SOME REMARKS ON QUASINEARLY SUBHARMONIC FUNCTIONS

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ABSTRACT. We prove some basic properties of quasi-nearly subharmonic functions and quasi-nearly subharmonic functions in the narrow sense.

1. Introduction

The class of nearly subharmonic functions, quasi-nearly subharmonic functions, and quasi-nearly subharmonic functions in the narrow sense are some generalizations of subharmonic functions (see Hervë [1] and Riihentaus [3]). These functions include but are not limited to subharmonic functions, functions satisfying natural growth conditions, especially certain eigenfunctions, and polyharmonic functions (see cited works and the reference therein). In this paper, we study some basic properties of these functions.

2. Notation and Definitions

Notation . In what follows D is a domain of $\mathbb{R}^N$ ($N \geq 2$). The ball of center $x \in D$ and radius $r > 0$ is noted $B(x, r)$. We write ν_N for the volume of the unit ball, and λ designates the $N-$dimensional Lebesgue measure.

Definition 2.1. A function $u : D \to [-\infty, +\infty)$ is called nearly subharmonic, if u is (Lebesgue) measurable and satisfies the mean value inequality; i.e, for all ball $B(x, r)$ relatively compact in D,

$$u(x) \leq \frac{1}{\nu_N r^N} \int_B (x, r) u(\xi) d\lambda(\xi).$$

This is a generalization of subharmonic functions, in the sense of, and given by J. Riihentaus that differs slightly from the standard definition of nearly subharmonic functions (see [3] and the references therein).

Theorem 2.1. *A function $u : D \to [-\infty, +\infty)$ is nearly subharmonic if and only if there exists a subharmonic function that is equal to u almost everywhere in D. Further, if such a function exists, it is unique and is given by the upper-semi regularization of u:*

$$u^*(x) = \limsup_{\zeta \to x} u(x).$$

See [1, pg 14]

Definition 2.2. A function $u : D \to [-\infty, +\infty)$ is called $K-$quasi-nearly subharmonic, if u is (Lebesgue) mearsurable, its positive part u^+ is locally integrable and there exists a constant $K = K(N, u, D) \geq 1$ such that for all ball $B(x, r)$ relatively compact in D,

$$u_M(x) \leq \frac{K}{\nu_N r^N} \int_{B(x,r)} u_M(\xi) d\lambda(\xi),$$

for all $M \geq 0$. Here, $u_M := \max\{u, -M\} + M$.

This and the following definition are generalizations of subharmonic function given by J. Riihentaus (see [3]).

Definition 2.3. A function $u : D \to [-\infty, +\infty)$ is called $K-$quasi-nearly subharmonic n.s. (in the narrow sense), if u is (Lebesgue) mearsurable, its positive part u^+ is locally integrable and there exists a constant $K = K(N, u, D) \geq 1$ such that for all ball $B(x, r)$ relatively compact in D,

$$u(x) \leq \frac{K}{\nu_N r^N} \int_{B(x,r)} u(\xi) d\lambda(\xi).$$

Theorem 2.2 (Riihentaus)**.** *Let u be a K-quasi-nearly subharmonic function n.s. on a domain D of $\mathbb{R}^{\mathbb{N}}$ ($N \geq 2$).*

   (i) *If $u \not\equiv -\infty$, then u is finite almost everywhere and is locally integrable on D;*
   (ii) *The function u is locally bounded above on D.*

See [3, Proposition 1]

3. MAIN RESULTS AND THEIR PROOFS

Theorem 3.1. *If u is K-quasi nearly subharmonic, then so is u^*.*

All we need to show is that $(u^*)_M$ satisfies the quasi mean inequality. We start by proving that

$$(3.1) \qquad\qquad (u^*)_M = (u_M)^*.$$

Lemma 3.2. *Let (a_n) be a convergent sequence of real numbers and k a constant. Then*

$$\lim_{n \to +\infty} \max\{a_n, k\} = \max\{\lim_{n \to +\infty} a_n, k\}.$$

Proof. Let $f(x) := \max\{x, k\}$. This is a continuous function. The left side of the above equation equals $\lim_{n \to +\infty} f(a_n)$ and the right side equals $f(\lim_{n \to +\infty} a_n)$. They are equal by continuity of f. ∎

Now we can prove (3.1). Let (r_n) be a sequence of real numbers that approaches 0, as $n \to +\infty$ and let $\overline{B(\zeta, r_n)} \subset D$. It is easy to check that

$$\sup_x \{\max\{u(x), -M\} + M\} = \sup_x \{\max\{u(x), -M\}\} + M$$
$$= \max\{\sup_x u(x), -M\} + M,$$

where the suprema are taken over the ball $B(\zeta, r_n)$. By letting $n \to +\infty$, the left side of the first equation approaches $(u_M)^*(\zeta)$, and the last expression is equal, according to Lemma 2.3, to

$$\max\{\lim_{n \to +\infty} \sup_x u(x), -M\} + M,$$

which is $(u^*)_M(\zeta)$. This proves (3.1).

