EXISTENCE OF SOLUTION OF DIFFERENTIAL AND RIEMANN-LIOUVILLE EQUATION VIA FIXED POINT APPROACH IN COMPLEX VALUED $b$-METRIC SPACES.

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ABSTRACT. In this paper, we establish some fixed point and common fixed point results for a new type of generalized contractive mapping using the notion of $C$-class function in the framework of complex valued $b$-metric spaces. As an application, we establish the existence and uniqueness of a solution for Riemann-Liouville integral and ordinary differential equation in the framework of a complete complex valued $b$-metric spaces. The obtained results generalize and improve some fixed point results in the literature.

Key words and phrases: Generalized contraction type mapping; Common fixed point; Fixed point; Complex valued $b$-metric space.

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1. INTRODUCTION AND PRELIMINARIES

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [8] in 1922 proved the well celebrated Banach contraction principle in the framework of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [1, 2, 14, 21] and the references therein). For example, Berinde [9, 10] introduced and studied a class of contractive mappings, which is defined as follows:

Definition 1.1. Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is said to be a generalized almost contraction if there exist \(\delta \in [0, 1)\) and \(L \geq 0\) such that
\[
d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},
\]
for all \(x, y \in X\).

Furthermore, in 2008, Suzuki [28] introduced a class of mappings satisfying condition \((C)\), known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.2. Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is said to satisfy condition \((C)\) if for all \(x, y \in X\),
\[
\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).
\]

Theorem 1.1. Let \((X, d)\) be a compact metric space and \(T : X \to X\) be a mapping satisfying
\[
\frac{1}{2} d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),
\]
for all \(x, y \in X\). Then \(T\) has a unique fixed point.

In 2014, Ansari [4] introduced the notion of \(C\)-class function, he proved some fixed point results using the concept of \(C\)-class function and also established that the \(C\)-class function is a generalization of a whole lot of contractive conditions.

Definition 1.3. [4] A mapping \(F : [0, \infty)^2 \to \mathbb{R}\) is called a \(C\)-class function if it is continuous and the following axioms hold:
\begin{enumerate}
  \item \(F(s, t) \leq s\) for all \(s, t \in [0, \infty)\);
  \item \(F(s, t) = s\) implies either \(s = 0\) or \(t = 0\).
\end{enumerate}

Example 1.1. The following functions \(F : [0, \infty)^2 \to \mathbb{R}\) defined for all \(s, t \in [0, \infty)\) by
\begin{enumerate}
  \item \(F(s, t) = s - t, F(s, t) = s\) implies \(t = 0\);
  \item \(F(s, t) = ms, 0 < m < 1, F(s, t) = s\) implies \(s = 0\);
  \item \(F(s, t) = s\beta(s), \beta : [0, \infty) \to [0, 1)\) is a continuous function, \(F(s, t) = s\) implies \(s = 0\).
\end{enumerate}

For details about \(C\)-class functions see [4], and \(C\) denote the class of \(C\)-functions. In 2016, Chandok et al. [12] introduced a new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces.

Definition 1.4. [12] Let \(T : X \to X\) be a mapping and let \(\alpha, \beta : X \to \mathbb{R}^+\) be two functions. Then \(T\) is called a cyclic \((\alpha, \beta)\)-admissible mapping, if
two given mappings. We say that $T$ is a TAC-contractive mapping, if for all $x, y \in X$,
\[
\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),
\]
where $\psi$ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, $\phi$ is continuous with $\lim_{n \to \infty} \phi(t_n) = 0 \Rightarrow \lim_{n \to \infty} t_n = 0$ and $f \in C$.

**Theorem 1.2.** [12] Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a cyclic $(\alpha, \beta)$-admissible mapping. Suppose that $T$ is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and either of the following conditions hold:

1. $T$ is continuous,
2. if for any sequence $\{x_n\}$ in $X$ with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of $T$), then $T$ has a unique fixed point.

