the rotation matrix between Serret–Frenet frame and Darboux frame is

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{2.2}$$

The variation of the Darboux frame through $\boldsymbol{\alpha}(s)$ is described using the following equations:

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{2.3}$$

where

$$\left.\begin{aligned} \kappa_n(s) &= \kappa \sin\theta = \kappa_n(u), \\ \kappa_g(s) &= \kappa \cos\theta, \\ \tau_g(s) &= \tau - \theta'. \end{aligned}\right\} \tag{2.4}$$

We call $\kappa_g = \kappa_g(s)$ a geodesic curvature, $\kappa_n = \kappa_n(s)$ a normal curvature, and $\tau_g = \tau + \theta'$ a geodesic torsion of $\boldsymbol{\alpha}(s)$, in the same order. Using these quantities, geodesics line of curvatures, and asymptotic lines on the smooth surface can be characterized, as loci along which $\kappa_g = 0$, $\tau_g = 0$, and $\kappa_n = 0$, in the same order.

3. SWEEPING SURFACES WITH DARBOUX FRAME

In this section, we give the parametric representations of sweeping surfaces through the spine curve $\boldsymbol{\alpha}(s)$ of the surface F in the following: The sweeping surface associated to $\boldsymbol{\alpha}(s)$, is the envelope of the family of unit spheres, with the center on the curve $\boldsymbol{\alpha}(s) \in F$.

Remark 3.1. Clearly, if $\boldsymbol{\alpha}(s)$ is a straight line, thus the sweeping surface is just a circular cylinder, having $\boldsymbol{\alpha}(s)$ as symmetry axis. If, on the other hand, $\boldsymbol{\alpha}(s)$ is a circle, then the corresponding sweeping surface is a torus.

Now, it is easy to see that the contact between the spheres from the family and the sweeping surface is a great circle of the unit sphere, lying in the subspace $Sp\{\mathbf{e}_2, \mathbf{e}_3\}$, of the spine curve $\boldsymbol{\alpha}(s)$. Let us denote by $\mathbf{Q}$ the position vector connecting the point from the curve $\boldsymbol{\alpha}(s)$ with the point from the sweeping surface. Then, clearly, we have

$$M : \mathbf{Q} = \boldsymbol{\alpha}(s) + \mathbf{x}, \tag{3.1}$$

where the unit vector $\mathbf{x}$ itself lies in the same subspace $Sp\{\mathbf{e}_2, \mathbf{e}_3\}$. Let us denote by the angle ϑ between the vectors $\mathbf{x}$ and $\mathbf{e}_2$. Then, as one can see immediately, we have

$$\mathbf{x}(\vartheta) = \cos\vartheta \mathbf{e}_2 + \sin\vartheta \mathbf{e}_3, \tag{3.2}$$

which is the characteristic circles of sweeping surface. Combining Eqs. (3.1) and (3.2), we see that we obtained a parameterization of the sweeping surface,

$$M : \mathbf{Q}(s, \vartheta) = \boldsymbol{\alpha}(s) + \cos\vartheta \mathbf{e}_2(s) + \sin\vartheta \mathbf{e}_3(s). \tag{3.3}$$

This parametrization of M excludes sweeping surfaces with stationary vector $\mathbf{e}_1$, because its geometrical properties that is not very important and very easy to be studied.

3.1. The Properties of sweeping surfaces.

Using the formulae in Eq. (3.3) , we calculate

$$\left.\begin{aligned} \mathbf{Q}_\vartheta(s, \vartheta) &= -\sin\vartheta \mathbf{e}_2 + \cos\vartheta \mathbf{e}_3 + \tau_g \mathbf{Q}_s(s, \vartheta), \\ \mathbf{Q}_s(s, \vartheta) &= (1 - \kappa_g \cos\vartheta - \kappa_n \sin\vartheta)\mathbf{e}_1, \end{aligned}\right\} \tag{3.4}$$

and

$$\mathbf{N}(s, \vartheta) := \frac{\mathbf{Q}_\vartheta \times \mathbf{Q}_s}{\|\mathbf{Q}_\vartheta \times \mathbf{Q}_s\|} = \cos\vartheta \mathbf{e}_2 + \sin\vartheta \mathbf{e}_3. \tag{3.5}$$

