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RESIDUAL-BASED A POSTERIORI ERROR ESTIMATES FOR A CONFORMING MIXED FINITE ELEMENT DISCRETIZATION OF THE MONGE-AMPÈRE EQUATION

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ABSTRACT. In this paper we develop a new a posteriori error analysis for the Monge-Ampère equation approximated by conforming finite element method on isotropic meshes in $\mathbb{R}^2$. The approach utilizes a slight variant of the mixed discretization proposed by Gérard Awanou and Hengguang Li in [4]. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient.

Key words and phrases: Monge-Ampère equation; Conforming finite element method; A posteriori error analysis.

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1. Introduction

The adaptive techniques have become indispensable tools and unavoidable in the field of study behavior of the error committed during solving partial differential equations (PDE). A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equi-distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [8, 7], adaptive finite element methods based on a posteriori error estimates have been extensively investigated. Several a-posteriori error analysis methods for PDE have been developed in the last five decades [24, 23, 1, 3, 5, 6, 9, 10, 11, 28, 16, 19, 2, 21].

We consider the Monge-Ampère equation on a convex domain of $\mathbb{R}^2$ with a smooth solution and our approach utilizes a slight variant of the mixed discretization proposed by Gérard Awanou and Hengguang Li in [4]. The purpose of these works is to determine to which extend the general framework for adaptivity for non linear problems of Gatica and his collaborators in [15, 19] can be applied to the Monge-Ampère equation. More precisely, we attempt to determine to which extent one can prove results analogous to the ones of Houédanou, Adetola and Ahounou [20]. Indeed, in [4] Gérard Awanou and Li study the mixed method for this equation and they gave a priori error estimator under the assumption of regularity for the solution of continuous problem. Omar Lakkis in his presentation of July 15, 2014, presents a family of reliable error indicators for a primal formulation [22]. However, to our best knowledge, they didn't talk about adaptative method for this mixed formulation. In this case our main objective is to perform an a posteriori error analysis by constructing reliable and efficiency indicators errors.

In [4] Gerard and Li have introduced a mixed finite element method formulation for the elliptic Monge-Ampère equation by puting $\sigma = D^2u$. The new unknowns in the formulation are u and σ which have been approached respectively by the discrete polynomials spaces of Lagrange. Finally, they give a result of error priori analysis with some numerical tests confirming the convergence rates. In this paper, we obtain a new family of a local indicator error Θ_K (see Definition 3.1, eq. 3.16) and global Θ (eq. 3.17) efficiency and reliability for the mixed method of Monge-Ampère model. We prove that our indicators error are efficient and reliable, and then, are optimal. The global inf-sup condition is the main tool yielding the reliability. In turn, The local efficiency result is derived using the technique of bubble function introduced by R. Verfürth [25] and used in similar context by C. Carstensen [13, 12].

The paper is organized as follows. Some preliminaries and notation are given in section 2. In section 3, the a posteriori error estimates are derived. We offer our conclusion and the further works in Section 4.

2. Preliminaries and Notation

2.1. **Model.** Let Ω be a convex polygonal domain of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following problem : find the unique strictly convex $C^3(\overline{\Omega})$ solution u (when it exists) of

$$\begin{aligned} \det(D^2u) &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

The given function $f \in C^1(\overline{\Omega})$ is assumed to satisfy $f > 0$ and the function $g \in C(\partial\Omega)$ is also given and assumed to extend to a $C^3(\overline{\Omega})$ function. Here $\det(D^2u)$ denotes the determinant of the Hessian matrix $D^2u = \big(\partial^2 u/(\partial x_i \partial x_j)\big)_{i,j=1,2}$.

