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EVALUATION OF A NEW CLASS OF DOUBLE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION $_4F_3$

JOOHYUNG KIM, INSUK KIM∗ AND HARSH V. HARSH

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DEPARTMENT OF MATHEMATICS EDUCATION, WONKWANG UNIVERSITY, IKSAN, 570-749, KOREA.

DEPARTMENT OF MATHEMATICS EDUCATION, WONKWANG UNIVERSITY, IKSAN, 570-749, KOREA.

DEPARTMENT OF MATHEMATICS, AMITY SCHOOL OF ENG. AND TECH., AMITY UNIVERSITY RAJASTHAN
NH-11C, JAIPUR-303002, RAJASTHAN, INDIA.
joohyung@wku.ac.kr
iki@wku.ac.kr
harshvardhanharsh@gmail.com

ABSTRACT. Very recently, Kim evaluated some double integrals involving a generalized hypergeometric function $_3F_2$ with the help of generalization of Edwards's well-known double integral due to Kim, *et al.* and generalized classical Watson's summation theorem obtained earlier by Lavoie, *et al.* In this research paper we evaluate one hundred double integrals involving generalized hypergeometric function $_4F_3$ in the form of four master formulas (25 each) viz. in the most general form for any integer. Some interesting results have also be obtained as special cases of our main findings.

Key words and phrases: Generalized hypergeometric function; Generalized Watson's summation theorem; Generalization of Edwards's double integrals.

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1. INTRODUCTION

In 1812, Gauss [4] defined his famous infinite series as follows: for a, b and $c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$,

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots . \tag{1.1}$$

The series (1.1) is defined by the notation

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; z\right], \quad \text{or} \quad {}_2F_1\left[a,\, b\,; c\,; z\right], $$

which is known as the Gauss's function or hypergeometric function. This is called the 'Hypergeometric series' because either $a = 1$ and $b = c$ or $b = 1$ and $a = c$, it becomes to the well-known 'Geometric seires'.

The Pochhammer symbol $(a)_n$ is defined for any complex number a by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0) \\ a(a+1)\cdots(a+n-1) & (n \in \mathbb{N}) \end{cases}, \tag{1.2}$$

where $\Gamma(z)$ is the well-known gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$$

for Re$(z) > 0$. Thus, the series (1.1) is represented as

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z}{n!}.$$

The generalized hypergeometric function ${}_pF_q$, where $p, q \in \mathbb{N}_0$ is defined by [1, 9].

$$ {}_pF_q\left[\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}. \tag{1.3}$$

The series (1.3) is convergent for any $p \leq q$. In fact, it converges in $|z| < 1$ for $p = q + 1$, converges everywhere for $p < q + 1$ and converges nowhere ($z \neq 0$) for $p > q + 1$. Moreover, if $p = q + 1$, the series converges absolutely for $|z| = 1$ if

$$A = \mathrm{Re}\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right) > 0,$$

holds and converges conditionally for $|z| = 1$ and $z \neq 1$ if $-1 < A \leq 0$ and finally diverges for $|z| = 1$ and $z \neq 1$ if $A \leq -1$ (we refer [9] for more details).

Whenever the generalized hypergeometric functions ${}_pF_q$ can be summed in terms of gamma function, the results are very important from the application point of view. In this sense, the classical summation theorems such as Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$, Dixon, Watson and Whipple for the series ${}_3F_2$, Whipple for the series ${}_4F_3$, Dougall for the series ${}_5F_4$ and others are well-known.

Among the above theorems, we are interested in the following classical Watson's summation theorem for the series ${}_3F_2$ [1], viz.

(1.4)
$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})},
$$

provided $\mathrm{Re}(2c-a-b) > -1$.

Moreover, Lavoie, *et al.* [8] have generalized the above mentioned classical Watson's summation theorem

(1.5)
$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix}; 1\right] \\
&= A_{i,j}\, 2^{a+b+i-2}\, \Gamma\left(c-\frac{1}{2}\left(a+b+|i+j|-j-1\right)\right) \\
&\quad\times \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\
&\quad\times \left\{ \frac{B_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)} \right. \\
&\qquad \left. + \frac{C_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1+(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\} \\
&= \Omega_{i,j},
\end{aligned}
$$

for $i,j = 0, \pm 1, \pm 2$.

Here, $[x]$ denotes the greatest integer less than or equal to x and it's modulus is denoted by $|x|$. The coefficients $A_{i,j}, B_{i,j}$ and $C_{i,j}$ are same as given in the paper [8].

