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RIEMANN-STIELTJES INTEGRALS AND SOME OSTROWSKI TYPE INEQUALITIES

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ABSTRACT. In this article, we investigate new integral inequalities of Ostrowski's type of various functional aspects. For mapping's second derivative, we assume two cases, namely, L_1 and L_∞ spaces. Moreover, for first derivative, we investigate two different characteristics, namely, bounded variation and locally Lipchitz continuity. Applications to special means and composite quadrature rules are also carried out.

Key words and phrases: Ostrowski inequality; Bounded variation; Riemann-Stieltjes integral; Locally Lipschitz continuous.

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1. INTRODUCTION

In 1938, A. Ostrowski [1] introduced his famous integral inequality:

Theorem 1.1. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous mapping on* $[a, b]$ *such that* $f' \in L_\infty(a, b)$ *then* $\forall x \in [a, b]$

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty . \tag{1.1}$$

Since that time, an extensive research history on obtaining companions of Ostrowski type inequalities has been conducted. Most early studies as well as current work utilize various characteristics of the function and/or its derivatives such as absolutely continuous, convexity, bounded variation, and Lipschitz continuous. The offspring of Ostrowski type inequalities has been a large toolbox of numerical integration theory employed by many researchers. They provide the numerical integration field with a large class of quadrature and cubature rules.

In 1999, Dragomir [2], [3] derived some interesting inequalities of Ostrowski type as follows:

Theorem 1.2. *Let* $f : [a, b] \to \mathbb{R}$ *be L-lipschitzian mapping on* $[a, b]$. *Then we have*

$$\left| \frac{b-a}{3} \left[\frac{f\ (a) + f\ (b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \int_a^b f\ (t)\ dt \right| \leq \frac{5}{36} L\ (b-a)^2 . \tag{1.2}$$

Theorem 1.3. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a, b]$. *Then* $\forall x \in [a, b]$ *we have*

$$\left| [(x-a)\ f(a) + (b-x)\ f(b)] - \int_a^b f\ (t)\ dt \right| \leq \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) . \tag{1.3}$$

Most recently in 2017, Budak. H, et al. [4] reported the following Ostrowski type inequality for mappings whose first derivatives are of bounded variation as follows:

Theorem 1.4. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* f' *is continuous of bounded variation on* $[a, b]$. *Then* $\forall x \in \left[a, \frac{a+b}{2}\right]$ *we have*

$$\begin{aligned} &\left| \frac{b-a}{2} f(\frac{a+b}{2}) + \frac{b-a}{4} \left[f(\frac{3a+b}{4}) + f(\frac{a+3b}{4}) \right] \right. \\ &\left. + \frac{(b-a)^2}{32} \left[f'(\frac{a+3b}{4}) - f'(\frac{3a+b}{4}) \right] - \int_a^b f\ (t)\ dt \right| \\ \leq\ & \frac{(b-a)^2}{32} \bigvee_a^b (f') . \end{aligned} \tag{1.4}$$

Related studies can be found in [5], [6], [7], and [8].

2. PRELIMINARIES

A review of recent literature shows that many integral inequalities of Ostrowski type are carried out for differentiable functions either in the case where the second derivatives belong to L_∞ or the case when the first derivatives are of bounded variation. For instance, in [9], Budak & Sarikaya presented the following integral inequalities of Ostrowski type:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) such that $f''\in L_\infty(a,b)$. Then*

$$\left|\frac{b-a}{2}\left[f(\frac{a+b}{2})+\frac{f(a)+f(b)}{2}\right]-\frac{(b-a)^2}{36}\left[f'(b)-f'(a)\right]-\int_a^b f(t)\,dt\right| \leq \frac{11}{6^4}(b-a)^3\|f''\|_\infty . \tag{2.1}$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on I such that $[a,b]\subset I$ where f' is of bounded variation on $[a,b]$. Then*

$$\left|\frac{b-a}{2}\left[f(\frac{a+b}{2})+\frac{f(a)+f(b)}{2}\right]-\frac{(b-a)^2}{36}\left[f'(b)-f'(a)\right]-\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{36}\bigvee_a^b(f') . \tag{2.2}$$

In this current paper, we utilize differentiable functions, namely, functions with bounded second derivatives and functions whose first derivatives are of bounded variation as well as locally Lipschitz to establish new inequalities of Ostrowski's type. Applications for special means and quadrature rules are also given.