2.2. **Modified continuous mixed weak formulation.** We begin this subsection by introducing some useful notations. If W is a bounded domain of $\mathbb{R}^2$ and m is a non negative integer, the

Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\|\cdot\|_{m,W}$ and semi-norm $|.|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\|\cdot\|_W$ for $\|\cdot\|_{0,W}$. Similarly we denote by $(\cdot,\cdot)_W$ the $L^2(W)$, $[L^2(W)]^2$ or $[L^2(W)]^{2\times 2}$ inner product. Now, we recall the continuous mixed weak formulation introduce by Gérard et al. [4]. The mixed weak formulation of (2.1) is : find $(\sigma, u) \in H^1(\Omega)^{2\times 2} \times H^2(\Omega)$ such that

$$\left\{\begin{array}{lcl} (\sigma,\mu)_\Omega + (\nabla\cdot\mu, Du)_\Omega - < Du, \mu n >_{\partial\Omega} & = & 0,\ \forall \mu \in H^1(\Omega)^{2\times 2} \\ (\det\sigma, v) & = & (f,v)_\Omega,\ \forall v \in H^1_0(\Omega) \\ u & = & g \text{ on } \partial\Omega. \end{array}\right. \tag{2.2}$$

When we introduce the Lagrange multiplier $\lambda = u|_{\partial\Omega}$, the modified mixed weak formulation of (2.1) is : find $(\sigma, u, \lambda) \in H^1(\Omega)^{2\times 2} \times H^2(\Omega) \times H^{1/2}(\partial\Omega)$ such that

$$\left\{\begin{array}{lcl} (\sigma,\mu)_\Omega + (\nabla\cdot\mu, Du)_\Omega - \langle Du, \mu n\rangle_{\partial\Omega} & = & 0,\ \forall \mu \in H^1(\Omega)^{2\times 2} \\ (\det\sigma, v)_\Omega & = & (f,v)_\Omega,\ \forall v \in H^1_0(\Omega) \\ (\lambda,\mu)_{\partial\Omega} & = & (g,\mu)_{\partial\Omega}\ \forall \mu \in H^{-1/2}(\partial\Omega). \end{array}\right. \tag{2.3}$$

Lemma 2.1. *(ref.* [4]*) The problem* (2.2) *is well defined, and if u is a smooth solution of* (2.1)*, then (u, D^2u) solves* (2.2)*.*

Remark 2.1. In this formulation the boundary condition $u = g$ on $\partial\Omega$ viewed as constraints and imposed via Lagrange multiplier.

We end this section with some notation. Let $\mathbb{P}_k$ be the space of polynomials of total degree not larger than k. In order to avoid excessive use of constants, the abbreviations $x \lesssim y$ and $x \sim y$ stand for $x \leq cy$ and $c_1 x \leq y \leq c_2 x$, respectively, with positive constants independent of x, y or $\mathcal{T}_h$ (meshes).

2.3. **Modified discrete formulation.** Let Ω be an open convex bounded subset of $\mathbb{R}^2$ with boundary $\partial\Omega$ and let $\mathcal{T}_h$ denote a triangulation of Ω into simplices K. We denote by h_K the diameter of the element K and $h = \max_{K\in\mathcal{T}_h} h_K$. We make the assumption that the triangulation is conforming and satisfies the usual shape regularity condition, i.e. there exists a constant $\sigma > 0$ such that: $\frac{h_K}{\rho_K} \leq \sigma$, for all $K \in \mathcal{T}_h$ where ρ_K denotes the radius of the largest ball inside K. (See Figs. 1, 2,3).

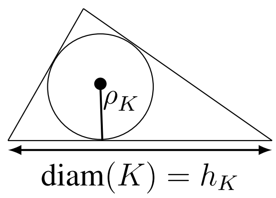

Figure 1: Isotropic element K in $\mathbb{R}^2$.

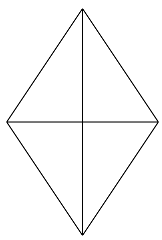

Figure 2: Example of conforming mesh in $\mathbb{R}^2$

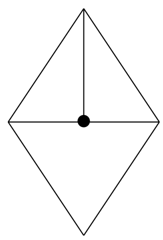

Figure 3: Example of nonconforming mesh in $\mathbb{R}^2$

To be able to use global inverse estimates, c.f. (2.2) and (2.3) of [4], we require the triangulation to be also quasi-uniform, i.e. there is a constant $C > 0$ such that $h \leq Ch_K$ for all $K \in \mathcal{T}_h$.