Proof of Theorem 2.1. Take $\overline{B(x, r)} \subset D$. We have

$$u_M(x) \le \frac{K}{\nu_N r^N} \int_{B(x,r)} u_M(\xi) d\lambda(\xi)$$
$$\le \frac{K}{\nu_N r^N} \int_{B(x,r)} (u_M)^*(\xi) d\lambda(\xi)$$
$$= \frac{K}{\nu_N r^N} \int_{B(x,r)} (u^*)_M(\xi) d\lambda(\xi),$$

by (3.1). We notice that the last integral is a continuous function of x, since the integrand is integrable; it majorizes u_M thus it also majorizes $(u_M)^*$, which equals $(u^*)_M$, by (3.1). We obtain

$$(u^*)_M \le \frac{K}{\nu_N r^N} \int_{B(x,r)} (u^*)_M(\zeta) d\lambda(\zeta),$$

as required. ∎

Theorem 3.3. *Let u be a K-quasi-nearly subharmonic function n.s. on D, and*

$$N := \{x \in D : u(x) < 0\}$$

be the negative set of u. If the interior of N is not empty, then u is nearly subharmonic.

Proof. We need to prove that u satisfies the mean value inequality everywhere on D. Take a ball $B(x, r)$ relatively compact in N. We have

$$u(x) \leq K \left(\frac{1}{\nu_N r^N} \int_{B(x,r)} u d\lambda \right) \leq 0.$$

We know that almost every x in the interior of N is a Lebesgue point, meaning that the above normalized integral within parenthesis converges to $u(x)$, as $r \rightarrow 0^+$. For such an x and by letting $r \rightarrow 0$ we get $u(x) \leq K u(x) \leq 0$ and thus $K \leq 1$. Since by definition $K \geq 1$, we obtain $K = 1$. Now, by Theorem 1.6 (i) u is locally integrable and satisfies the mean value inequality. Thus u is nearly subharmonic.

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Corollary 3.4. *Let u be a K-quasi-nearly subharmonic function n.s. on D. Then, either u^* is subharmonic on D, or $u^* \geq 0$ and is K-quasi-nearly subharmoic n.s. on D.*

Proof. If the negative set of u has interior points, then by Theorem 2.6 u^* is subharmonic on D. Next, assume that N has empty interior and let us prove that

(3.2) $$u^*(x) \geq 0$$

for all $x \in D$. If $x \in D \setminus N$ or if N is empty, there is nothing to prove. Let $x \in N$. There exits a sequence $\{x_n\} \subset D \setminus N$ converging to x, as $n \rightarrow +\infty$. We have

$$0 \leq \limsup_{n \rightarrow +\infty} u(x_n) \leq \limsup_{n \rightarrow +\infty} u^*(x_n) \leq u^*(x),$$

since u^* is by construction upper semi-continuous.

To prove that if u is $K-$quasi-nearly subharmonic n.s., then so is u^*, we follow J. Riihentaus [4, p. 5-6] and make maybe some minor adjustments. First notice that u is locally bounded above, according to Theorem 2.1., and so u^* is well-defined. Next, being upper semi-continuous, it is also measurable and integrable. Thus we just need to prove that u^* satisfies the quasi-mean inequality everywhere. Let $B(\zeta, \rho)$ be an arbitrary ball relatively compact in D. There exists $0 > \delta$ such that $B(\zeta, \rho + 2\delta)$ is steel relatively compact in D. We have

$$u(x) \leq \frac{K}{\nu_N \rho^N} \int_{B(x,\rho)} u(\xi) d\lambda(\xi),$$

for all $x \in B(\zeta, \delta)$. By taking the limit superior, we obtain

$$\limsup_{x \rightarrow \zeta} u(x) \leq K \limsup_{x \rightarrow \zeta} \left(\frac{1}{\nu_N \rho^N} \int_{B(x,\rho)} u(\xi) d\lambda(\xi) \right).$$

Let $\phi(x)$ designate the function defined by the integral within parenthesis. Since u is integrable, the function ϕ is continuous in $B(\zeta, \delta)$, according to a classic theorem of measure theory. Thus the limit of the right side is in fact $\phi(\zeta)$. The left side limit is the upper semi-continuous regularization of u. We thus get

$$u^*(\zeta) \leq \frac{K}{\nu_N \rho^N} \int_{B(\zeta,\rho)} u(\xi) d\lambda(\xi) \leq \frac{K}{\nu_N \rho^N} \int_{B(\zeta,\rho)} u^*(\xi) d\lambda(\xi).$$

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