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SOME PROPERTIES OF COSINE SERIES WITH COEFFICIENTS FROM CLASS OF GENERAL MONOTONE SEQUENCES ORDER r

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ABSTRACT. The coefficient of sine series from general monotone class has been generalized by Bogdan Szal to the new class which is called class of general monotone order r. This coefficient class is more general than class of general monotone introduced by Tikhonov. By special case, we study properties of cosine series with coefficient of sine series from the class of general monotone order r.

Key words and phrases: Cosine series; Coefficient of sine series; Monotone class.

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1. INTRODUCTION

One of the interesting problems in the Fourier sine series is the monotonicity of the coefficients of the series that is decreasing monotone and converging to zero. The necessary and sufficient condition of uniform convergence of sine series is

$$\lim_{n\to\infty} na_n = 0,$$

if $a = a_i$ is decreasing monotone and tending to zero [1, 6]. The decreasing monotone coefficients of sine series is written by MS and it has been extended to General Monotone written GM [5]. If $a = a_i$ sequences of complex numbers, a sequence a is in general monotone (GM) class, if there exists positive constant C, if inequality

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C|a_n|,$$

holds for each $n \in \mathbb{N}$. This class has been extended by Tikonov [5] to Definition 1.1 as follows:

Definition 1.1. Let $a = a_i$ sequences of complex numbers and $\beta = \beta_i$ is positive numbers. A sequence (a, β) is in β general monotone, if there exists positive constant C, such that

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C\beta_n,$$

holds for each $n \in \mathbb{N}$.

Later Bogdan Szal [2] extended to the class sequence number which is called class general monotone order r in Definition below.

Definition 1.2. Let $a = a_i$ complex sequence numbers and $\beta = \beta_i$ is positive numbers. A sequence (a, β) is in class β general monotone order r, written $(a, \beta) \in GM(\beta, r)$, if there exists constant $C > 0$, such that

$$\sum_{k=n}^{2n-1} |a_k - a_{k+r}| \leq C\beta_n,$$

holds for each $n \in \mathbb{N}$ and $r \in \mathbb{N}$.

For

$$\beta^* = \beta^*(r) = \sum_{k=n}^{n+r-1} |a_k| + \sum_{k=\left[\frac{n}{c}\right]}^{[cn]} \frac{|a_k|}{k}$$

with $c > 1$ and Definition 1.2 obtained some result at follows:

Theorem 1.1. *If $r_1, r_2 \in \mathbb{N}$ and r_1/r_2, then $GM(\beta^*, r_1) \subsetneq GM(\beta^*, r_2)$.*

Theorem 1.2. *If non-negative $a = a_i$ in class of $GM(\beta^*, r)$ where $r \in \mathbb{N}, r \geq 1$ and sine series is converges uniformly to continuous function, then $\lim_{n\to\infty} n\beta_n = 0$.*

Theorem 1.3. *If $a = a_i$ is element class of $GM(\beta^*, r)$, where $r = 2$ and $\lim_{n\to\infty} n|a_n| = 0$, then sine series coverges uniformly.*

In this paper, by idea definition of new class of general monotone [2] and uniform convergence [3] and [4] in class of supremum bounded variation sequences, we will discuss properties of coefficient from new class of general monotone sequences of order r in cosine series.

2. RESULTS

In this section, we will discuss some results about class of general monotone order r associate with uniform convergence of cosine series.

2.1. Some Properties Class of General monotone sequences order r.

Theorem 2.1. *If $a = a_i$ is element class of $GM(|a|, r)$ and $n|a_n|$ decreasing monotone, then*

$$\sum_{s=n}^{n+r} |a_s - a_{s+1}| \leq 4Ca_n. \tag{2.1}$$

Proof. Let $a = a_i$ is element class of $GM(|a|, r)$

$$\begin{aligned}\sum_{s=n}^{\infty} |a_s - a_{s+1}| &= \sum_{s=n}^{\infty} |a_s - a_{s+1} - a_{s+r} + a_{s+r} - a_{s+r+1} + a_{s+r+1}| \\ &\leq \sum_{s=n}^{\infty} |a_s - a_{s+r}| + \sum_{s=n}^{\infty} |a_{s+1} - a_{s+r+1}| + \sum_{s=n}^{\infty} |a_{s+r} - a_{s+r+1}|.\end{aligned}$$

Then we have

$$\begin{aligned}n\sum_{s=n}^{\infty} |a_s - a_{s+1}| &\leq n\sum_{s=n}^{\infty} |a_s - a_{s+r}| + n\sum_{s=n}^{\infty} |a_{s+1} - a_{s+r+1}| + n\sum_{s=n}^{\infty} |a_{s+r} - a_{s+r+1}| \\ &\leq 2n\sum_{s=n}^{\infty} |a_s - a_{s+r}| \\ &= 2\sum_{s=0}^{\infty} \sum_{v=2^s n}^{2^{s+1}n-1} |a_v| = 2C\sum_{s=0}^{\infty} \frac{2^s n}{2^s} |a_{2^s n}| \leq 4Cn|a_n|.\end{aligned}$$

Then (2.1) is proved. ∎

Theorem 2.2. *If $a = a_i$ is element class of $GM(|a|, r)$, $n|a_n|$ decreasing monotone and $\lim_{n\to\infty} na_n = 0$ then $a = a_i$ is bounded variation.*

Proof. Given $n \in \mathbb{N}$, by Theorem 2.1 we can write

$$\sum_{s=n}^{n+r} |a_s - a_{s+1}| \leq 4Ca_n.$$

Since $\lim_{n\to\infty} |a_n| = 0$ is convergence, then $a = a_i$ is bounded variation. ∎