For any $K \in \mathcal{T}_h$, we denote by $\mathcal{E}_h(K)$ (resp. $\mathcal{N}_h(K)$) the set of its edges (resp. vertices) and set $\mathcal{E}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{E}(K), \mathcal{N}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{N}(K)$. For $\mathcal{A} \subset \overline{\Omega}$ we define:

$$\mathcal{E}_h(\mathcal{A}) = \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}.$$

Let V_h denote the standard Lagrange finite element space of degree $k \geq 3$ and $\Sigma_h = V_h^{2\times 2}$; that is we consider the following discrete spaces :

$$V_h := \{v_h \in C^0(\overline{\Omega}) : v_{|K} \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h, \quad k \geq 3\}, \tag{2.4}$$

$$\Sigma_h := \{\tau_h \in \left[C^0(\overline{\Omega})\right]^{2\times 2} : \tau_{|K} \in [\mathbb{P}_k(K)]^{2\times 2} \quad \forall K \in \mathcal{T}_h, \quad k \geq 3\} \tag{2.5}$$

The discrete formulation of (2.2) is given by : find $(u_h, \sigma_h) \in V_h \times \Sigma_h$ such that

$$\left\{ \begin{array}{lcl} (\sigma_h, \tau_h) + (\nabla \cdot \tau_h, Du_h) - \langle Du_h, \tau_h n\rangle & = & 0 \quad \forall \tau_h \in \Sigma_h \\ (det\, \sigma_h, v_h) & = & (f, v_h) \quad \forall v_h \in V_h \\ u_h & = & g_h \quad \text{on} \quad \partial\Omega, \end{array} \right. \tag{2.6}$$

We recall that $H_0^1(\Omega)$ is the subset of $H^1(\Omega)$ of elements with vanishing trace on $\partial\Omega$. Let I_h denote the standard Lagrangian interpolation operator from $C^0(\partial\Omega)$ into the space $L_h := \{v_{h|\partial\Omega} : v_h \in V_h\}$. The modified discrete formulation of (2.3) is given by find : For $\rho > 0$, we define by : $(\sigma_h, u_h, \lambda_h) \in V_h \times \Sigma_h \times L_h$ such that

$$\left\{ \begin{array}{lcl} (\sigma_h, \tau_h)_\Omega + (\nabla \cdot \tau_h, Du_h)_\Omega - \langle Du_h, \tau_h n\rangle_{\partial\Omega} & = & 0 \quad \forall \tau_h \in \Sigma_h \\ (det\, \sigma_h, v_h)_\Omega & = & (f, v_h)_\Omega \quad \forall v_h \in V_h \\ \langle \lambda_h, \mu\rangle_{\partial\Omega} & = & \langle g_h, \mu\rangle_{\partial\Omega} \quad \forall \mu \in L_h, \end{array} \right. \tag{2.7}$$

with

$$g_h = I_h g.$$

Now, we define

$$\bar{B}_h(\rho) = \left\{(\eta_h, w_h) \in \Sigma_h \times V_h, \|w_h - I_h u\|_{H^1} \leq \rho, \|\eta_h - I_h\sigma\|_{L^2} \leq h^{-1}\rho\right\}$$

and

$$B_h(\rho) = \bar{B}_h(\rho) \cap Z_h,$$

where

$$\begin{gathered} Z_h := \{(w_h, \eta_h) \in H_h \times Q_h, w_h = g_h \text{ on } \partial\Omega, \\ (\eta_h, \tau_h) + (\nabla \cdot \tau_h, Dw_h) - \langle Dw_h, \tau_h n\rangle = 0 \quad \forall \tau_h \in Q_h\}. \end{gathered}$$

By simple calculations, the problem (2.7) is logically equivalent to (2.6) and we have the result:

Lemma 2.2. *(cf.* [4]*) The problem* (2.7) *as unique solution in* $B_h(\rho)$.

Theorem 2.3. [4] *Let* $(u, \sigma) \in H^{k+1}(\Omega) \times H^k(\Omega)^{2\times 2}$ *be the convex solution of non linear problem* (2.2) *with* $k \geq 3$. *The discrete non linear problem* (2.6) *has a unique solution* (u_h, σ_h) *in* $\mathcal{B}_h(\rho) \subset V_h \times \Sigma_h$. *Moreover the following estimate holds.*

$$\|u - u_h\|_{H^1} \leq Ch^k. \tag{2.8}$$

$$\|\sigma - \sigma_h\|_{L^2} \leq Ch^{k-1}. \tag{2.9}$$

3. A-POSTERIORI ERROR ANALYSIS

In order to solve Monge-Ampère problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, first the local and global indicators are defined then the lower and upper error bounds are derived.