One of the interesting generalization of metric spaces is the concept of $b$-metric spaces introduced by Czerwik in [13]. He established the Banach contraction principle in this framework with the fact that $b$ need not be continuous. Thereafter, several results has been extended from metric spaces to $b$-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the frame work of $b$-metric spaces has been established by different researchers in this area (see [11], [13], [19] and the references therein). Yamaod and Sintunawarat [20] introduced the notion of $(\alpha, \beta)$-$(\psi, \varphi)$-contraction mapping in the frame work of $b$-metric spaces as follows:

**Definition 1.6.** Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $T : X \to X$ is an $(\alpha, \beta)$-$(\psi, \varphi)$-contraction mapping if the following conditions holds: for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ implies that
\[
\psi(s^3d(Tx, Ty)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)),
\]
where $M_s(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\}$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are alternating distance functions.

**Theorem 1.3.** Let $(X, d)$ be a complete $b$-metric space with coefficient $s \geq 1$ and $T : X \to X$ an $(\alpha, \beta)$-$(\psi, \varphi)$-contraction mapping. Suppose that one of the following conditions holds:

1. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$,
2. there exists $y_0 \in X$ such that $\alpha(y_0) \geq 1$,

and the following holds:

1. $T$ is continuous,
2. $T$ is cyclic $(\alpha, \beta)$-admissible.

Then $T$ has a fixed point.

Recently, Babu et al. [6] generalized the result of Chandok et al. [12] by introducing a generalized TAC-contractive mapping in the frame work of $b$-metric spaces.

**Definition 1.7.** Let $(X, d)$ be a $b$-metric space, $\alpha, \beta : X \to [0, \infty)$ be two given mappings and $T$ be a self map on $X$. The mapping $T$ is said to be generalized TAC-contractive map in $b$-metric spaces, if for all $x, y \in X$,
\[
\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(s^3d(Tx, Ty)) \leq f(\psi(M_s(x, y)), \phi(M_s(x, y))),
\]
where $M_{\psi}(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, $\psi$ is an alternating distance function, $\phi$ is continuous with $\lim_{n \to \infty} \phi(t_n) = 0 \Rightarrow \lim_{n \to \infty} t_n = 0$ and $f \in C$.

**Theorem 1.4.** Let $(X,d)$ be a complete b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a generalized TAC-contraction mapping. Suppose the following conditions hold:

1. $T$ is a cyclic $(\alpha, \beta)$-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
3. $T$ is continuous,
4. if for any sequence $\{x_n\}$ in $X$ with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.

Then $T$ has a fixed point.

In mathematics researchers try to come up with new algebraic structures in order to improve and extend results obtained in the literature. In \cite{5} Azam et al. introduce the notion of complex valued metric space and established some common fixed point results for mapping satisfying generalized contractive conditions. Thereafter, several results and applications has been extended from metric spaces to complex valued metric spaces, more so, a lot of results on the fixed point theory and common fixed point results of various classes of mappings in the framework of complex valued metric spaces has been established by different researchers in this area (see \cite{25,26,27} and the references therein).

The following symbols, notation and definition can be found in \cite{5} will be useful in this study. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), \text{ Im}(z_1) \leq \text{Im}(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

1. $Re(z_1) = Re(z_2), \text{ Im}(z_1) < \text{Im}(z_2)$;
2. $Re(z_1) < Re(z_2), \text{ Im}(z_1) = \text{Im}(z_2)$;
3. $Re(z_1) < Re(z_2), \text{ Im}(z_1) < \text{Im}(z_2)$;
4. $Re(z_1) = Re(z_2), \text{ Im}(z_1) \leq \text{Im}(z_2)$.

In particular, we write $z_1 \not\preceq z_2$ if $z_1 \neq z_2$ and one of (1), (2) and (3) is satisfied and we write $z_1 \prec z_2$ if only (3) is satisfied. Note that

1. $a, b \in \mathbb{R}$ and $a \leq b$ implies that $az \preceq bz$ for all $z \in \mathbb{C}$;
2. $0 \preceq z_1 \not\preceq z_2$ implies that $|z_1| < |z_2|$;
3. $z_1 \preceq z_2$ and $z_2 \not\preceq z_1$ implies that $z_1 \prec z_2$.

**Definition 1.8.** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

1. $0 \not\preceq d(x,y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then $d$ is called a complex valued metric and $(X,d)$ is called a complex valued metric space.