Theorem 2.3. *If $a \in GM(|a|, r)$ and $\lim_{n\to\infty} n|a_n| = 0$ then*

$$\lim_{n\to\infty} n\sum_{v=n}^{\infty} |a_v - a_{v+r}| = 0. \tag{2.2}$$

Proof. Let

$$\rho_n = \sup_{v\geq n} v\,|a_v|$$

for each $n \in \mathbb{N}$ and $a \in GM(|a|, r)$ by Theorem 2.1 we have

$$n\sum_{v=n}^{\infty} |a_v - a_{v+r}| \leq \rho_n.$$

So, we obtain

$$\lim_{n\to\infty} n\sum_{v=n}^{\infty}|a_v - a_{v+r}| \le \lim_{n\to\infty}\rho_n.$$

Since $\lim\limits_{n\to\infty} n|a_n| = 0$, we have (2.2). ∎

2.2. **Uniform Convergence of Cosine Series.** In this section, we discuss uniform convergence of cosine series. Let the series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx \tag{2.3}$$

which $a = a_k$ is sequence of complex numbers with $a_k \to 0$ as $k \to \infty$. We define by $f(x)$ the sums of series (2.3) at the point where the series converges.

Theorem 2.4. *Let non-negative $a \in GM(|a|, r)$, $\lim\limits_{n\to\infty} n|a_n| = 0$ and $\lim\limits_{n\to\infty} |a_n| = 0$, cosine series (2.3) convergence uniformly on $[0, \pi]$ if $\sum\limits_{k=1}^{\infty} a_k$ converge.*

$$|f(x) - S_{m-1}(f,x)| \le M\left(a_m + 2Ca_m\right),$$

where C non-negative constant only depending on $GM(2), m \ge 2, x = \frac{\pi}{M}$ and $x \in (0, \pi]$.

Proof. Let $f(x) - S_{m-1}(f,x)$ with $S_m(f,x) = \sum\limits_{j=1}^{m} a_j \cos jx$, we have

$$\begin{aligned} f(x) - S_{m-1}(f,x) &= \sum_{k-m}^{\infty} a_k \cos kx \\ &= \sum_{k=m}^{\infty} \Delta a_k D_k(x) - a_m D_{m-1}(x) = A + B \end{aligned} \tag{2.4}$$

with $D_m(x) = \sum\limits_{j=1}^{m} \cos jx$, $A = \sum\limits_{k=m}^{\infty} \Delta a_k D_k(x)$, and $B = -a_m D_{m-1}(x)$.

$$|B| = |a_m D_{m-1}(x)| \le m\,|a_m| \le m\sum_{v=m}^{\infty}|\Delta a_m|.$$

By Theorem (2.1) we have

$$|B| \le m\sum_{v=m}^{\infty}|\Delta a_m| = 4mC\,|a_m|\,. \tag{2.5}$$

To estimate A, then we split A to

$$A = \sum_{k=m}^{m+N-1} \Delta a_k D_m(x) + \sum_{k=m+N}^{\infty} \Delta a_k D_{m+N}(x) = P + Q.$$

Let $x \in (0, \pi]$, for any x we can find $M \in \mathbb{N}$ such that $x \in (\frac{\pi}{M}, \frac{\pi}{M-1}]$. Since

$$D_m(x) = \left|\sum_{v=1}^{m} \cos vx\right| = \left|\frac{\sin(m+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right| \le \frac{1}{\sin\frac{1}{2}x} \le \frac{\pi}{x},$$

then we have

$$|Q| = \left|\sum_{k=m+N}^{\infty} \Delta a_k D_{m+N}(x)\right| \le \frac{\pi}{x}\sum_{k=m+N}^{\infty}|\Delta a_k|.$$

By using Theorem 2.1, we obtain

$$|Q| \leq 4MC\,|a_n|\,. \tag{2.6}$$

Further

$$|P| = \left|\sum_{k=m}^{m+N-1} \Delta a_k D_k(x)\right| \leq \frac{\pi}{x} \sum_{k=m}^{m+N-1} |\Delta a_k|.$$

By using Theorem 2.1 again, we have

$$|P| \leq 4MC\,|a_m|\,. \tag{2.7}$$

From, (2.6) and (2.7) , we get

$$A = P + Q \leq C\,|a_m| + 4MC\,|a_n|\,. \tag{2.8}$$

From (2.5) and (2.8), we have

$$|f(x) - S_m(f,x)| = A + B \leq C\,|a_m| + 4MC\,|a_n| + 4MC\,|a_m|\,.$$

Using Theorem 2.2 and Theorem 2.3, then series (2.3) converges uniformly on $(0,\pi]$.
For $x = 0$, we have

$$f(0) = \sum_{k=1}^{\infty} a_k.$$

Finally we have converges uniformly on $[0,\pi]$. ∎

Corolarry 2.1. *Let non-negative* $a \in GM(r)$ *and* $\lim_{n\to\infty} n(a_n) = 0$*, if cosine series (2.3) convergence uniformly on* $[0,\pi]$*, then* $\sum_{k=1}^{\infty} a_k$ *converge .*

Proof. It is obvious, for $x = 0$ we have

$$f(0) = \sum_{k=1}^{\infty} a_k$$

is converges. ∎

3. CONCLUSION

In this paper we have concluded that the cosine series with coefficient from new class of general monotone order r is corveges uniformly if $\sum_{k=1}^{\infty} a_k$ converges.

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