3.1. **Residual Error Estimators.** The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involves the data at hand.

Let $\mathbf{U} = (\sigma, u, \lambda)$ and $\mathbf{W} = (\tau, v, \mu)$. We define the operator $\mathbf{B}$ by

$$\langle \mathbf{B}(\mathbf{U}), \mathbf{W}\rangle := (\sigma, \tau)_\Omega + (\nabla \cdot \tau, Du)_\Omega - < Du, \tau n >_{\partial\Omega} - (\det \sigma, v)_\Omega + \langle \lambda, \mu\rangle_{\partial\Omega},$$

and

$$\langle \mathcal{F}, \mathbf{W}\rangle := (-f, v)_\Omega + (g, \mu)_{\partial\Omega}.$$

We also define by $\mathbf{H} = H^1(\Omega)^{2\times 2} \times H^2(\Omega) \times H^{1/2}(\partial\Omega)$ and $\mathbf{M} = H^1(\Omega)^{2\times 2} \times H^1(\Omega) \times H^{1/2}(\partial\Omega)$. Then, the continuous problem (2.3) is equivalent to : find $\mathbf{U} \in \mathbf{H}$, such that

$$\langle \mathbf{B}(\mathbf{U}), \mathbf{W}\rangle = \langle \mathcal{F}, \mathbf{W}\rangle, \quad \forall\ \mathbf{W} \in \mathbf{M} \tag{3.1}$$

We define the discrete version in the same way.

Then, let $\mathbf{U}_h = (\sigma_h, u_h, \lambda_h)$ and $\mathbf{W} = (\tau, v, \mu)$. We define by:

$$\begin{aligned} \langle \mathbf{B}(\mathbf{U}_h), \mathbf{W}\rangle &= (\sigma_h, \mu)_\Omega + (\nabla \cdot \mu, Du_h)_\Omega - < Du_h, \mu n >_{\partial\Omega} \\ &+ (\det \sigma_h, v) + \langle \lambda_h, \mu\rangle_{\partial\Omega} \end{aligned} \tag{3.2}$$

and

$$\langle \mathcal{F}, \mathbf{W}\rangle := (f, v)_\Omega + (g_h, \mu)_{\partial\Omega}. \tag{3.3}$$

We also define by $\mathbf{H}_h = V_h \times \Sigma_h \times L_h$. The discrete problem (2.7) is equivalent to : find $\mathbf{U}_h \in \mathbf{H}_h$, such that,

$$\langle \mathbf{B}(\mathbf{U}_h), \mathbf{W}\rangle = \langle \mathcal{F}, \mathbf{W}\rangle, \quad \forall\ \mathbf{W} \in \mathbf{H}_h. \tag{3.4}$$

We recall the following Lemma

Lemma 3.1. *(cf.* [4, Section 3.1]*) : **Fréchet derivate of the determinant**. For $F(v) = \det D^2 v$, we have $F'(v)(u) = (cof D^2 v) : D^2 u$.*

By using the Lemma 3.1, we deduce that, the operator B is differentiable and for all $\mathbf{U} = (\sigma, u, \lambda) \in \mathbf{H}$, $\mathbf{W} = (\tau, v, \mu) \in \mathbf{M}$ and $\mathbf{V} = (\tau', v', \mu') \in \mathbf{M}$, its differential at $\mathbf{V} = (\tau', v', \mu')$ is given by:

$$\begin{aligned} \langle D_{\mathbf{V}}\mathbf{B}(\mathbf{U}), \mathbf{W}\rangle &= (\sigma, \tau)_\Omega + (\nabla \cdot \tau, Du)_\Omega - < Du, \tau n >_{\partial\Omega} \\ &+ (cof D^2 v' : D^2 u, v) + \langle \lambda, \mu\rangle_{\partial\Omega}. \end{aligned} \tag{3.5}$$

We deduce the existence of a positive constant C_m, independent of $\xi \in \mathbf{H}$ and the continuous and discrete solutions, such that the following global inf-sup condition holds:

$$C_m \|\boldsymbol{\xi}\|_{\mathbf{H}} \le \sup_{\mathbf{W} \in \mathbf{M}\setminus\{\mathbf{0}\}} \frac{|\langle D\mathbf{B}_{\mathbf{V}}(\boldsymbol{\xi}), \mathbf{W}\rangle|}{\|\mathbf{W}\|_{\mathbf{M}}} \tag{3.6}$$