**Example 1.2.** Let $X = \mathbb{C}$ and $d_i : X \times X \to \mathbb{C}, i = 1, 2, 3$ be defined as

1. $d_1(z_1, z_2) = |z_1 - z_2|$ for all $z_1, z_2 \in X$;
2. $d_2(z_1, z_2) = e^{i|k|} |z_1 - z_2|$ for all $z_1, z_2 \in X$ and $k \in \mathbb{R}$;
3. $d_3(z_1, z_2) = e^{i\theta} |z_1 - z_2|$ for all $z_1, z_2 \in X$ and $\theta \in (0, \frac{\pi}{2})$.

Motivated by the concept of b-metric spaces and complex valued metric spaces \cite{13,5}, Rao et al. in \cite{18}, introduced the notion of complex valued b-metric spaces and established some common fixed point results. Thereafter, several results and applications has been extended from metric spaces, $b$-metric spaces and complex valued metric spaces to complex valued $b$-metric
spaces (see [18] and the reference therein). The notion of complex valued $b$-metric spaces generalize, improves and unifies results in metric spaces, $b$-metric spaces and complex valued metric spaces.

**Definition 1.9.** Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d_b : X \times X \to \mathbb{C}$, satisfies:

1. $0 \preceq d_b(x, y)$ for all $x, y \in X$ and $d_b(x, y) = 0$ if and only if $x = y$;
2. $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$;
3. $d_b(x, y) \preceq s[d_b(x, z) + d_b(z, y)]$ for all $x, y, z \in X$.

Then $d_b$ is called a complex valued $b$-metric and $(X, d_b)$ is called a complex valued $b$-metric space.

**Example 1.3.** [18] Let $X = \mathbb{C}$ defined the mapping $d_b : X \times X \to \mathbb{C}$ by $d_b(z_1, z_2) = |z_1 - z_2|^2 + i|z_1 - z_2|^2$ for all $z_1, z_2 \in X$.

**Definition 1.10.** Suppose that $(X, d_b)$ is a complex valued $b$-metric space and $\{z_n\}$ is a sequence in $X$, then the sequence $\{z_n\}$

1. converges to an element to and element $z_0 \in X$ if for every $0 \prec c \in \mathbb{C}$, there exist an integer $\mathbb{N}$ such that $d_b(z_n, z_0) \prec c$ for all $n \in \mathbb{N}$.
2. is a Cauchy sequence if for every $0 \prec c \in \mathbb{C}$, there exist an integer $\mathbb{N}$ such that $d_b(z_n, z_m) \prec c$ for all $n, m \in \mathbb{N}$.

**Definition 1.11.** Suppose that $(X, d_b)$ is a complex valued $b$-metric space, the space $(X, d_b)$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$.

**Definition 1.12.** [16] Let $X$ be a nonempty set and $S, T : X \to X$ be any two mappings.

1. A point $x \in X$ is called:
   a. coincidence point of $S$ and $T$ if $Sx = Tx$,
   b. common fixed point of $S$ and $T$ if $x = Sx = Tx$.
2. If $y = Sx = Tx$ for some $x \in X$, then $y$ is called the point of coincidence of $S$ and $T$.
3. A pair $(S, T)$ is said to be:
   a. commuting if $TSx = STx$ for all $x \in X$,
   b. weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.

Motivated by the current research interest in this direction, the purpose of this work is to further develop the concept of $C$-class function and establish some fixed point and common fixed point results for a new type of generalized contractive mapping using the notion of $C$-class function in the framework of complex valued $b$-metric spaces. As an application, we establish the existence of a solution for Riemann-Liouville integral and ordinary differential equation in the framework of a complete complex valued $b$-metric spaces.

2. **MAIN RESULT**

In this section, we define a complex $C$-class function, established some common fixed point and fixed point results. Throughout this work, we will use $d$ instead of $d_b$ to denoted a complex valued $b$ metric. We define

$$S = \{z \in \mathbb{C} : 0 \preceq z\}$$

**Definition 2.1.** A mapping $F : S \times S \to \mathbb{C}$ is called a $C$-class function if it is continuous and the following axioms holds:

1. $F(s, t) \preceq s$;
2. $F(s, t) = s$ implies either $s = 0$ or $t = 0$
In the course of this work, $C$ denote the class of $C$-functions. Let $\Psi$ denote the class of functions $\psi : S \to S$ satisfying the following condition:

1. $\psi$ is continuous;
2. $\psi(t) > t$ for all $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$.