In this paper we evaluate one hundred double integrals in the form of the following four master integrals (twenty-five each):

(i)
$$
\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d-1}(1-x)^{d-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-1} \times {}_4F_3\left[\begin{matrix} a, & b, & c, & 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j \end{matrix}; \frac{(1-x)y}{1-xy}\right] dxdy,
$$

(ii)
$$
\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d+\ell}(1-x)^{d+\ell}(1-y)^{d-1}\,(1-xy)^{\delta-2d-\ell-1} \times {}_4F_3\left[\begin{matrix} a, & b, & c, & 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j \end{matrix}; \frac{1-y}{1-xy}\right] dxdy,
$$

(iii)
$$
\int_0^1\int_0^1 x^{d-1}y^{d+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{d+\ell-\alpha-\beta+1} \times {}_4F_3\left[\begin{matrix} a, & b, & c, & 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j \end{matrix}; xy\right] dxdy,
$$

and

(iv)
$$\int_0^1 \int_0^1 x^{d+\ell} y^{d+\ell+\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d-\alpha-\beta} \times {}_4F_3\left[\begin{matrix} a, \ b, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), \ d, \ 2c+j \end{matrix}; 1-xy\right] dxdy$$

in the most general form for any $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$.

The results are obtained with the help of the following double integral established by Kim, *et al.* [7], viz.

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\delta-\alpha-\beta} dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}, \tag{1.6}$$

provided Re$(\alpha) > 0$, Re$(\beta) > 0$ Re$(\gamma) > 0$ and Re$(\delta) > 0$. Furthermore, we give more than two hundred special cases of our main findings.

The equation (1.6) is the generalization of well-known double integral by Edwards [3]

$$\int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

provided Re$(\alpha) > 0$ and Re$(\beta) > 0$.

2. MAIN RESULTS

Four new classes of double integrals involving generalized hypergeometric functions containing one hundred results (twenty-five each) to be established in this paper are given in the following four theorems.

Theorem 2.1.

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+d-1} (1-x)^{d-1} (1-y)^{d+\ell} (1-xy)^{\delta-2d-\ell-1} \times {}_4F_3\left[\begin{matrix} a, \ b, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), \ d, \ 2c+j \end{matrix}; \frac{(1-x)y}{1-xy}\right] dxdy = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \cdot \frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \Omega_{i,j}, \tag{2.1}$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$, provided Re$(\gamma) > 0$, Re$(\delta) > 0$, Re$(d) > 0$ and Re$(d+\ell+1) > 0$.

Theorem 2.2.

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+d+\ell} (1-x)^{d+\ell} (1-y)^{d-1} (1-xy)^{\delta-2d-\ell-1} \times {}_4F_3\left[\begin{matrix} a, \ b, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), \ d, \ 2c+j \end{matrix}; \frac{1-y}{1-xy}\right] dxdy = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \cdot \frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \Omega_{i,j}, \tag{2.2}$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$, provided Re$(\gamma) > 0$, Re$(\delta) > 0$, Re$(d) > 0$ Re$(d+\ell+1) > 0$.

Theorem 2.3.

$$\begin{aligned}(2.3)\qquad &\int_0^1\int_0^1 x^{d-1}y^{d+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{d+\ell-\alpha-\beta+1}\\ &\quad\times\ {}_4F_3\left[\begin{matrix} a,\ \ b,\ \ c,\ \ 2d+\ell+1\\ \frac{1}{2}(a+b+i+1),\ d,\ 2c+j\end{matrix};xy\right]dxdy\\ &=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\,\Omega_{i,j}\end{aligned}$$

for $\ell\in\mathbb{Z}$ *and* $i,j=0,\pm1,\pm2$*, provided Re*$(\alpha)>0$*, Re*$(\beta)>0$*, Re*$(d)>0$ *and Re*$(d+\ell+1)>0$*.*

Theorem 2.4.

$$\begin{aligned}(2.4)\qquad &\int_0^1\int_0^1 x^{d+\ell}y^{d+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{d-\alpha-\beta}\\ &\quad\times\ {}_4F_3\left[\begin{matrix} a,\ \ b,\ \ c,\ \ 2d+\ell+1\\ \frac{1}{2}(a+b+i+1),\ d,\ 2c+j\end{matrix};1-xy\right]dxdy\\ &=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\,\Omega_{i,j},\end{aligned}$$

for $\ell\in\mathbb{Z}$ *and* $i,j=0,\pm1,\pm2$*, provided Re*$(\alpha)>0$*, Re*$(\beta)>0$*, Re*$(d)>0$ *and Re*$(d+\ell+1)>0$*.*

Here in all four cases, $\Omega_{i,j}$ *are the same as given in* (1.5)*.*

Proof. The proofs of our results are quite straight forward. In order to establish the result (2.1) in Theorem 2.1, let's denote the left-hand side in (2.1) by S. If we express ${}_4F_3$ as a series and change the order of integration and summation due to the uniform convergence of the series, then we have

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n(2d+\ell+1)_n}{n!\left(\frac{1}{2}(a+b+i+1)\right)_n(d)_n(2c+j)_n}\\ &\times\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d+n-1}(1-x)^{d+n-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-n-1}\,dxdy.\end{aligned}$$

Using the result (1.6) in the above double integral and using (1.2) with some simplification, we obtain

$$S=\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{n!\left(\frac{1}{2}(a+b+i+1)\right)_n(2c+j)_n}.$$

Next, summing up the series, we get

$$S=\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\,{}_3F_2\left[\begin{matrix} a,\ \ b,\ \ c\\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};1\right].$$

Finally, we observe that ${}_3F_2$ appearing the above can be evaluated with the help of the result (1.5) and we arrive at the right-hand side of (2.1).