We have the following lemma:

Lemma 3.2. *There exists a positive constant C independent of the mesh sizes such that:*

$$\|\boldsymbol{U}-\boldsymbol{U}_h\|_{\boldsymbol{H}} \leq C\left(\|R\|_{\boldsymbol{H}'}+\|f-\det\sigma_h - A : D^2u_h\|_{L^2(\Omega)}\right), \tag{3.7}$$

where R is the residual functional defined by $R(\boldsymbol{W}) = [\mathcal{F}-\mathbf{B}(\mathbf{U}_h),\mathbf{W}] \quad \forall \boldsymbol{W}\in\mathbf{M}$, which satisfies:

$$R(\mathbf{W}_h) = 0 \quad \forall \mathbf{W}_h \in \boldsymbol{H}_h.$$

Proof. Let $\mathbf{U}=(\sigma,u,\lambda)$, $\mathbf{W}=(\tau,v,\mu)$ and $\mathbf{V}=(\tau',v',\mu')$, we have

$$\begin{aligned}\langle D_{\mathbf{V}}\mathbf{B}(\mathbf{U}),\mathbf{W}\rangle &= (\sigma,\tau)_\Omega+(\nabla\cdot\tau,Du)_\Omega\\ &- <Du,\tau n>_{\partial\Omega} -(cofD^2v' : D^2u,v)+\langle\lambda,\mu\rangle_{\partial\Omega}\end{aligned}$$

and

$$\begin{aligned}\langle D_{\mathbf{V}}\mathbf{B}(\mathbf{U}-\mathbf{U}_h),\mathbf{W}\rangle &= (\sigma,\tau)_\Omega+(\nabla\cdot\tau,Du)_\Omega\\ &- <Du,\tau n>_{\partial\Omega} -(cofD^2v' : D^2u,v)\\ &+ \langle\lambda,\mu\rangle_{\partial\Omega}\\ &- [(\sigma_h,\tau)_\Omega+(\nabla\cdot\tau,Du_h)_\Omega\\ &- <Du_h,\tau n>_{\partial\Omega} +(cofD^2v' : D^2u_h,v)\\ &+ \langle\lambda_h,\mu\rangle_{\partial\Omega}]\end{aligned}$$

Particularly for $\mathbf{V}=\mathbf{U}$, we have

$$\begin{aligned}\langle D_{\mathbf{V}}\mathbf{B}(\mathbf{U}-\mathbf{U}_h),\mathbf{W}\rangle &= \langle\mathcal{F},\mathbf{W}\rangle-[(\sigma_h,\tau)_\Omega\\ &+ (\nabla\cdot\tau,Du_h)_\Omega- <Du_h,\tau n>_{\partial\Omega} +(A : D^2u_h,v)\\ &+ (\det\sigma_h,v)-(\det\sigma_h,v)+(\det\sigma,v)+\langle\lambda_h,\mu\rangle_{\partial\Omega}],\end{aligned}$$

where $A := cofD^2u$. Therefore,

$$\begin{aligned}\langle D_{\mathbf{V}}\mathbf{B}(\mathbf{U}-\mathbf{U}_h),\mathbf{W}\rangle &= \langle\mathcal{F}-\mathbf{B}(\mathbf{U}_h),\mathbf{W}\rangle+(f-\det\sigma_h,v)-(A : D^2u_h,v)\\ &= \langle\mathcal{F}-\mathbf{B}(\mathbf{U}_h),\mathbf{W}\rangle+(f-\det\sigma_h-A : D^2u_h,v)\\ &= R(\mathbf{W})+(f-\det\sigma_h-A : D^2u_h,v) \qquad (3.8)\end{aligned}$$