Let $\Phi$ denote the class of functions $\phi : S \to S$ satisfying the following condition:

1. $\phi$ is continuous;
2. $\phi(t) > t$ for all $t > 0$ and $\phi(0) \geq 0$.

**Theorem 2.1.** Let $(X, d_0)$ be a complex valued $b$-metric space with $s > 1$ and $S, T$ be a self map on $X$ satisfying

\begin{equation}
\frac{1}{2s} d(x, Sx) \leq d(x, y) \Rightarrow \psi(s^2 d(Tx, Sy)) \leq F(\psi(M(x, y)), \phi(M(x, y)))
\end{equation}

for all $x, y \in X$, where $\epsilon \geq 1$, $F \in C$, $\phi \in \Phi$, $\psi \in \Psi$ and

\[ M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}. \]

Then the pair $S$ and $T$ have a unique common fixed point.

**Proof.** Let $x_0$ be any arbitrary point in $X$ and we define $x_{2n+1} = Tx_n$ and $x_{2n+2} = Sx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. Since $\frac{1}{2s}d(x_{2n+1}, Sx_{2n+1}) = \frac{1}{2s}d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2})$, using (2.1) and the properties of $F$, we have

\begin{equation}
\psi(s^2 d(x_{2n+1}, x_{2n+2})) = \psi((s^2 d(Tx_n, Sx_{2n+1}))) \\
\leq F(\psi(M(x_{2n}, x_{2n+1})), \phi(M(x_{2n}, x_{2n+1}))) \\
\leq \psi(M(x_{2n}, x_{2n+1})),
\end{equation}

where

\[ M(x_{2n}, x_{2n+1}) = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Tx_{2n})d(x_{2n+1}, Sx_{2n+1})}{s + d(x_{2n}, x_{2n+1})}\right\} \]

\[ = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{s + d(x_{2n}, x_{2n+1})}\right\}, \]

since $\frac{d(x_{2n+1}, x_{2n+2})}{s + d(x_{2n}, x_{2n+1})} < 1$, we have that $\frac{d(x_{2n+1}, x_{2n+2})}{s + d(x_{2n}, x_{2n+1})} < d(x_{2n+1}, x_{2n+2})$, as such, we have that

\[ M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \]

If we suppose that $M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, we then have (2.2) becomes

\[ \psi(s^2 d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n+1}, x_{2n+2})) \]

\[ \Rightarrow s^2 d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}), \]

which is a contradiction, as such we must have that

\[ M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}), \]
we then have (2.2) becomes

\[ \psi(s^t d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n}, x_{2n+1})) \]
\[ \Rightarrow s^t d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}), \]
\[ \Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s^t} d(x_{2n}, x_{2n+1}). \]

Inductively, we have that

\[ d(x_n, x_{n+1}) \leq \left( \frac{1}{s^t} \right)^n d(x_0, x_1) \]
\[ |d(x_n, x_{n+1})| \leq \left( \frac{1}{s^t} \right)^n |d(x_0, x_1)|. \]

Now for any \( m > n, \) where \( m, n \in \mathbb{N}, \) observe that

\[ d(x_n, x_m) \leq s d(x_n, x_{n+1}) + s d(x_{n+1}, x_m) \]
\[ \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \]
\[ = s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_m) \]
\[ \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) \]
\[ + \cdots + s^{m-n} d(x_{m-1}, x_m) \]
\[ \leq s \left( \frac{1}{s^t} \right)^n d(x_0, x_1) + s^2 \left( \frac{1}{s^t} \right)^{n+1} d(x_0, x_1) + s^2 \left( \frac{1}{s^t} \right)^{n+2} d(x_0, x_1) \]
\[ + \cdots + s^{m-n} \left( \frac{1}{s^t} \right)^{m-1} d(x_0, x_1) \]
\[ \leq s \left( \frac{1}{s^t} \right)^n \left[ 1 + \frac{1}{s^t-1} + \cdots + \left( \frac{1}{s^t-1} \right)^{m-n-1} \right] d(x_0, x_1) \]
\[ \leq s \left( \frac{1}{s^t} \right)^n \left( \frac{s^t-1}{s^t-1} \right) d(x_0, x_1). \]