3. MAIN RESULTS

3.1. **The case when $f''\in L_\infty[a,b]$.** We commence our main results with the following Ostrowski's type inequality for differentiable function with bounded second derivative:

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a twice continuously differentiable function on $[a,b]$ i.e., $(f\in C^2[a,b])$. Suppose that $f''\in L_\infty[a,b]$ i.e., $\left(\|f''\|_\infty=\sup_{t\in[a,b]}|f''(t)|<\infty\right)$. Then*

$$\left|\frac{b-a}{16}\left[3\left[f(a)+f(b)\right]+10f(\frac{a+b}{2})\right]-\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\int_a^b f(t)\,dt\right| \leq \frac{17}{3.2^9}(b-a)^3\|f''\|_\infty . \tag{3.1}$$

Proof. First, we consider the following

$$\int_a^b K(t)f''(t)\,dt = \int_a^{\frac{a+b}{2}} \left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)f''(t)\,dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)f''(t)\,dt \tag{3.2}$$

which can be reduced as,

$$\int_a^b K(x,t)f''(t)\,dt = \frac{(b-a)^2}{32}\left[f'(b)-f'(a)\right]-\frac{b-a}{8}\left[3f(a)+5f(\frac{a+b}{2})+3f(b)\right]+2\int_a^b f(t)\,dt \tag{3.3}$$

Now, using both (3.2) and (3.3) yields,

$$\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]+\int_a^b f(t)\,dt = \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} \left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)f''(t)\,dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)f''(t)\,dt\right]. \tag{3.4}$$

Further, since $f'' \in L_\infty[a,b]$, then imposing Hölder's integral inequality on (3.4) gives

$$\left|\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]+\int_a^b f(t)\,dt\right| = \frac{1}{2}\left[\|f''\|_{\substack{\infty \\ t\in\left[a,\frac{a+b}{2}\right]}} \int_a^{\frac{a+b}{2}} \left|\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)\right|dt + \|f''\|_{\substack{\infty \\ t\in\left[\frac{a+b}{2},b\right]}} \int_{\frac{a+b}{2}}^b \left|\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)\right|dt\right]. \tag{3.5}$$

But,

$$\int_a^{\frac{a+b}{2}} \left|\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)\right| dt = \int_{\frac{a+b}{2}}^{b} \left|\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)\right| dt = \frac{17}{3.2^9}(b-a)^3. \tag{3.6}$$

Now, considering both (3.5) and (3.6) completes the proof. ∎

Corollary 3.2. *Under the assumptions of Theorem* (3.1) *and assuming* $f'(a)=f'(b)$, *we have the following Bullen type inequality* [10]

$$\left|\frac{b-a}{16}\left[3\left[f(a)+f(b)\right]+10f(\frac{a+b}{2})\right]-\int_a^b f(t)\,dt\right| \le \frac{17}{3.2^9}(b-a)^3\left\|f''\right\|_\infty. \tag{3.7}$$

3.2. **The case when f' is of bounded variation.** For differentiable function whose first derivative is of bounded variation, the following Ostrowski's type inequality holds:

Theorem 3.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuously differentiable function on* $[a,b]$, *i.e.,* $(f\in C^1[a,b])$. *Suppose that* f' *is of bounded variation on* $[a,b]$. *Then*

$$\left|\frac{b-a}{16}\left[3\left[f(a)+f(b)\right]+10f(\frac{a+b}{2})\right]-\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\int_a^b f(t)\,dt\right| \le \frac{3}{2^6}(b-a)^2\bigvee_a^b(f'). \tag{3.8}$$

Proof. We start by rewriting (3.4) using Riemann-Stieltjes integrals of $\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)$ and $\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)$ with respect to f' over $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, respectively, as follows

$$\begin{aligned}&\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]+\int_a^b f(t)\,dt\\ &= \frac{1}{2}\left[\int_a^{\frac{a+b}{2}}\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)df'(t)\right.\\ &\quad\left.+\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)df'(t)\right].\end{aligned} \tag{3.9}$$

Now, applying Hölder's integral inequality on (3.9) yields

$$\begin{aligned} &\left|\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]+\int_a^b f(t)\,dt\right| \qquad (3.10)\\ &\leq \frac{1}{2}\left[\left|\int_a^{\frac{a+b}{2}}\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)df'(t)\right|\right.\\ &\quad\left.+\left|\int_{\frac{a+b}{2}}^b\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)df'(t)\right|\right]. \end{aligned}$$

