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RATIONAL EXPRESSIONS OF ARITHMETIC AND GEOMETRIC MEANS FOR THE SEQUENCE $(n^p)_{n\in\mathbb{N}}$ AND THE GEOMETRIC PROGRESSION

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ABSTRACT. In this paper, we consider the arithmetic and geometric means for the sequence $(n^p)_{n\in\mathbb{N}}$ and the geometric progression. We obtain the results associated with the rational expressions of the means.

Key words and phrases: Rational expression; Arithmetic mean; Geometric mean; Sequence of $(n)_n \in \mathbb{N}$; Geometric progression.

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1. INTRODUCTION

We assume that (a_n) is a sequence of the positive real number, then the arithmetic mean of $a_1, a_2, \cdots, a_n$ is defined by

$$A(a_1, a_2, \cdots, a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

and the geometric mean of $a_1, a_2, \cdots, a_n$ is defined by

$$G(a_1, a_2, \cdots, a_n) = \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}}.$$

It is known that McCartin [4] asserts $\lim_{n\to\infty} \frac{A(1,2,\cdots,n)}{G(1,2,\cdots,n)} = \frac{e}{2}$ and Knopp [2] in Problem 96, presented $\lim_{n\to\infty} \frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)} = \frac{e}{2}$, where the sequence (a_n) is the arithmetical progression with the positive equal difference. Recently, Hassani [6] proved that the rational expression $\frac{A(1,2,\cdots,n)}{G(1,2,\cdots,n)}$ is strictly increasing for the integer $n > 1$, the inequality $\frac{A(1,2,\cdots,n)}{G(1,2,\cdots,n)} < \frac{e}{2}$ and $\frac{A(1,2,\cdots,n)}{G(1,2,\cdots,n)} = \frac{e}{2} + O\left(\frac{\ln n}{n}\right)$ hold for the integer $n \geq 1$. Moreover, Hassani [7] showed that $\frac{A(p_1,p_2,\cdots,p_n)}{G(p_1,p_2,\cdots,p_n)} = \frac{e}{2} + O\left(\frac{1}{\ln n}\right)$, where p_n denotes the nth prime number. As mentioned above, the rational expression $\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}$ has interesting properties. In this paper, we consider the two rational expressions $\frac{A(1^p,2^p,\cdots,n^p)}{G(1^p,2^p,\cdots,n^p)}$ and $\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}$, where the sequence (a_n) is the geometric progression with the positive geometric ratio. Here, we have the following theorems.

Theorem 1.1. *We have*

$$\lim_{n\to\infty} \frac{A_{n,p}}{G_{n,p}} = \frac{e^p}{1+p}$$

for $p > -1$ *and*

$$\lim_{n\to\infty} \frac{A_{n,p}}{G_{n,p}} = \infty$$

for $p \leq -1$*, where* $A_{n,p} = A(1^p, 2^p, \cdots, n^p)$ *and* $G_{n,p} = G(1^p, 2^p, \cdots, n^p)$.

Kubelka [3] proved that $\lim_{n\to\infty} \frac{A(1^p,2^p,\cdots n^p)}{G(1^p,2^p,\cdots n^p)} = \frac{e^p}{1+p}$ for $p \geq 0$. From the Kubelka's result, Hassani [5] showed that there exists a real positive sequence (a_n) such that $\lim_{n\to\infty} \frac{A(a_1,a_2,\cdots a_n)}{G(a_1,a_2,\cdots a_n)} = c$ for any $c \geq 1$. Since we consider the case of the any real number p, Theorem 1.1 is an extended the Kubelka's result. From Theorem 1.1, we have the following corollary immediately.

Corollary 1.2. *Let* c *be the real number with* $c > 1$*, then there exists two real positive sequences* (a_n) *such that* $\lim_{n\to\infty} \frac{A(a_1,a_2,\cdots a_n)}{G(a_1,a_2,\cdots a_n)} = c$.

Theorem 1.3. *Let* n *be the integer with* $n > 1$*, then the rational expression* $\frac{A_{n,p}}{G_{n,p}}$ *is strictly increasing for* $p > 0$*, where* $A_{n,p} = A(1^p, 2^p, \cdots, n^p)$ *and* $G_{n,p} = G(1^p, 2^p, \cdots, n^p)$.