We then have

\[ |d(x_n, x_m)| \leq s \left( \frac{1}{s^t} \right)^n \left( \frac{s^t-1}{s^t-1} \right) |d(x_0, x_1)|, \]

since \( s > 1, \epsilon \geq 1, \) taking limit as \( m, n \rightarrow \infty, \) we have

(2.3) \[ \lim_{n,m \rightarrow \infty} |d(x_n, x_m)| = 0. \]

Thus, then sequence \( \{x_n\} \) is a complex valued \( b \)-Cauchy sequence. Since \( X \) is complete there exists \( x \in X \) such that \( \lim_{n \rightarrow \infty} |d(x_n, x)| = 0. \) We also have \( d(x_{n+1}, x) \leq s d(x_{n+1}, x_n) + s d(x_n, x) \)

using (2.3) and the fact that \( \lim_{n \rightarrow \infty} |d(x_n, x)| = 0, \) we have \( \lim_{n \rightarrow \infty} |d(x_{n+1}, x)| = 0. \) Using a similar approach, we have \( \lim_{n \rightarrow \infty} |d(x_{n+2}, x)| = 0. \) Now, we claim that

\[ \frac{1}{2s} d(x_n, x_{n+1}) \leq d(x_n, x) \text{ or } \frac{1}{2s} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x). \]

Suppose on the contrary that there exists \( m \geq 0, \) such that

(2.4) \[ \frac{1}{2s} d(x_m, x_{m+1}) \geq d(x_m, x) \text{ or } \frac{1}{2s} d(x_{m+1}, x_{m+2}) \geq d(x_{m+1}, x). \]
Now observe that
\[ 2sd(x_m, x) < d(x_m, x_{m+1}) \]
(2.5)
\[ \leq sd(x_m, x) + sd(x, x_{m+1}) \]
\[ \Rightarrow sd(x_m, x) < sd(x, x_{m+1}). \]

It follows from (2.4) and (2.5) that
\[ sd(x_m, x) < sd(x, x_{m+1}) < \frac{1}{2}d(x_{m+1}, x_{m+2}). \]
(2.6)

Since \( \frac{1}{2}d(x_{m+1}, x_{m+2}) \leq d(x_{m+1}, x_{m+2}) \), we have that
\[ \psi(s^d d(x_{m+1}, x_{m+2})) = \psi(s^d (Tx_m, Sx_{m+1})) \]
(2.7)
\[ \leq F(\psi(M(x_{m+1})), \phi(M(x_{m+1}))) \]
\[ \leq \psi(M(x_{m+1})) \]
\[ \leq \psi(d(x_m, x_{m+1})). \]

We then have that (2.7) becomes
\[ \psi(s^d d(x_{m+1}, x_{m+2})) \leq \psi(d(x_m, x_{m+1})). \]
(2.8)
Using the properties of \( \psi \), we have that
\[ s^d d(x_{m+1}, x_{m+2}) \leq d(x_m, x_{m+1}). \]
(2.9)
Using this fact, (2.4) and (2.5), we have
\[ s^d d(x_{m+1}, x_{m+2}) \leq d(x_m, x_{m+1}) \]
\[ \leq sd(x_m, x) + sd(x, x_{m+1}) \]
\[ \leq sd(x, x_{m+1}) + sd(x, x_{m+1}) \]
\[ \leq \frac{1}{2}d(x_{m+1}, x_{m+2}) + \frac{1}{2}d(x_{m+1}, x_{m+2}) \]
\[ = d(x_{m+1}, x_{m+2}), \]
(2.10)
which is a contradiction. Thus, we must have that
\[ \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x) \text{ or } \frac{1}{2}d(x_n, x_{n+2}) \leq d(x_{n+1}, x). \]