Recall that

$$\left|\int_a^b g(t)df(t)\right| \leq \sup_{t\in[a,b]}|g(t)| \bigvee_a^b (f), \qquad (3.11)$$

where g, $f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$. Now, by (3.11), the inequality (3.10) can be written in the following form

$$\begin{aligned} &\left|\frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right]-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]+\int_a^b f(t)\,dt\right| \qquad (3.12)\\ &\leq \frac{1}{2}\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\left|\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)\right|\bigvee_a^{\frac{a+b}{2}}(f')\right.\\ &\quad\left.+\sup_{t\in\left[\frac{a+b}{2},b\right]}\left|\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)\right|\bigvee_{\frac{a+b}{2}}^b(f')\right]. \end{aligned}$$

But,

$$\begin{aligned} &\sup_{t\in\left[a,\frac{a+b}{2}\right]}\left|\left(t-\frac{3a+b}{4}\right)\left(t-\frac{7a+b}{8}\right)\right| \qquad (3.13)\\ &= \sup_{t\in\left[\frac{a+b}{2},b\right]}\left|\left(t-\frac{a+3b}{4}\right)\left(t-\frac{a+7b}{8}\right)\right|\\ &= \frac{3}{2^5}(b-a)^2. \end{aligned}$$

Finally, by substituting (3.13) into (3.12), the proof is completed. ∎

3.3. **The case when $f'' \in L_1[a,b]$.** For twice differentiable function whose second derivative belongs to $L_1[a,b]$, the following Ostrowski's type inequality holds:

Theorem 3.4. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice continuously differentiable function on* $[a,b]$ *i.e.,* $(f \in C^2[a,b])$. *Suppose that* $f'' \in L_1[a,b]$ *i.e.,* $\left(\|f''\|_1 = \int_a^b |f''(t)|\, dt < \infty\right)$. *Then*

$$\left| \frac{b-a}{16}\left[3\left[f(a)+f(b)\right]+10f(\frac{a+b}{2})\right] - \frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right] - \int_a^b f(t)\, dt \right| \leq \frac{3}{2^5}(b-a)^2 \|f''\|_1 . \tag{3.14}$$

Proof. Imposing Hölder's inequality on (3.4), where $f'' \in L_1[a,b]$, and using the result (3.13) completes the proof. ∎

3.4. **The case when f' is locally Lipschitz on $[a,b]$.** For differentiable function whose first derivative is locally Lipschitz on $[a,b]$, the following Ostrowski's type inequality holds:

Theorem 3.5. *Let* $f : [a,b] \to \mathbb{R}$ *be a continuously differentiable function on* $[a,b]$, *i.e.,* $(f \in C^1[a,b])$. *Suppose that* f' *is locally Lipschitz on* $[a,b]$ *with Lipschitz constant* L. *Then*

$$\left| \frac{b-a}{16}\left[3\left[f(a)+f(b)\right]+10f(\frac{a+b}{2})\right] - \frac{(b-a)^2}{64}\left[f'(b)-f'(a)\right] - \int_a^b f(t)\, dt \right| \leq \frac{3L}{2^6}(b-a)^3 . \tag{3.15}$$

Proof. Recall that if f satisfies the local Lipschitz condition on $[a,b]$ with Lipschitz constant L then, f is of bounded variation on $[a,b]$ such that

$$\sup\left\{ \bigvee_a^b (f,P) \mid P \text{ is a partition of } [a,b] \right\} \leq L(b-a). \tag{3.16}$$

Therefore, by Theorem (4) and since f' is locally Lipschitz on $[a,b]$ with Lipschitz constant L, then (3.15) can be obtained. ∎

4. APPLICATIONS

4.1. **Applications to special means.** Before we start, we introduce the following means:

(1) The arithmetic mean,

$$A(a,b) = \frac{a+b}{2}, \quad a,b \in \mathbb{R}$$

(2) The geometric mean,

$$G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

(3) The harmonic mean,

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b > 0$$

(4) The logarithmic mean,

$$L(a,b) = \begin{cases} a, & a=b \\ \frac{b-a}{\ln b - \ln a}, & a \neq b \end{cases}, \quad a,b > 0$$

(5) The p-logarithmic mean,

$$L_p(a,b) = \begin{cases} a, & a=b \\ \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b \end{cases}, \quad a,b > 0, p \in \mathbb{R}\setminus\{-1,0\}$$