Theorem 1.4. *Let the sequence* (a_n) *be the geometric progression with the geometric ratio* $r > 0$*, then we have*

$$r^{\frac{n-1}{2}} n^{\alpha} < \frac{A(a_1, a_2, \cdots, a_n)}{G(a_1, a_2, \cdots, a_n)} < r^{\frac{n-1}{2}} n^{\beta}$$

for $r > 1$ *and*

$$r^{\frac{1-n}{2}} n^{\tilde{\alpha}} < \frac{A(a_1, a_2, \cdots, a_n)}{G(a_1, a_2, \cdots, a_n)} < r^{\frac{1-n}{2}} n^{\tilde{\beta}}$$

for $0 < r < 1$, *where* $\alpha = \tilde{\alpha} = -1$ *and* $\beta = \tilde{\beta} = 0$ *are the best possible constants. Moreover we have*

$$\lim_{n\to\infty} \frac{A(a_1, a_2, \cdots, a_n)}{G(a_1, a_2, \cdots, a_n)} = \infty$$

Theorem 1.5. *Let the sequence* (a_n) *be the geometric progression with the positive geometric ratio* $r \neq 1$, *then the rational expression* $\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}$ *is strictly increasing for the integer* $n > 1$ *and the convex sequence.*

2. PROOF OF MAIN RESULTS

The following Lemma 2.1 is proved by Hummel in [1].

Lemma 2.1. *We have*

$$e^{\frac{11}{12}-n} n^{n+\frac{1}{2}} < n! < e^{1-n} n^{n+\frac{1}{2}}$$

for the integer $n > 1$.

Proof of Theorem 1.1. We have

$$\frac{A_{n,p}}{G_{n,p}} = \frac{\sum_{k=1}^{n} k^p}{n \cdot n!^{\frac{p}{n}}}.$$

First, we consider the case of $p > 0$ and the integer $n > 1$. From Lemma 2.1 and the inequality

$$\int_1^n x^p dx + 1 < \sum_{k=1}^{n} k^p < \int_1^{n+1} x^p dx$$

holds for $p > 0$ and the integer $n > 1$, we can get

$$\frac{A_{n,p}}{G_{n,p}} < \frac{\int_1^{n+1} x^p dx}{n\left(e^{\frac{11}{12}-n} n^{n+\frac{1}{2}}\right)^{\frac{p}{n}}} = \frac{e^p}{1+p} \cdot \frac{\left(1+\frac{1}{n}\right)^{1+p} - \frac{1}{n^{1+p}}}{e^{\frac{11p}{12n}} n^{\frac{p}{2n}}} = F(n,p)$$

and

$$\frac{A_{n,p}}{G_{n,p}} > \frac{\int_1^{n} x^p dx + 1}{n\left(e^{1-n} n^{n+\frac{1}{2}}\right)^{\frac{p}{n}}} = \frac{e^p}{1+p} \cdot \frac{1 + \frac{p}{n^{1+p}}}{e^{\frac{p}{n}} n^{\frac{p}{2n}}} = G(n,p)$$

for $p > 0$ and the integer $n > 1$. Next, we consider the case of $p < 0$ and the integer $n > 1$. From Lemma 2.1 and the inequality

$$\int_1^{n+1} x^p dx < \sum_{k=1}^{n} k^p < \int_1^{n} x^p dx + 1$$

holds for $p < 0$ and the integer $n > 1$, we can get

$$\frac{A_{n,p}}{G_{n,p}} < \frac{\int_1^{n} x^p dx + 1}{n\left(e^{\frac{11}{12}-n} n^{n+\frac{1}{2}}\right)^{\frac{p}{n}}} = H(n,p) = \begin{cases} \frac{e^p}{1+p} \cdot \frac{1+\frac{p}{n^{1+p}}}{e^{\frac{11p}{12n}} n^{\frac{p}{2n}}} & (p \neq -1) \\ (\ln n + 1)\, e^{\frac{11}{12n}-1} n^{\frac{1}{2n}} & (p = -1) \end{cases}$$

and

$$\frac{A_{n,p}}{G_{n,p}} > \frac{\int_1^{n+1} x^p dx}{n\left(e^{1-n} n^{n+\frac{1}{2}}\right)^{\frac{p}{n}}} = I(n,p) = \begin{cases} \frac{e^p}{1+p} \cdot \frac{\left(1+\frac{1}{n}\right)^{1+p} - \frac{1}{n^{1+p}}}{e^{\frac{p}{n}} n^{\frac{p}{2n}}} & (p \neq -1) \\ (\ln(1+n))\, e^{\frac{1}{n}-1} n^{\frac{1}{2n}} & (p = -1) \end{cases}$$

for the integer $n > 1$. By $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, we obtain

$$\lim_{n\to\infty} F(n,p) = \lim_{n\to\infty} G(n,p) = \frac{e^p}{1+p} \quad \text{for} \quad p > 0,$$