Using Cauchy-Schwarz inequality and the inequality (3.6), the result follows. ∎

Now we define the residual equation:

$$R(\mathbf{W}) = [\mathcal{F}-\mathbf{B}(\mathbf{U}_h),\mathbf{W}], \tag{3.9}$$

hence,

$$\begin{aligned}R(\mathbf{W}) = &= (f,v)_\Omega+(g,\mu)_{\partial\Omega}-(\sigma_h,\tau)_\Omega-(\nabla\tau,Du_h)_\Omega+\langle\tau n,Du_h\rangle\\ &- (A : D^2u_h,v)_\Omega-(\lambda_h,\mu)_{\partial\Omega}\\ &= (f-A : D^2u_h,v)_\Omega-(\sigma_h,\tau)_\Omega-(\nabla\tau,Du_h)_\Omega\\ &+ \langle\tau n,Du_h\rangle_{\partial\Omega}+(g-\lambda_h,\mu)_{\partial\Omega}.\end{aligned}$$

By integrating by parts, we obtain for $\mathbf{W}=(v,\tau,\mu)\in\mathbf{M}$, the equation:

$$R(\mathbf{W}) = R_1(v)+R_2(\tau)+R_3(\mu), \tag{3.10}$$

where:

$$\begin{aligned}R_1(v) &:= (f-A : D^2u_h,v)_\Omega,\ v\in\mathcal{H} := H^1(\Omega)\\ R_2(\tau) &:= (D^2u_h-\sigma_h,\tau)_\Omega,\ \tau\in\Sigma := [H^1(\Omega)]^{2\times 2}\\ R_3(\mu) &:= (g-\lambda_h,\mu)_{\partial\Omega},\ \mu\in H^{1/2}(\partial\Omega).\end{aligned}$$

In this way, it follows that:

$$\|R\|_{\mathbf{M}'}\leq C\{\|R_1\|_{\mathcal{H}'}+\|R_2\|_{\Sigma'}+\|R_3\|_{H^{-1/2}(\partial\Omega)}\} \tag{3.11}$$

and hence our next purpose is to derive suitable upper bounds for each one of the terms on the right hand side of (3.11). We start with the following lemma, which is a direct consequence of the Cauchy-Schwarz inequality.

Lemma 3.3. *There exist C_1, C_2 and $C_3>0$, independent of the mesh sizes, such that:*

$$\|R_1\|_{\mathcal{H}'}\leq C_1\left\{\sum_{K\in\mathcal{T}_h}\left(\|f_h-A:D^2u_h\|_K^2+\|f-f_h\|_K^2\right)\right\}^{1/2} \tag{3.12}$$

and

$$\|R_2\|_{\Sigma'}\leq C_2\left\{\sum_{K\in\mathcal{T}_h}\|\sigma_h-D^2u_h\|_K^2\right\}^{1/2}. \tag{3.13}$$

In addition there holds

$$\|R_3\|_{H^{-1/2}(\Omega)}\leq C_3\left\{\sum_{E\in\mathcal{E}_h(\partial\Omega)}\|g-\lambda_h\|_E^2\right\}^{1/2}. \tag{3.14}$$

Lemma 3.4. *There exist a positive constant C, such that*

$$\|R\|_{\boldsymbol{M}'}\leq C\left\{\left(\sum_{K\in\mathcal{T}_h}\|f_h-A:D^2u_h\|_K^2\right)^{1/2}+\left(\sum_{K\in\mathcal{T}_h}\|\sigma_h-D^2u_h\|_K^2\right)^{1/2}+\left(\sum_{E\in\mathcal{E}_h(\partial\Omega)}\|g-\lambda_h\|_E^2\right)^{1/2}+\left(\sum_{K\in\mathcal{T}_h}\|f_h-f\|_K^2\right)^{1/2}\right\}. \tag{3.15}$$

Proof. By using (3.11),(3.12),(3.13),(3.14) and Cauchy-Schwarz inequality, the result follows. ∎

3.1.1. *A-posteriori error indicators.* Now, we define the error indicators:

Definition 3.1 (A-posteriori error indicators). Let $\mathrm{U}_h=(\sigma_h,u_h,\lambda_h)\in\mathbf{H}_h$ be the finite element solution. Then, the residual error estimator is locally defined by:

$$\begin{aligned}\Theta_K^2(\mathbf{U}_h) &:= \|f_h-A:D^2u_h\|_K^2+\|\sigma_h-D^2u_h\|_K^2\\ &+ \|f_h-\det\sigma_h-A:D^2u_h\|_K^2\\ &+ \sum_{E\in\mathcal{E}_h(K\cap\partial\Omega)}\|g_h-\lambda_h\|_E^2.\end{aligned} \tag{3.16}$$