We then have
\[ \psi(s^d d(x_{n+1}, Sx)) \]
\[ = \psi(s^d d(Tx_n, Sx)) \]
\[ \leq F(\psi(M(x_n, x)), \phi(M(x_n, x))) \]
\[ \leq \psi(M(x_n, x)) \]
\[ = \psi(\max\{d(x_n, x), d(x_n, x_{n+1}), d(x, Sx), \frac{d(x_n, Tx_n)d(x, Sx)}{s + d(x, x)}\}) \]
\[ \leq \psi(\max\{d(x_n, x), sd(x_n, x) + sd(x, x_{n+1}), d(x, Sx), \frac{d(x_n, Tx_n)d(x, Sx)}{s + d(x, x)}\}) \],

this implies that \( s^d d(x_{n+1}, Sx) \leq \max\{d(x_n, x), sd(x_n, x) + sd(x, x_{n+1}), d(x, Sx), \frac{d(x_n, Tx_n)d(x, Sx)}{s + d(x, x)}\} \)
and taking limit as \( n \to \infty \), we have
\[ |d(x, Sx)| \leq \frac{1}{s^e}|d(x, Sx)|, \]
(2.11)
which is a contradiction, as such we must have that

\[(2.12) \quad Sx = x.\]

More so, we have

\[
\psi(s'd(Tx, x_{n+1})) = \psi(s'd(Tx, Sx_{n+1})) \\
\leq F(\psi(M(x, x_{n+1})), \phi(M(x, x_{n+1}))) \\
\leq \psi(M(x, x_{n+1})) \\
= \psi(\max\{d(x, x_{n+1}), d(x, Tx), d(x_{n+1}, Sx_{n+1}), \frac{d(x, Tx)d(x_{n+1}, Sx_{n+1})}{s + d(x, x)}\}) \\
\leq \psi(\max\{d(x, x_{n+1}), d(x, Tx), d(x_{n+1}, Sx_{n+1}), \frac{d(x, Tx)d(x_{n+1}, Sx_{n+1})}{s + d(x, x)}\}),
\]

this implies that

\[(2.13) \quad s'd(Tx, x_{n+2}) \leq \max\{d(x, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x, Tx), \frac{d(x, Tx)d(x_{n+1}, Sx_{n+1})}{s + d(x, x)}\},
\]

and taking limit as \(n \to \infty\), we have

\[(2.14) \quad Tx = x.
\]

From (2.12) and (2.14), we have that

\[(2.15) \quad x = Tx = Sx.
\]

Hence \(x\) is the common fixed point for the pair \(S\) and \(T\).

Let \(x\) and \(y\) be two common fixed points for the pair \(S\) and \(T\), such that \(x \neq y\). Now observe that

\[
\psi(s'd(x, y)) = \psi(s'd(Tx, Ty)) \\
\leq F(\psi(M(x, y)), \phi(x, y)) \\
\leq \psi(M(x, y)) \\
= \psi(d(x, y)),
\]

which implies that \(|d(x, y)| < \frac{1}{s'}|d(x, y)|\), we get a contradiction, as such, we have \(x = y\).}

**Theorem 2.2.** Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(T\) be a self map on \(X\) satisfying

\[(2.15) \quad \frac{1}{2s}d(x, Tx) \leq d(x, y) \Rightarrow \psi(s'd(Tx, Ty)) \leq F(\psi(M(x, y)), \phi(M(x, y)))
\]

for all \(x, y \in X\), where \(\epsilon \geq 1, F \in \mathcal{C}, \phi \in \Phi, \psi \in \Psi\) and

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Ty)d(x, Ty)}{s + d(x, y)}\}.
\]

Then \(T\) has a unique fixed point.

**Proof.** The prove follows a similar approach as in Theorem 2.1 as such we omit it.
Corollary 2.3. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(S, T\) be a self map on \(X\) satisfying
\[
\psi(s'd(Tx, Sy)) \leq \psi(M(x, y)) - \phi(M(x, y))
\]
for all \(x, y \in X\), where \(\epsilon \geq 1, \phi \in \Phi, \psi \in \Psi\) and
\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}.
\]
Then the pair \(S\) and \(T\) have a unique common fixed point.