$$\lim_{n\to\infty} H(n,p) = \lim_{n\to\infty} I(n,p) = \frac{e^p}{1+p} \quad \text{for} \quad -1 < p < 0,$$

$$\lim_{n\to\infty} H(n,p) = \lim_{n\to\infty} I(n,p) = \infty \quad \text{for} \quad p < -1$$

and

$$\lim_{n\to\infty} H(n,-1) = \lim_{n\to\infty} I(n,-1) = \infty\,.$$

If $p = 0$ then $\frac{A_{n,0}}{G_{n,0}} = 1 = \frac{e^0}{1+0}$, hence we can prove $\lim_{n\to\infty} \frac{A_{n,p}}{G_{n,p}} = \frac{e^p}{1+p}$ for $p > -1$ and $\lim_{n\to\infty} \frac{A_{n,p}}{G_{n,p}} = \infty$ for $p \leq -1$. ∎

Proof of Corollary 1.2. We set $F(p) = \frac{e^p}{1+p}$ and the derivative of $F(p)$ is $F'(p) = \frac{e^p p}{(1+p)^2}$. If $-1 < p < 0$ then $F'(p) < 0$ and $\lim_{p\to -1+0} F(p) = \infty$. If $p > 0$ then $F'(p) > 0$ and $\lim_{p\to\infty} F(p) = \infty$. Hence, for any $c > 1$, we have the two real numbers s and t with $-1 < s < 0 < t$ such that $F(s) = F(t) = c$. Therefore we have

$$\lim_{n\to\infty} \frac{A(1^s, 2^s, \cdots, n^s)}{G(1^s, 2^s, \cdots, n^s)} = \lim_{n\to\infty} \frac{A(1^t, 2^t, \cdots, n^t)}{G(1^t, 2^t, \cdots, n^t)} = c$$

and the proof of Corollary 1.2 is complete. ∎

Proof of Theorem 1.3. We set $F(n,p) = \frac{\sum_{k=1}^n k^p}{n \cdot n!^{\frac{p}{n}}}$ and the derivative of $F(n,p)$ is

$$\frac{\partial F}{\partial p}(n,p) = \frac{n\sum_{k=1}^n k^p \ln k - \sum_{k=1}^n k^p \sum_{k=1}^n \ln k}{n^2 n!^{\frac{p}{n}}}.$$

If $p \geq 0$ then we have $1^p \leq 2^p \leq \cdots \leq n^p$ and $\ln 1 < \ln 2 < \cdots < \ln n$. From Chebyshev inequality, we can obtain

$$n\sum_{k=1}^n k^p \ln k \geq \sum_{k=1}^n k^p \sum_{k=1}^n \ln k$$

with equality if and only if $p = 0$. Therefore, we have $\frac{\partial F}{\partial p}(n,p) > 0$ for $p > 0$ and $\frac{A_{n,p}}{G_{n,p}}$ is strictly increasing for $p > 0$. ∎

Proof of Theorem 1.4. We have

$$\frac{A(a_1, a_2, \cdots, a_n)}{G(a_1, a_2, \cdots, a_n)} = \frac{r^{\frac{1-n}{2}}(-1+r^n)}{n(-1+r)}\,.$$

First, we consider the equation

$$\frac{r^{\frac{1-n}{2}}(-1+r^n)}{n(-1+r)} = n^{F(n,r)} r^{\frac{n-1}{2}}$$

for $r > 1$ and the integer $n > 1$. The equation is equivalent to

$$F(n,r) = \frac{(1-n)\ln r + \ln(-1+r^n) - \ln n - \ln(-1+r)}{\ln n}$$

and the derivative of $F(n,r)$ is

$$\frac{\partial F}{\partial r}(n,r) = \frac{1-n+nr-r^n}{(-1+r)r(-1+r^n)\ln n} = \frac{G(n,r)}{(-1+r)r(-1+r^n)\ln n}\,.$$

The derivative of $G(n,r)$ is $\frac{\partial G}{\partial r}(n,r) = n - nr^{-1+n} < 0$ for $r > 1$ and the integer $n > 1$. Since $G(n,r)$ is strictly decreasing for $r > 1$, we have $G(n,r) < G(n,1) = 0$ for $r > 1$ and the integer $n > 1$ Thus, $F(n,r)$ is strictly decreasing for $r > 1$ and we obtain

$$\lim_{r\to\infty} F(n,r) < F(n,r) < \lim_{r\to 1+0} F(n,r)$$

for the integer $n > 1$. From

$$\lim_{r\to\infty} ((1-n)\ln r + \ln(-1+r^n) - \ln(-1+r)) = \lim_{r\to\infty} \ln\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = 0$$

and

$$\lim_{r\to 1} (\ln(-1+r^n) - \ln(-1+r)) = \lim_{r\to 1} \ln(1+r+r^2+\cdots+r^{n-1}) = \ln n,$$

we have

$$\lim_{r\to\infty} F(n,r) = -1 \quad \text{and} \quad \lim_{r\to 1+0} F(n,r) = 0.$$