This completes the proof of (2.1). ∎

In exactly the same manner, Theorem 2.2 to Theorem 2.4 can be also proven .

We conclude this section by remarking that more than two hundred interesting special cases will be given in the next section in the form of eight general results.

3. SPECIAL CASES

In this section, more than two hundred interesting special cases of our main results are given.

(a) In (2.1) to (2.4), replace a by $a+2n$ and let $b=-2n$ or replace a by $a+2n+1$ and let $b=-2n-1$ for $n\in\mathbb{N}_0$. In both cases, we notice that one of the two terms on the right-hand sides of (2.1) to (2.4) will vanish and hence we get the following two hundred interesting special cases in the form of eight general integrals. These are

$$\tag{3.1} \begin{aligned} &\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d-1}(1-x)^{d-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-1} \\ &\qquad\times {}_4F_3\left[\begin{matrix} a+2n,\ -2n,\ c,\ 2d+\ell+1 \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j \end{matrix}\ ; \frac{(1-x)y}{1-xy}\right] dxdy \\ &= D_{i,j}\,\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\qquad\times\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n}=\Omega^{(1)}_{i,j}, \end{aligned}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, where the coefficients, $D_{i,j}$ are same as given in the paper [8].

$$\tag{3.2} \begin{aligned} &\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d-1}(1-x)^{d-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-1} \\ &\qquad\times {}_4F_3\left[\begin{matrix} a+2n+1,\ -2n-1,\ c,\ 2d+\ell+1 \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j \end{matrix}\ ; \frac{(1-x)y}{1-xy}\right] dx \\ &= E_{i,j}\,\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\qquad\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n}=\Omega^{(2)}_{i,j}, \end{aligned}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. The coefficients, $E_{i,j}$ are same as given in the paper [8].

$$\tag{3.3} \begin{aligned} &\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d+\ell}(1-x)^{d+\ell}(1-y)^{d-1}\,(1-xy)^{\delta-2d-\ell-1} \\ &\qquad\times {}_4F_3\left[\begin{matrix} a+2n,\ -2n,\ c,\ 2d+\ell+1; \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j; \end{matrix}\ \frac{1-y}{1-xy}\right] dxdy \\ &= \Omega^{(1)}_{i,j}, \end{aligned}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. Here, $\Omega^{(1)}_{i,j}$ is the same as defined in (3.1).

$$\tag{3.4} \begin{aligned} &\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d+\ell}(1-x)^{d+\ell}(1-y)^{d-1}\,(1-xy)^{\delta-2d-\ell-1} \\ &\qquad\times {}_4F_3\left[\begin{matrix} a+2n+1,\ -2n-1,\ c,\ 2d+\ell+1 \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j \end{matrix}\ ; \frac{1-y}{1-xy}\right] dxdy \\ &= \Omega^{(2)}_{i,j}, \end{aligned}$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$. Here, $\Omega_{i,j}^{(2)}$ is the same as defined in (3.2).

(3.5)
$$\int_0^1 \int_0^1 x^{d-1} y^{d+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d+\ell-\alpha-\beta+1} \times {}_4F_3 \left[\begin{matrix} a+2n, \ -2n, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+i+1), \ d, \ 2c+j \end{matrix} ; xy \right] dxdy$$
$$= D_{i,j} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \times \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + \frac{3}{4} - \frac{(-1)^i}{4} - \left[\frac{j}{2} + \frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j}{2}\right]\right)_n \left(\frac{1}{2}a + \frac{1}{4}\left(1+(-1)^i\right)\right)_n} = \Omega_{i,j}^{(3)},$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$, where the coefficients, $D_{i,j}$ are same as given in the paper [8].

(3.6)
$$\int_0^1 \int_0^1 x^{d-1} y^{d+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d+\ell-\alpha-\beta+1} \times {}_4F_3 \left[\begin{matrix} a+2n+1, \ -2n-1, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+i+1), \ d, \ 2c+j \end{matrix} ; xy \right] dx$$
$$= E_{i,j} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \times \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a - c + \frac{5}{4} + \frac{(-1)^i}{4} - \left[\frac{j}{2} + \frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j+1}{2}\right]\right)_n \left(\frac{1}{2}a + \frac{1}{4}\left(3-(-1)^i\right)\right)_n} = \Omega_{i,j}^{(4)},$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$. The coefficients, $E_{i,j}$ are same as given in the paper [8].