Eq.(3.5) shows that the surface normal $\mathbf{N}(s,\vartheta)$ is included in the subspace $Sp\{\mathbf{e}_2, \mathbf{e}_3\}$, because it is perpendicular to $\mathbf{e}_1$. Also Eqs. (3.4), it is easily checked that the coefficients of the first fundamental form $g_{11} =< \mathbf{Q_s}, \mathbf{Q_s} >$, $g_{12} =< \mathbf{Q}_s, \mathbf{Q}_\vartheta >$ and $g_{22} =< \mathbf{Q}_\vartheta, \mathbf{Q}_\vartheta >$ are given by

$$g_{11}{=}(1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta)^2+\tau_g^2,\ g_{12}=\tau_g,\ \ g_{22}=1. \tag{3.6}$$

To compute the second fundamental form of M, we have to calculate the following

$$\left.\begin{aligned}
&\mathbf{Q}_{ss}=\left[-\left(\kappa_g^{'}\cos\vartheta+\kappa_n^{'}\sin\vartheta\right)+\tau_g(\kappa_g\cos\vartheta-\kappa_n\sin\vartheta)\right]\mathbf{e}_1\\
&+\left[\kappa_g\left(1-\kappa_n\sin\vartheta\right)-\left(\kappa_g^2+\tau_g^2\right)\cos\vartheta-\tau_g^{'}\sin\vartheta\right]\mathbf{e}_2\\
&+\left[\kappa_n\left(1-\kappa_g\cos\vartheta\right)-\left(\kappa_g^2+\tau_g^2\right)\sin\vartheta+\tau_g^{'}\cos\vartheta\right]\mathbf{e}_3,\\
&\mathbf{Q}_{s\vartheta}=(\kappa_g\sin\vartheta-\kappa_n\cos\vartheta)\mathbf{e}_1+\tau_g\mathbf{e}_3,\\
&\mathbf{Q}_{\vartheta\vartheta}=-\cos\vartheta\mathbf{e}_2-\sin\vartheta\mathbf{e}_3.
\end{aligned}\right\}$$

This gives the second fundamental form elements as $h_{11} =< \mathbf{Q}_{ss}, \mathbf{N} >$, $h_{12} =< \mathbf{Q}_{s\vartheta}, \mathbf{N} >$, and $h_{22} =< \mathbf{Q}_{\vartheta\vartheta}, \mathbf{N} >$ are given by

$$\left.\begin{aligned}
&h_{11}=(1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta)(\kappa_g\ \cos\vartheta+\kappa_n\sin\vartheta)+\tau_g^2,\\
&h_{12}=\tau_g,\ h_{22}=-1.
\end{aligned}\right\} \tag{3.7}$$

The Gaussian and mean curvature of the sweeping surface at a regular point can be calculated, respectively, as

$$K(s,\vartheta):=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{\kappa_g\ \cos\vartheta+\kappa_n\sin\vartheta}{1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta}. \tag{3.8}$$

and

$$H(s,\vartheta):=\frac{g_{22}h_{11}-2g_{12}h_{12}+g_{11}h_{22}}{2\left(g_{11}g_{22}-g_{12}^2\right)}=\frac{2\left(\kappa_g\ \cos\vartheta+\kappa_n\sin\vartheta\right)-1}{2\left(1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta-1\right)}. \tag{3.9}$$

Proposition 3.1. *For the sweping surface M represented by Eq.* (3.3) **,** *the values of $K(s,\vartheta)$, and $H(s,\vartheta)$ are independent of the geodesic torsion of the spine curve $\boldsymbol{\alpha}(s)$.*

Proposition 3.2. *For the sweping surface M represented by Eq.*(3.3)*, then we state the following:*
(1) *If $\boldsymbol{\alpha}$ is a geodesic on F, then the Gaussian and mean curvature of the sweeping surface M are:*

$$K(s,\vartheta)=\frac{\kappa_n\sin\vartheta}{1-\kappa_n\sin\vartheta},\ \textit{and}\ H(s,\vartheta)=\frac{2\kappa_n\sin\vartheta-1}{2\left(1-\kappa_n\sin\vartheta-1\right)}.$$