(6) The identric mean,

$$I(a,b)=\left\{\begin{array}{ll} a, & a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a\neq b \end{array}\right. ,\quad a,b>0.$$

Now, we deduce some inequalities for the above means, by considering (3.1) and the use of particular functions, as follows:

Corollary 4.1. *Let $a,b\in\mathbb{R}^+$ such the $a<b$, and $n\in\mathbb{Z}\setminus\{-1,0,1,2\}$. Then the following inequality holds:*

$$\left|L_n^n(a,b)+\frac{(b-a)^2}{64}n\,(n-1)\,L_{n-2}^{n-2}(a,b)-\frac{3A(a^n,b^n)+5A^n(a,b)}{8}\right| \leq \frac{17}{3.2^9}\,(b-a)^2\,\delta_n(a,b), \tag{4.1}$$

where

$$\delta_n(a,b)=\left\{\begin{array}{ll} n\,(n-1)\,a^{n-2}, & n\in(-\infty,2)\setminus\{-1,0,1\} \\ n\,(n-1)\,b^{n-2}, & n\in(2,\infty). \end{array}\right.$$

Proof. Consider $f(x)=x^n$, where $x\in[a,b]\subset(0,\infty)$ and $n\in\mathbb{Z}\setminus\{-1,0,1,2\}$. Then we have

$$\left[3\,[f(a)+f(b)]+10f(\frac{a+b}{2})\right]=6A(a^n,b^n)+10A^n(a,b), \tag{4.2}$$

$$[f'(b)-f'(a)]=(b-a)\,n\,(n-1)\,L_{n-2}^{n-2}(a,b),\qquad \int_a^b f(t)\,dt=(b-a)\,L_n^n(a,b), \tag{4.3}$$

and

$$\|f''\|_\infty=\left\{\begin{array}{ll} n\,(n-1)\,a^{n-2}, & n\in(-\infty,2)\setminus\{-1,0,1\} \\ n\,(n-1)\,b^{n-2}, & n\in(2,\infty). \end{array}\right. \tag{4.4}$$

Now, substituting equations (4.2), (4.3) and (4.4) into inequality (3.1) completes the proof. ∎

Similarly, the following can be obtained by considering, respectively, $f(x)=\frac{1}{x}$ and $f(x)=\ln x$ for $x\in[a,b]\subset(0,\infty)$:

Corollary 4.2. *Let $a,b\in\mathbb{R}^+$ such the $a<b$, then the following inequality holds:*

$$\left|L^{-1}(a,b)-\frac{3H^{-1}(a,b)+5A^{-1}(a,b)}{8}+\frac{(b-a)^2\,H^{-1}(a,b)}{32ab}\right| \leq \frac{17}{3.2^8}\frac{(b-a)^2}{a^3}, \tag{4.5}$$

Corollary 4.3. *Let $a,b\in\mathbb{R}^+$ such the $a<b$, then the following inequality holds:*

$$\left|\ln\left(\frac{A^5(a,b).G^3(a,b)}{I^8(a,b)}\right)+\frac{(b-a)^2\,G^{-2}(a,b)}{8}\right| \leq \frac{17}{3.2^6}\left(\frac{b-a}{a}\right)^2. \tag{4.6}$$

4.2. **Applications to quadrature rules.** Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$ and define $\delta_i = x_{i+1} - x_i$ for $i = 0, 1, \cdots, n-1$ such that $\upsilon(\delta) := \max\{\delta_i \mid i = 0, 1, ..., n-1\}$. Now, consider the following general quadrature rule:
(4.7)

$$Q_n(f, I_n) = \sum_{i=0}^{n-1} \left(\frac{1}{16} \left[3\left[f(x_i) + f(x_{i+1})\right] + 10 f(\frac{x_i + x_{i+1}}{2}) \right] - \frac{\delta_i}{64} \left[f'(x_{i+1}) - f'(x_i)\right] \right) \delta_i$$

Then the following Theorems holds:

Theorem 4.4. *Let $f : [a,b] \to \mathbb{R}$ such that $f \in C^2[a,b]$ and $f'' \in L_\infty[a,b]$. Then*

$$\int_a^b f(x)\,dx = Q_n(f, I_n) + R_n(f, I_n), \tag{4.8}$$

where $Q_n(f, I_n)$ is defined by formula (4.7), and the remainder $R_n(f, I_n)$ satisfies the estimates

$$|R_n(f, I_n)| \le \frac{17n}{3.2^9} (\upsilon(\delta))^3 \|f''\|_\infty . \tag{4.9}$$