Corollary 2.4. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(S, T\) be a self map on \(X\) satisfying
\[
\psi(s'd(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y))
\]
for all \(x, y \in X\), where \(\epsilon \geq 1, \phi \in \Phi, \psi \in \Psi\) and
\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{s + d(x, y)}\}.
\]
Then \(T\) has a unique fixed point.

Corollary 2.5. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(S, T\) be a self map on \(X\) satisfying
\[
\frac{1}{2s}d(x, Sx) \leq d(x, y) \Rightarrow \psi(s'd(Tx, Sy)) \leq k\psi(M(x, y))
\]
for all \(x, y \in X\), where \(k \in (0, 1), \epsilon \geq 1, \psi \in \Psi\) and
\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Tx)d(y, Sy)}{s + d(x, y)}\}.
\]
Then the pair \(S\) and \(T\) have a unique common fixed point.

Corollary 2.6. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(T\) be a self map on \(X\) satisfying
\[
\frac{1}{2s}d(x, Tx) \leq d(x, y) \Rightarrow \psi(s'd(Tx, Ty)) \leq k\psi(M(x, y))
\]
for all \(x, y \in X\), where \(k \in (0, 1), \epsilon \geq 1, \psi \in \Psi\) and
\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx)d(y, Ty)}{s + d(x, y)}\}.
\]
Then \(T\) has a unique fixed point.

Corollary 2.7. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(S, T\) be a self map on \(X\) satisfying
\[
sd(Tx, Sy) \leq kd(x, y)
\]
for all \(x, y \in X\), where \(k \in (0, 1)\). Then the pair \(S\) and \(T\) have a unique common fixed point.

Corollary 2.8. Let \((X, d_b)\) be a complex valued \(b\)-metric space with \(s > 1\) and \(T\) be a self map on \(X\) satisfying
\[
s^2d(Tx, Ty) \leq d(x, y)
\]
for all \(x, y \in X\). Then \(T\) has a unique fixed point.
3. Application

3.1. Application to Riemann-Liouville Equation. In this section, we establish the existence of a solution of a Riemann-Liouville of the form:

\[ R_L I_t^u x(t) = \Gamma(u) \int_c^t (t - s)^{u-1} x(s) ds, \mathcal{R}(u) > 0 \]  

(3.1)

where \( u \in \mathbb{C}, x(t) \in X = C([0, 1], \mathbb{R}) \) and \( t, s \in [0, 1] \) which is the fractional integral. Let \( X = C([0, 1], \mathbb{R}) \) be the space of continuous function, and \( d : X \times X \to \mathbb{C} \) be defined as

\[
d(u, v) = \left[ \max_{t \in [0, 1]} \| u(t) - v(t) \| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2
\]

where \( s = 2 \). It is well-known that \((X, d)\) is a complete complex valued \( b\)-metric space. Define \( T : X \to X \) by

\[ Tx(t) = \Gamma(u) \int_c^t (t - s)^{u-1} x(s) ds. \]

(3.2)

**Theorem 3.1.** \( X = C([0, 1], \mathbb{R}) \) and suppose that

\[
\left[ \max_{t \in [0, 1]} \frac{1}{\Gamma(u+1)} \frac{(t - s)^{u-1}(t - c)^u}{|(t - s)^{u-1}|} \right]^2 \leq \frac{1}{4}
\]

then Equation (3.1) has a solution.
Proof. It is well-known that \( x \in X \) is a fixed point of \( T \) if and only if \( x \) is a solution of problem (5.1). Note that at some point Now observe that for all \( u, v \in X \), we have that
\[
d(Tx, Ty) \\
= \left[ \max_{t \in [0,1]} \left| Tx(t) - Ty(t) \right| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \\
= \left[ \max_{t \in [0,1]} \frac{1}{\Gamma (u)} \int_{c}^{t} (t-s)^{u-1} x(s) \frac{1}{\Gamma (u)} \int_{c}^{t} (t-s)^{u-1} y(s) \sqrt{1 + a^2 e^{t \tan^{-1} a}} \right]^2 \\
\leq \left[ \max_{t \in [0,1]} \frac{1}{\Gamma (u)} \int_{c}^{t} (t-s)^{u-1} ds \left( \left| x(s) - y(s) \right| \sqrt{1 + a^2 e^{t \tan^{-1} a}} \right)^2 \right. \\
\leq \left. \max_{t \in [0,1]} \frac{1}{\Gamma (u)} \int_{c}^{t} (t-s)^{u-1} ds \left( \left| x(s) - y(s) \right| \sqrt{1 + a^2 e^{t \tan^{-1} a}} \right)^2 \right] \\
\leq \frac{1}{4} d(x, y) \\
= d(x, y) \frac{1}{s^2}.
\]