(2) *If $\boldsymbol{\alpha}$ is an asymptotic on F, then the Gaussian and mean curvature of the sweeping surface M are*

$$K(s,\vartheta)=\frac{\kappa_g\ \cos\vartheta}{1-\kappa_g\cos\vartheta},\ \textit{and}\ H(s,\vartheta)=\frac{2\kappa_g\ \cos\vartheta-1}{2\left(1-\kappa_g\cos\vartheta-1\right)}.$$

On the other hand, from Eq.(3.3) *it is easily checked that the isoparametric curve*

$$\boldsymbol{\zeta}(\vartheta):=\mathbf{Q}(\vartheta,s_0)=\boldsymbol{\alpha}(s_0)+\cos\vartheta\mathbf{e}_2(s_0)+\sin\vartheta\mathbf{e}_3(s_0), \tag{3.10}$$

is a planar unit speed curve. The unit tangent vector to $\boldsymbol{\zeta}(\vartheta)$ is

$$\mathbf{T}_\zeta(\vartheta)=-\sin\vartheta\mathbf{e}_2(s_0)+\cos\vartheta\mathbf{e}_3(s_0),$$

and therefore the unit principal normal vector of $\boldsymbol{\zeta}(\vartheta)$ *is calculated as*

$$\mathbf{N}_\zeta = \mathbf{e}_1(s_0) \times \mathbf{T}_\zeta(u) = \cos\vartheta \mathbf{e}_2 + \sin\vartheta \mathbf{e}_3 = \mathbf{N}(s_0, \vartheta).$$

Hence, the surface normal $\mathbf{N}(s_0, \vartheta)$ *is parallel to the principal normal* $\mathbf{N}_\zeta$*, i.e., the curve* $\boldsymbol{\zeta}(\vartheta)$ *is a geodesic, and cannot be asymptotic curve on* M*.*

Proposition 3.3. *For the sweping surface* M *represented by Eq.*(3.3)*, then the* s*-parameter curves are asymptotic curves on* M *if and only if*

$$\vartheta = \tan^{-1}\left(\frac{\kappa_n \pm \kappa_g\sqrt{\kappa_n^2 + \kappa_g^2 - \sigma}}{\kappa_g \pm \kappa_n\sqrt{\kappa_n^2 + \kappa_g^2 - \sigma}}\right), \tag{3.11}$$

where

$$\sigma(s) = \frac{1}{2}\left[1 \pm \sqrt{1 - 4\tau_g^4}\right].$$

Proof. The s- parameter curves are asymptotic curves on M if and only if

$$< \mathbf{N}, \mathbf{Q}_{ss} >= 0 \Leftrightarrow (\kappa_g \cos\vartheta + \kappa_n \sin\vartheta)^2 - (\kappa_g \cos\vartheta + \kappa_n \sin\vartheta) - \tau_g^2 = 0.$$

It follows that

$$\sin\vartheta = \frac{\kappa_n \pm \kappa_g\sqrt{\kappa_n^2 + \kappa_g^2 - \sigma^2}}{\kappa_n^2 + \kappa_g^2}, \text{ and } \cos\vartheta = \frac{\kappa_g \pm \kappa_n\sqrt{\kappa_n^2 + \kappa_g^2 - \sigma^2}}{\kappa_n^2 + \kappa_g^2}, \tag{3.12}$$

we therefore obtain Eq.(3.11) which leads to the end of the proof ∎

3.1.1. **Singularity and lines of curvature.** Singularities and lines of curvature are very important to understand the properties and they are studied as in the following: M has singular points if and only if their first derivatives are linearly dependent, that is,

$$\mathbf{Q}_\vartheta \times \mathbf{Q}_s = (1 - \kappa_g \cos\vartheta - \kappa_n \sin\vartheta)\mathbf{N} = \mathbf{0}. \tag{3.13}$$

Since $\mathbf{N}$ is a nonzero unit vector, then $1 - \kappa_g \cos\vartheta - \kappa_n \sin\vartheta = 0$, that is,

$$\sin\vartheta = \frac{\kappa_n \pm \kappa_g\sqrt{\kappa_n^2 + \kappa_g^2 - 1}}{\kappa_n^2 + \kappa_g^2}, \text{ and } \cos\vartheta = \frac{\kappa_g \pm \kappa_n\sqrt{\kappa_n^2 + \kappa_g^2 - 1}}{\kappa_n^2 + \kappa_g^2}. \tag{3.14}$$