The global residual error estimator is given by:

$$\Theta(\mathbf{U}_h) := \left[\sum_{K\in\mathcal{T}_h}\Theta_K^2(\mathbf{U}_h)\right]^{1/2}. \tag{3.17}$$

Furthermore, we denote the local and global approximation terms by:

$$\zeta_K^2 := \|f-f_h\|_K^2+\sum_{E\in\mathcal{E}_h(K\cap\partial\Omega)}\|g-g_h\|_E^2,$$

and

$$\zeta := \left(\sum_{K\in\mathcal{T}_h} \zeta_K^2\right)^{1/2}.$$

3.1.2. *Reliability of* Θ. The first main result is given by the following theorem.

Theorem 3.5 (**Reliability of the a-posteriori error estimator**). *Let $\boldsymbol{U} = (\sigma, u, \lambda) \in \boldsymbol{H}$ be the exact solution and $\boldsymbol{U}_h = (\sigma_h, u_h, \lambda_h) \in \boldsymbol{H}_h$ be the finite element solution. Then, there exist a positive constant C_{rel} such that:*

$$\|\mathbf{U} - \mathbf{U}_h\|_{\boldsymbol{H}} \;\leq\; C_{rel}\left[\Theta(\boldsymbol{U}_h) + \zeta\right]. \tag{3.18}$$

Proof. By using Lemma 3.2 and the estimate (3.15), the result follows. ∎

3.1.3. *Efficiency of* Θ. The second main result of this paper is the efficiency of the a posteriori error estimator Θ. In order to derive the local lower bounds, we proceed similarly as in [12, 20, 14] (see also [18]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$, and $E \in \mathcal{E}_h(K)$, we let b_K and b_E be the usual simplex-bubble and face-bubble functions respectively (see (1.5) and (1.6) in [27]). In particular, b_K satisfies $b_K \in \mathbb{P}^2(K)$, $supp(b_K) \subseteq K$, $b_K = 0$ on ∂K, and $0 \leq b_K \leq 1$ on K. Similarly, $b_E \in \mathbb{P}^2(K)$, $supp(b_E) \subseteq \omega_E := \{K' \in \mathcal{T}_h : E \in \mathcal{E}_h(K')\}$, $b_E = 0$ on $\partial K \setminus E$ and $0 \leq b_E \leq 1$ in ω_E. We also recall from [26] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \longrightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)_{|E} = p, \forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L, that is, the component wise application of L, is denoted by $\mathbf{L}$. Additional properties of b_K, b_E and L are collected in the following lemma (see [26]):

Lemma 3.6. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplexe K and $E \in \mathcal{E}_h(K)$ the following holds*

$$\| q \|_K \;\lesssim\; \| q b_K^{1/2} \|_K \lesssim \| q \|_K, \forall q \in \mathbb{P}^k(K) \tag{3.19}$$

$$|q b_K|_{1,K} \;\lesssim\; h_K^{-1} \| q \|_K, \forall q \in \mathbb{P}^k(K) \tag{3.20}$$

$$\| p \|_E \;\lesssim\; \| b_E^{1/2} p \|_E \lesssim \| p \|_E, \forall p \in \mathbb{P}^k(E) \tag{3.21}$$

$$\| L(p) \|_K + h_E |L(p)|_{1,K} \;\lesssim\; h_E^{1/2} \| p \|_E \; \forall p \in \mathbb{P}^k(E). \tag{3.22}$$

To this end, we recall some notation. We define the error respect to σ, u and λ respectively by $e_\sigma = \sigma - \sigma_h$, $e_u = u - u_h$ and $\mathbf{e}_\lambda = \lambda - \lambda_h$. Then we recall the global error defined by

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathbf{H}} := \left(\|e_\sigma\|_{1,\Omega}^2 + \|e_u\|_{2,\Omega}^2 + \|e_\lambda\|_{1/2,\partial\Omega}^2\right)^{1/2}$$

To prove local efficiency for $w \subset \Omega$, let us denote:

$$\| \mathbf{V} \|_{h,w} := \left[\sum_{K\subset\bar{w}} \left(\| v \|_{2,K}^2 + \| \tau \|_{0,K}^2 + \sum_{E\in\mathcal{E}_h(\bar{K})} \| \mu \|_{0,E}^2 \right)\right]^{1/2}, \text{ where } \mathbf{V} = (\tau, v, \mu) \in \mathbf{H}.$$