Thus, we have that \( s^2 d(Tx, Ty) \lesssim d(x, y) \). Clearly, all conditions in Corollary 2.8 are satisfied and guarantees the existence of the fixed point \( x \in X \). Thus, \( x \) is the solution of the integral equation (3.1).

3.2. Application to Differential Equation. In this section, we establish the existence of a solution of a differential equation of the form:

\[
u'(t) = f(t, u(t)), \quad t \in I = [0,1] \\
\]

(3.3)

\[
u(0) = u(1),
\]

where \( f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function. It is easy to see that (3.3) is can be rewritten as

\[
u'(t) + 2u(t) = f(t, u(t)) + 2u(t), \quad t \in I = [0,1] \\
\]

(3.4)

\[
u(0) = u(1),
\]

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which is equivalent to

\[ u(t) = \int_0^1 G(t, s)[f(s, u(s)) + 2u(s)]ds. \]

The Green function \( G(t, s) \) associated with (3.3) is given by

\[
G(t, s) = \begin{cases} 
\frac{e^{2(t+s-t)}}{c^2-1} & \text{if } 0 \leq s \leq t \leq 1 \\
\frac{e^{2(s-t)}}{c^2-1} & \text{if } 0 \leq t \leq s \leq 1.
\end{cases}
\]

It is easy to see that \( \max_{t \in [0, 1]} \int_0^1 G(t, s)ds = \frac{1}{2} \). Let \( X = C([0, 1], \mathbb{R}^n) \) be the space of continuous function, \( u : [0, 1] \to \mathbb{R}^n \) and \( \|(u_1, u_2, \cdots, u_n)\| = \max \{|u_1|, |u_2|, \cdots, |u_n|\} \) and \( d : X \times X \to \mathbb{C} \) be defined as

\[ d(u, v) = \left[ \max_{t \in [0, 1]} \|u(t) - v(t)\| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \]

where \( s = 2 \). It is well-known that \( (X, d) \) is a complete complex valued \( b \)-metric space. Define \( T : X \to X \) as

\[ Tu(t) = \int_0^1 G(t, s)[f(s, u(s)) + 2u(s)]ds \]

**Theorem 3.2.** \( X = C([0, 1], \mathbb{R}^n) \) and suppose that

\[ \|f(t, u) + 2u(s) - f(t, v) + 2v(s)\| \lesssim \|u(s) - v(s)\|, \]

then Equation (3.3) has a solution.

**Proof.** It is well-known that \( u \in X \) is a fixed point of \( T \) if and only if \( u \) is a solution of problem (3.3). Now observe that for all \( u, v \in X \), we have that

\[
d(Tu, Tv)
= \left[ \max_{t \in [0, 1]} \|Tu(t) - Tv(t)\| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2
\]

\[
= \left[ \max_{t \in [0, 1]} \int_0^1 G(t, s)[f(s, u(s)) + 2u(s) - f(s, v(s)) - 2v(s)]ds \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2
\]

\[
\lesssim \left[ \max_{t \in [0, 1]} \int_0^1 G(t, s)|u(s) - v(s)|ds\sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2
\]

\[
= \left[ \max_{t \in [0, 1]} |u(s) - v(s)|\sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \left( \max_{t \in [0, 1]} \int_0^1 G(t, s)ds \right)^2
\]

\[
= d(u, v) \frac{1}{4}
\]

\[
= d(u, v) \frac{1}{s^2}.
\]

Thus, we have that \( s^2 d(Tu, Tv) \lesssim d(u, v) \). Clearly, all conditions in Corollary 2.8 are satisfied and guarantees the existence of the fixed point \( x \in X \). Thus, \( x \) is the solution of the integral equation (3.3). 

\[ \square \]
REFERENCES