Therefore we can get the best possible constants $\alpha = -1$ and $\beta = 0$. Next, we consider the equation

$$\frac{r^{\frac{1-n}{2}}(-1+r^n)}{n(-1+r)} = n^{H(n,r)} r^{\frac{1-n}{2}}$$

for $0 < r < 1$ and the integer $n > 1$. The equation is equivalent to

$$H(n,r) = \frac{\ln(1-r^n) - \ln n - \ln(1-r)}{\ln n}$$

and the derivative of $H(n,r)$ is

$$\frac{\partial H}{\partial r}(n,r) = \frac{r - nr^n - r^{1+n} + nr^{1+n}}{(-1+r)r(-1+r^n)\ln n}.$$

Since we have

$$\begin{aligned} r - nr^n - r^{1+n} + nr^{1+n} &= r\left(1 - r^n - nr^{n-1}(1-r)\right) \\ &= r(1-r)\left(1 + r + \cdots + r^{n-1} - nr^{n-1}\right) > 0, \end{aligned}$$

$\frac{\partial H}{\partial r}(n,r) > 0$ for $0 < r < 1$ and the integer $n > 1$. Since $H(n,r)$ is strictly increasing for $0 < r < 1$, we have

$$\lim_{r\to 0} H(n,r) = -1 \quad \text{and} \quad \lim_{r\to 1-0} H(n,r) = 0.$$

Therefore we can get the best possible constants $\tilde{\alpha} = -1$ and $\tilde{\beta} = 0$. By l'Hopital's rule, we have

$$\lim_{n\to\infty} \frac{r^{\frac{n-1}{2}}}{n} = \lim_{n\to\infty} \frac{r^{\frac{n-1}{2}}\ln r}{2} = \infty$$

for $r > 1$ and

$$\lim_{n\to\infty} \frac{r^{\frac{1-n}{2}}}{n} = \lim_{n\to\infty} \frac{-\ln r}{2r^{\frac{n-1}{2}}} = \infty$$

for $0 < r < 1$. Therefore, we can get $\lim_{n\to\infty} \frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)} = \infty$ and the proof of Theorem 1.4 is complete. ∎

Proof of Theorem 1.5. We have

$$\begin{aligned}&\frac{A(a_1,a_2,\cdots,a_{n+1})}{G(a_1,a_2,\cdots,a_{n+1})}-\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}\\=&\frac{r^{\frac{1-(n+1)}{2}}(-1+r^{n+1})}{(n+1)(-1+r)}-\frac{r^{\frac{1-n}{2}}(-1+r^n)}{n(-1+r)}\\=&\frac{-n+r^{\frac{1}{2}}+nr^{\frac{1}{2}}-r^{\frac{1}{2}+n}-nr^{\frac{1}{2}+n}+nr^{1+n}}{n(1+n)(r-1)r^{\frac{n}{2}}}\\=&\frac{F(n,r)}{n(1+n)(r-1)r^{\frac{n}{2}}}\end{aligned}$$

and the derivative of $F(n,r)$ is

$$\frac{\partial F}{\partial r}(n,r)=\frac{(1+n)(r^{\frac{1}{2}}-r^{\frac{1}{2}+n}-2nr^{\frac{1}{2}+n}+2nr^{1+n})}{2r}.$$

We set $G(n,r)=\ln(2nr^{1+n}+r^{\frac{1}{2}})-\ln(2nr^{\frac{1}{2}+n}+r^{\frac{1}{2}+n})$ and the derivative of $G(n,r)$ is

$$\frac{\partial G}{\partial r}(n,r)=\frac{n(-r+r^{\frac{3}{2}+n})}{r^2(1+2nr^{\frac{1}{2}+n})}.$$