Hence there exist two singular points on every generating circle. Joining them gives two curves that contain all the singular points of a sweeping surface. Using Eq. (3.3), the two singular curves are

$$\left.\begin{aligned} C_1 : \boldsymbol{\gamma}(s) &= \boldsymbol{\alpha} + \tfrac{\kappa_g + \kappa_n\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_2 + \tfrac{\kappa_n + \kappa_g\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_3, \\ C_2 : \boldsymbol{\gamma}(s) &= \boldsymbol{\alpha} + \tfrac{\kappa_g - \kappa_n\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_2 + \tfrac{\kappa_n - \kappa_g\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_3. \end{aligned}\right\} \tag{3.15}$$

From the above analysis the following conclusions can be reached:

Corollary 3.1. The sweeping surface M represented by Eq. (3.3), has no singular points if the condition

$$1 - \kappa_g\frac{\kappa_g \pm \kappa_n\sqrt{\kappa_n^2 + \kappa_g^2 - 1}}{\kappa_n^2 + \kappa_g^2} - \kappa_n\frac{\kappa_n \pm \kappa_g\sqrt{\kappa_n^2 + \kappa_g^2 - 1}}{\kappa_n^2 + \kappa_g^2} \neq 0,$$

is satisfied.

According to theorem of line of curvature, it is well known that for the generating circles to be lines of curvature it must be $\mathbf{N}_\vartheta = \lambda(\vartheta)\mathbf{Q}_\vartheta$, where $\lambda(\vartheta)$ is a differentiable function of ϑ.

Using algebraic manipulations, it is founded that the generating circles are lines of curvature if and only if

$$\tau_g(1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta)=0.$$

for all the values of s, and ϑ. Clearly, there are two major cases, we present them as:
Case (1) happens if $\tau_g\neq 0$, and $1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta=0$. Therefore, two singular points on the generating circle occur at

$$\vartheta=\tan^{-1}\left(\frac{\kappa_n\pm\kappa_g\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_g\pm\kappa_n\sqrt{\kappa_n^2+\kappa_g^2-1}}\right). \tag{3.16}$$

Case (2) occurs when $\tau_g=0$, and $1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta\neq 0$. Therefore, from Eqs.(3.6) and (3.7) it can be found that $g_{12}=h_{12}=0$. Thus, the ϑ-and s curves of M are lines of curvature.

Surfaces whose parametric curves are lines of curvature have several applications in geometric designs ([2]-[4]). In the case of sweeping surfaces, one has to compute the offset surfaces $\mathbf{Q}_f(s,\vartheta)=\mathbf{Q}(s,\vartheta)+f\ \mathbf{N}(s,\vartheta)$ of a given surface $\mathbf{Q}(s,\vartheta)$ at a certain distance f. In consequence of this equation, the offsetting operation for sweeping surface is reduced to the offsetting of planar profile curve, which is easier. Hence, the following proposition is given:

Proposition 3.4. *Consider a sweeping surface M represented by Eq.*(3.3)*. Let $\mathbf{x}_f(\vartheta)$ be the planar offset of the profile $\mathbf{x}(\psi)$ at distance f. Therfore the offset surface $\mathbf{Q}_f(s,\vartheta)$ is a sweeping surface, generated by spine curve $\boldsymbol{\alpha}(s)$ and profile curve $\mathbf{x}_f(\vartheta)$.*

Through the reminder of this work we will study sweeping surfaces characterized by $\tau_g=0$, and $1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta\neq 0$. Therefore, the value of one principal curvature is

$$\chi_1:=\left\|\frac{d\mathbf{x}}{d\vartheta}\times\frac{d^2\mathbf{x}}{d\vartheta^2}\right\|\left\|\frac{d\mathbf{x}}{d\vartheta}\right\|^{-3}=1. \tag{3.17}$$