(3.7)
$$\int_0^1 \int_0^1 x^{d+\ell} y^{d+\ell+\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d-\alpha-\beta} \times {}_4F_3 \left[\begin{matrix} a+2n, \ -2n, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+i+1), \ d, \ 2c+j \end{matrix} ; 1-xy \right] dxdy = \Omega_{i,j}^{(3)},$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$. Here, $\Omega_{i,j}^{(3)}$ are the same as defined in (3.5).

(3.8)
$$\int_0^1 \int_0^1 x^{d+\ell} y^{d+\ell+\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d-\alpha-\beta} \times {}_4F_3 \left[\begin{matrix} a+2n+1, \ -2n-1, \ c, \ 2d+\ell+1 \\ \frac{1}{2}(a+i+1), \ d, \ 2c+j \end{matrix} ; 1-xy \right] dxdy = \Omega_{i,j}^{(4)},$$

for $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$. Here, $\Omega_{i,j}^{(4)}$ are the same as defined in (3.6).
Similarly, other results can be obtained.

(b) In particular, in (3.1) and (3.2), if we take $i = j = 0$, we get

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d-1}(1-x)^{d-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-1}\\ &\qquad\times {}_4F_3\left[\begin{matrix}a+2n,\ -2n,\ c,\ 2d+\ell+1\\ \frac{1}{2}(a+1),\quad d,\quad 2c\end{matrix};\frac{(1-x)y}{1-xy}\right]dxdy\\ &=\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\,\frac{\left(\frac{1}{2}\right)_n\,\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\,\left(\frac{1}{2}a+\frac{1}{2}\right)_n}\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+d-1}(1-x)^{d-1}(1-y)^{d+\ell}\,(1-xy)^{\delta-2d-\ell-1}\\ &\times\ {}_4F_3\left[\begin{matrix}a+2n+1,\ -2n-1,\ c,\ 2d+\ell+1\\ \frac{1}{2}(a+1),\quad d,\quad 2c\end{matrix};\frac{(1-x)y}{1-xy}\right]dxdy=0.\end{aligned}\tag{3.10}$$

(c) In (2.1) and (2.2), if we set $d = c$, we get two general results obtained recently by Kim [5].

(d) In (2.3) and (2.4), if we set $d = c$, we get two general results obtained very recently by Kim [6].

(e) In (2.1) and (2.2), if we set $d = c$ and $\gamma = \delta = 1$, we get the two results obtained by Choi and Rathie [2].

4. CONCLUDING REMARK

In this paper, we have evaluated one hundred interesting double integrals involving generalized hypergeometric function ${}_4F_3$ in the form of four general integrals (twenty-five each). The results are obtained with the help of generalization of Edwards's double integral and generalization of Watson's summation theorem available in the literature. The results established in this paper may be useful in applied mathematics, mathematical physics and engineering mathematics.

REFERENCES

[1] W. N. BAILEY, *Generalized Hypergeometric Series*, Stechert-Hafner, New York, (1964).

[2] J. CHOI, and A. K. RATHIE, A new class of double integrals involving generalized hypergeometric functions, *Advanced Studies in Contemporary Mathematics*, 27(2), pp. 189-198, (2017).

[3] J. EDWARDS, *A Treaties on the Integral Calculus with Applications*, Examples and Problems, Vol. II, Chelsea Publishing Campany, New York, (1954).

[4] C. F. GAUSS, *"Disquisitiones Generales Circa Serium Infinitum, Thesis, Gottingen"*, Ges. Werke Gottingen, Vol. II, pp. 437-445; III pp. 123-163; III pp. 207-229; III pp. 446-460, (1866).

[5] I. KIM, On a new class of double integrals involving generalized hypergeometric function ${}_3F_2$, *Honam Math. J.*, 40(4), pp. 809-816, (2018).

[6] I. KIM, *A new class of double integrals involving generalized hypergeometric function* ${}_3F_2$, *Tamkang J. Math.*, 51(1), pp. 69-80, (2020).

[7] I. KIM, S. JUN, Y. VYAS, and A. K. RATHIE, On an extension of Edwards's double integrals with applications, *Australian J. Math. Analysis and Application*, 16(2), Article 2, pp. 1-13, (2019).

[8] J. L. LAVOIE, F. GRONDIN, and A. K. RATHIE, Generalizations of Watson's theorem on the sum of a ${}_3F_2$, *Indian J. Math.*, 34, pp. 23-32, (1992).

[9] E. D. RAINVILLE, *Special Functions*, Macmillan Company, New York, (1960); Reprinted by Chelsea Publishing Company, Bronx, New York, (1971).