From $-r+r^{\frac{3}{2}}<0$ for $0<r<1$ and $-r+r^{\frac{3}{2}}>0$ for $r>1$, $G(r)$ is strictly decreasing for $0<r<1$ and strictly increasing for $r>1$. Hence, we have $G(n,r)>G(n,1)=0$ for $r>0$ and $n>1$. Hence, we have $\frac{\partial F}{\partial r}(n,r)>0$ for $r>0$ and $F(n,r)$ is strictly increasing for $r>0$. By $F(n,1)=0$, we have $F(n,r)<0$ for $0<r<1$ and $F(n,r)>0$ for $r>1$. Therefore, we can get

$$\frac{A(a_1,a_2,\cdots,a_{n+1})}{G(a_1,a_2,\cdots,a_{n+1})}>\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}$$

for the positive $r\neq1$ and the integer $n>1$. We have

$$\begin{aligned}&\ln\left(\frac{A(a_1,a_2,\cdots,a_{n+2})}{G(a_1,a_2,\cdots,a_{n+2})}+\frac{A(a_1,a_2,\cdots,a_n)}{G(a_1,a_2,\cdots,a_n)}\right)-\ln\left(2\frac{A(a_1,a_2,\cdots,a_{n+1})}{G(a_1,a_2,\cdots,a_{n+1})}\right)\\&=\frac{H(n,r)}{2},\end{aligned}$$

where

$$\begin{aligned}H(n,r)=&-2\ln2-2\ln n+2\ln(1+n)-2\ln(2+n)-\ln r-2\ln(-1+r^{1+n})\\&+2\ln(-n-2r-nr+2r^{1+n}+nr^{1+n}+nr^{2+n})\end{aligned}$$

for $r>1$ and

$$\begin{aligned}H(n,r)=&-2\ln2-2\ln n+2\ln(1+n)-2\ln(2+n)-\ln r-2\ln(1-r^{1+n})\\&+2\ln(n+2r+nr-2r^{1+n}-nr^{1+n}-nr^{2+n})\end{aligned}$$

for $0<r<1$. The derivative of $H(n,r)$ is

$$\frac{\partial H}{\partial r}(n,r)=\frac{(1+r^{1+n})I(n,r)}{2r(-1+r^{1+n})(-n-2r-nr+2r^{1+n}+nr^{1+n}+nr^{2+n})},$$

where $I(n,r) = -n + 2r + nr - 2r^{1+n} - nr^{1+n} + nr^{2+n}$. The derivative of $I(n,r)$ is

$$\begin{aligned}\frac{\partial I}{\partial r}(n,r) &= (2+n)(1 - r^n - nr^n + nr^{1+n}) \\ &= (2+n)(1-r)\left(1 + r + r^2 + \cdots + r^{n-1} - nr^n\right).\end{aligned}$$

From $\frac{\partial I}{\partial r}(n,r) > 0$ for $0 < r < 1$ and $r > 1$, $I(n,r)$ is strictly increasing for $r > 0$. By $I(n,1) = 0$, we have $I(n,r) < 0$ for $0 < r < 1$ and $I(n,r) > 0$ for $r > 1$. Thus, we can get $\frac{\partial H}{\partial r}(n,r) < 0$ for $0 < r < 1$ and $\frac{\partial H}{\partial r}(n,r) > 0$ for $r > 1$. Therefore, $H(n,r)$ is strictly decreasing for $0 < r < 1$ and $H(n,r)$ is strictly increasing for $r > 1$. By l'Hopital's rule, we have

$$\begin{aligned}&\lim_{r\to 1}\frac{-n - 2r - nr + 2r^{1+n} + nr^{1+n} + nr^{2+n}}{-1 + r^{1+n}} \\ &= \lim_{r\to 1}\frac{-2 - n + 2(1+n)r^n + n(1+n)r^n + n(2+n)r^{1+n}}{(1+n)r^n} = \frac{2n(2+n)}{1+n}.\end{aligned}$$

Hence, we obtain $H(n,r) > \lim_{r\to 1} H(n,r) = 0$ and

$$\frac{A(a_1, a_2, \cdots, a_{n+2})}{G(a_1, a_2, \cdots, a_{n+2})} + \frac{A(a_1, a_2, \cdots, a_n)}{G(a_1, a_2, \cdots, a_n)} > 2\frac{A(a_1, a_2, \cdots, a_{n+1})}{G(a_1, a_2, \cdots, a_{n+1})}$$

for the positive $r \neq 1$ and the integer $n > 1$, so the proof of Theorem 1.5 is complete. ∎

Finally, we propose an open problem.

Problem 1. *Let p be the real number with $p > -1$, then the rational expression $\frac{A_{n,p}}{G_{n,p}}$ is strictly increasing for the integer $n > 1$, where $A_{n,p} = A(1^p, 2^p, \cdots, n^p)$ and $G_{n,p} = A(1^p, 2^p, \cdots, n^p)$.*

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