The other principal curvature is easy to get

$$\chi_2=\frac{K(s,\vartheta)}{\chi_1}=-\frac{\kappa_g\ \cos\vartheta+\kappa_n\sin\vartheta}{1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta}. \tag{3.18}$$

To analyze the shape of $M(\vartheta,s)$ we investigate the Gaussian curvature $K(s,\vartheta)$ in the following: The curvature of the isoparametric s-curves (u-constant) is

$$\chi:=\frac{\|\mathbf{Q}_s\times\mathbf{Q}_{ss}\|}{\|\mathbf{Q}_s\|^3}=\frac{\kappa}{1-\kappa_g\cos\vartheta-\kappa_n\sin\vartheta}. \tag{3.19}$$

Furthermore, from Eqs (2.2) and (3.5) we see that

$$\mathbf{N}(s,\varphi)=\cos\varphi\mathbf{n}+\sin\varphi\mathbf{b},\ \text{with}\ \varphi=\vartheta-\theta. \tag{3.20}$$

Here φ is the angle from $\mathbf{n}$ to $\mathbf{N}$ in the orientation of the tangent plane $T(M)$. In addition, the principal curvature χ_2 relates to the curvature $\chi(s,\vartheta)$ via Meusnier's Theorem ([14],[15]):

$$\chi_2=\chi(s,\vartheta)\cos\varphi. \tag{3.21}$$

Therefore, the Gaussian curvature $K(s,\vartheta)$ can be rewritten as

$$K(s,\vartheta)=\chi(s,\vartheta)\cos\varphi. \tag{3.22}$$

To fined curves on M that are generated by parabolic points (where Gaussian curvature vanishing). Those curves give the separation of the surface as elliptic parts ($K>0$, locally convex) and hyperbolic parts ($K<0$, hence non-convex). In computer aided design, conditions which insure convexity of the surface are essential in many applications (such as manufacturing of sculptured surfaces, or layered manufacturing). In the case of the sweeping surface M, the convexity is controlled by the differential geometric properties as:

$$K(s,\vartheta)=0 \Leftrightarrow \chi(s,\vartheta)\cos\varphi=0. \tag{3.23}$$

It can be seen that there are two potential cases that cause parabolic points:
Case (1) occurs in the case of $\chi(s,\vartheta)=0$. Using Eq. (3.19), it is clear that if $\kappa(s)=0$. In other words, the spine curve $\alpha=\alpha(s)$ is degenerate to a straight line. Therefore, an inflection or flat point of the spine curve gives a parabolic curve $\vartheta=$ const.
Case (2) occurs in the case of $\varphi=\pi/2$, this means that $N(s,\vartheta)\parallel \mathbf{b}$, and so $\cos\varphi=0$. Hence, the curve $\alpha(s)$ is a line of curvature as well as an asymptotic of the sweeping surface. Also, for existence of the parabolic points, the condition

$$\cos\varphi=0 \Leftrightarrow \vartheta-\theta=\frac{\pi}{2}, \tag{3.24}$$

is satisfied. In fact we have the following:

Corollary 3.2. Consider a sweeping surface M represented by Eq.(3.3) with spine and profile curves have nonzero curvatures everywhere. If the normal $\mathbf{N}(s,\vartheta)$ is never parallel to the principal normal $\mathbf{n}(s)$ of the spine curve $\boldsymbol{\alpha}(s)$, then M has no parabolic points.

According to Proposition 3.3, with attention to $\tau_g=0$, Eqs.(3.3) and (3.11) the expression of the two parabolic curves is

$$\left.\begin{aligned}\Gamma_1:\mathbf{P}(s)&=\boldsymbol{\alpha}-\tfrac{\kappa_n+\kappa_g\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_2+\tfrac{\kappa_g+\kappa_n\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_3,\\ \Gamma_2:\mathbf{P}(s)&=\boldsymbol{\alpha}-\tfrac{\kappa_n-\kappa_g\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_2+\tfrac{\kappa_g-\kappa_n\sqrt{\kappa_n^2+\kappa_g^2-1}}{\kappa_n^2+\kappa_g^2}\mathbf{e}_3.\end{aligned}\right\} \tag{3.25}$$

Corollary 3.3. Consider a sweeping surface M represented by Eq.(3.3) with spine and profile curves have nonzero curvatures everywhere. Therefore, M has exactly two parabolic curves if and only if the spine curve is an asymptotic curve.