Proof. Applying (3.1) to the interval $[x_i, x_{i+1}]$, we get

$$\left| \frac{\delta_i}{16} \left[3\left[f(x_i) + f(x_{i+1})\right] + 10 f(\frac{x_i + x_{i+1}}{2}) \right] - \frac{\delta_i^2}{64} \left[f'(x_{i+1}) - f'(x_i)\right] - \int_{x_i}^{x_{i+1}} f(t)\,dt \right| \tag{4.10}$$

$$\le \frac{17}{3.2^9} \delta_i^3 \|f''\|_\infty .$$

for all $i = 0, 1, ..., n-1$. Therefore, by summing (4.10) over i from 0 to $n-1$, we get (4.8). ∎

Theorem 4.5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^1[a,b]$. Suppose that f' is of bounded variation on $[a,b]$. Then*

$$\int_a^b f(x)\,dx = Q_n(f, I_n) + S_n(f, I_n), \tag{4.11}$$

where $Q_n(f, I_n)$ is defined by formula (4.7), and the remainder $S_n(f, I_n)$ satisfies the estimates

$$|S_n(f, I_n)| \le \frac{3}{2^6} (\upsilon(\delta))^2 \bigvee_a^b (f') . \tag{4.12}$$

Proof. Applying (3.8) to the interval $[x_i, x_{i+1}]$, we get

$$\left| \frac{\delta_i}{16} \left[3\left[f(x_i) + f(x_{i+1})\right] + 10 f(\frac{x_i + x_{i+1}}{2}) \right] - \frac{\delta_i^2}{64} \left[f'(x_{i+1}) - f'(x_i)\right] - \int_{x_i}^{x_{i+1}} f(t)\,dt \right| \tag{4.13}$$

$$\le \frac{3}{2^6} \delta_i^2 \bigvee_{x_i}^{x_{i+1}} (f') .$$

for all $i = 0, 1, ..., n-1$. Therefore, by summing (4.13) over i from 0 to $n-1$, we get (4.11). ∎

Theorem 4.6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f\in C^2[a,b]$ and $f''\in L_1[a,b]$. Then*

$$\int_a^b f(x)\,dx = Q_n(f,I_n)+M_n(f,I_n), \tag{4.14}$$

where $Q_n(f,I_n)$ is defined by formula (4.7), and the remainder $M_n(f,I_n)$ satisfies the estimates

$$|M_n(f,I_n)|\le \frac{3n}{2^5}(\upsilon(\delta))^2\|f''\|_1. \tag{4.15}$$

Proof. Applying (3.14) to the interval $[x_i,x_{i+1}]$, we get

$$\left|\frac{\delta_i}{16}\left[3[f(x_i)+f(x_{i+1})]+10f(\frac{x_i+x_{i+1}}{2})\right]-\frac{\delta_i^2}{64}[f'(x_{i+1})-f'(x_i)]-\int_{x_i}^{x_{i+1}} f(t)\,dt\right| \tag{4.16}$$
$$\le \frac{3}{2^5}\delta_i^2\|f''\|_1.$$

for all $i=0,1,.....,n-1$. Therefore, by summing (4.16) over i from 0 to $n-1$, we get (4.14). ∎

Theorem 4.7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f\in C^1[a,b]$. Suppose that f' is locally Lipschitz on $[a,b]$ with Lipschitz constant L. Then*

$$\int_a^b f(x)\,dx = Q_n(f,I_n)+N_n(f,I_n), \tag{4.17}$$

where $Q_n(f,I_n)$ is defined by formula (4.7), and the remainder $N_n(f,I_n)$ satisfies the estimates

$$|N_n(f,I_n)|\le \frac{3nL}{2^6}(\upsilon(\delta))^3. \tag{4.18}$$

Proof. Applying (3.15) to the interval $[x_i,x_{i+1}]$, we get

$$\left|\frac{\delta_i}{16}\left[3[f(x_i)+f(x_{i+1})]+10f(\frac{x_i+x_{i+1}}{2})\right]-\frac{\delta_i^2}{64}[f'(x_{i+1})-f'(x_i)]-\int_{x_i}^{x_{i+1}} f(t)\,dt\right| \tag{4.19}$$
$$\le \frac{3L}{2^6}\delta_i^3.$$

for all $i=0,1,.....,n-1$. Therefore, by summing (4.19) over i from 0 to $n-1$, we get (4.17). ∎

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