3.2. **Developable surfaces.** This part discuss in what conditions the sweeping surfaces are developable surfaces. Therefore, in the case of the profile curve $\mathbf{x}$ degenerates to a straight line, we have the following developable surface

$$S:\mathbf{P}(s,u)=\boldsymbol{\alpha}(s)+u\mathbf{e}_3(s),\ \ u\in\mathbb{R}. \tag{3.26}$$

Similarly, from Eq.(3.3), we have the following developable surface

$$S^{\perp}:\mathbf{P}^{\perp}(s,u)=\boldsymbol{\alpha}(s)+u\mathbf{e}_2(s),\ u\in\mathbb{R}. \tag{3.27}$$

It is possible to show $\mathbf{P}(s,0)=\boldsymbol{\alpha}(s)$ (resp. $\mathbf{P}^{\perp}(s,0)=\boldsymbol{\alpha}(s)$), $0\leq s\leq L$, that is the surface S (resp. $S^{\perp}$) interpolate the curve $\boldsymbol{\alpha}(s)$. Furthermore, since

$$\mathbf{P}_s\times\mathbf{P}_u:=-\left(1-u\kappa_n\right)\mathbf{e}_2(s), \tag{3.28}$$

then $S^{\perp}$ is the normal developable surface of S along $\boldsymbol{\alpha}(s)$. Therefore, the surface S (resp. $S^{\perp}$) interpolates the curve $\boldsymbol{\alpha}(s)$, and $\boldsymbol{\alpha}(s)$ is a line of curvature of S (resp. $S^{\perp}$).

Proposition 3.5. *Consider a sweeping surface M represented by Eq.*(3.3) *Thus we have the following:*
(1) *The developable surfaces S and $S^{\perp}$ intersect along $\boldsymbol{\alpha}(s)$ at a right angle,*
(2) *The curve $\boldsymbol{\alpha}(s)$ is a line of curvature on S and $S^{\perp}$.*

As an application of the developable surface S we can associate it with the Darbox frame through the motion , so we can fined family of cylindrical cutter equations that is defined along $\boldsymbol{\alpha}(s)$ in the following:

$$S_f : \overline{\mathbf{P}}(s,u) = \mathbf{P}(s,u) + f\mathbf{e}_2(s), \tag{3.29}$$

where f defines cylindrical cutter radius. This surface is a developable surface offset of the surface $\mathbf{P}(s,u)$. Then the equation of S_f is

$$S_f : \overline{\mathbf{P}}(s,u) = \boldsymbol{\alpha}(s) + u\mathbf{e}_3(s) + f\mathbf{e}_2(s). \tag{3.30}$$

The normal vector of cylindrical cutter is presented as

$$\mathbf{U}_f(s,0) = \frac{\overline{\mathbf{P}}_s \times \overline{\mathbf{P}}_u}{\left\|\overline{\mathbf{P}}_s \times \overline{\mathbf{P}}_u\right\|} = \mathbf{e}_2(s). \tag{3.31}$$

Also, from Eq.(3.30), we have

$$S : \mathbf{P}(s,u) = \overline{\mathbf{P}}(s,u) - f\mathbf{e}_2(s). \tag{3.32}$$

The derivation of Eq. (3.32) respect with s is written as

$$\overline{\mathbf{P}}_s(s,u) = \mathbf{P}_s(s,u) - (f\boldsymbol{\omega}) \times \mathbf{e}_2. \tag{3.33}$$

From Eq.(3.33) it is clear that the vector $\overline{\mathbf{P}}_s(s,u)$ is perpendicular to the normal vector $\mathbf{e}_2$. Additionally, the vector $\mathbf{e}_2$ is perpendicular to the tool axis vector $\mathbf{e}_1(s)$. As a consequence, the envelope surface of the cylindrical cutter and the developable surface $\mathbf{P}(s,u)$ have the common normal vector and the length between the two surfaces is cylindrical cutter radius f. Hence, the following conclusion is presented:

Proposition 3.6. *Consider a developable surface S as in Eq.* (3.26)*. Let S_f be the envelope surface of cylindrical cutter at distance f. Therefor, the two surfaces S and S_f are offset developable surfaces.*

3.3. **Application.** Now, as an application of our main results, we give the following example.

Example 3.1. *Let F be a surface define by*

$$F : \mathbf{X}(s,v) = (\cos s - \frac{v}{\sqrt{2}}\cos s, \sin s - \frac{v}{\sqrt{2}}\sin s, \frac{v}{\sqrt{2}}),$$

where $I \subseteq \mathbb{R}$, and $v \in \mathbb{R}$. According to

$$\mathbf{e}_3(s,v) = \frac{\mathbf{X}_s \times \mathbf{X}_v}{\|\mathbf{X}_s \times \mathbf{X}_v\|}, \text{ and } \mathbf{e}_3(s,0) = \mathbf{e}_3(s),$$

we get

$$\mathbf{e}_3(s) = (\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}).$$

Moreover, we have

$$\mathbf{e}_1(s) = (-\sin s, \cos s, 0).$$

Since $\mathbf{e}_2(s) = \mathbf{e}_3(s) \times \mathbf{e}_1(s)$,

$$\mathbf{e}_2(s) = (-\frac{1}{\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}).$$

Also, we can calculate

$$\kappa_g = \frac{1}{\sqrt{2}}, \kappa_n = \frac{-1}{\sqrt{2}}, \text{ and } \tau_g = 0.$$

So, the parametric form of the sweeping surface family can be written as

$$M : \mathbf{Q}(s, \psi) = (\cos s, \sin s, 0) + (0, \cos\psi, \sin\psi) \begin{pmatrix} -\sin s & \cos s & 0 \\ -\frac{1}{\sqrt{2}}\cos s & -\frac{1}{\sqrt{2}}\sin s & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\cos s & \frac{1}{\sqrt{2}}\sin s & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The graphs of the surfaces F, M, and $F \cup M$ are shown in Figs. 1, 2 and 3; $0 \leq \psi,\, s \leq 2\pi$.

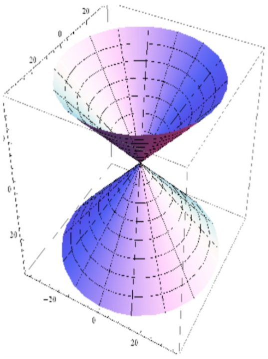

Figure 1: The surface F.

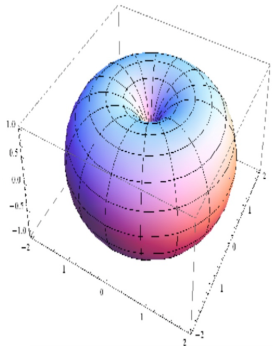

Figure 2: The surface M.

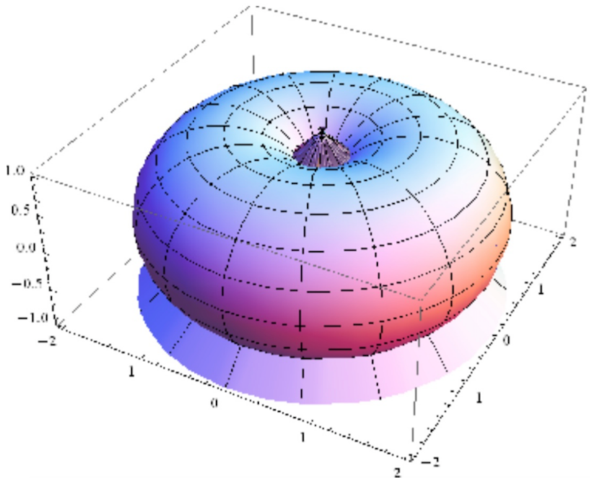

Figure 3: The surface $F \cup M$.

4. CONCLUSION

This paper studied the Darboux frames that are associated with a curve on surface and the sweeping surface that is generated by these frames. Moreover, the requirements of the surfaces to be both sweeping and developable surfaces at the same time are investigated. It is clear that there are many possibilities for extended studies. Analogously, the study in this paper can be considered for 3-surfaces in 4-space and we plane to do it in the future work.

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