The main result of this subsection can be given as follows:

Theorem 3.7. *Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Let (σ, u, λ) be the unique solution of the continuous problem and $(\sigma_h, u_h, \lambda_h)$ be the unique solution of discrete problem. Then, the local error estimator Θ_K satisfies:*

$$\Theta_K \;\lesssim\; \|(e_\sigma, e_u, e_\lambda)\|_{h,\tilde{w}_K} + \sum_{K'\subset\tilde{w}_K} \zeta'_K \quad \forall K \in \mathcal{T}_h, \tag{3.23}$$

where $\tilde{w}_K$ is a finite union of neighbouring elements of K.

Proof. To establish the lower error bound (3.23), we will make extensive use of the original system of equations given by (2.1) and (2.3), which is recovered from the mixed formulation (2.6) by choosing suitable test functions and integrating by parts backwardly the corresponding equations. Thereby, we bound each term of the residual separately.

(1) **Residual element** $(f_h - A : D^2 u_h)$:
Let us define by $v_K = (f_h - A : D^2 u_h) b_K$ where b_K is the bubble function defined in the Section 3.1.3. We have

$$(f_h - A : D^2 u_h, v_K)_K = \int_K (f_h - A : D^2 u_h).v_K \tag{3.24}$$

Introduce f and use the modified continuous formulation (2.3) to get:

$$\begin{aligned}(f_h - A : D^2 u_h, v_K)_K &= \int_K (f - f_h).v_K - \int_K (A : D^2 u_h).v_K + \int_K (\det \sigma).v_K \\ &= \int_K (f - f_h).v_K + \int_K ((A : D^2 u) - \int_K (A : D^2 u_h)).v_K \\ &= \int_K (f - f_h).v_K + \int_K A : ((D^2 u) - (D^2 u_h)).v_K\end{aligned}$$

Using Cauchy-Schwarz inequality we obtain

$$\|(f_h - A : D^2 u_h) v_K^{\frac{1}{2}}\| \leq C((\|f - f_h\|_K \|v_K\| + \|e_u\|_{2,K} \|v_K\|_K) . \tag{3.25}$$

Using an inverses inequalities we have

$$\|f_h - A : D^2 u_h\|_K \lesssim \|e_u\|_K + \zeta_K. \tag{3.26}$$

Thus,

$$\|f_h - A : D^2 u_h\|_K \quad \lesssim \quad \|(e_\sigma, e_u, e_\lambda)\|_{h,\tilde{w}_K} + \zeta_K. \tag{3.27}$$

(2) **Residual element** $(\sigma_h - D^2 u_h)$: Let $K \in \mathcal{T}_h$. We have

$$\begin{aligned}\sigma_h - D^2 u_h &= (\sigma_h - \sigma) + (\sigma - D^2 u_h) \\ &= \sigma_h - \sigma + (D^2 u - D^2 u_h) \\ &= e_\sigma + (D^2 e_u).\end{aligned}$$

The triangular inequality leads to

$$\| \sigma_h - D^2 u_h \|_K \quad \lesssim \quad \| e_\sigma \|_K + \| e_u \|_{2,K} \ .$$

Hence,

$$\| \sigma_h - D^2 u_h \|_K \quad \lesssim \quad \|(e_\sigma, e_u, e_\lambda)\|_{h,\tilde{w}_K} + \zeta_K. \tag{3.28}$$

(3) **Residual element** $(f_h + \det \sigma_h - A : D^2 u_h)$: From the residual equation of differential form (3.8), we obtain for $\mathbf{W} = (\tau, v, \mu)$:

$$(f - \det \sigma_h - A : D^2 u_h, v) = \langle D_{\mathbf{U}} \mathbf{B}(\mathbf{U} - \mathbf{U}_h), \mathbf{W} \rangle - R(\mathbf{W})$$

and we deduce for $\mathbf{W} = (0, v, 0)$:

$$\begin{aligned}(f_h - \det \sigma_h - A : D^2 u_h, v)_K &= -(A : D^2 u, v) + (A : D^2 u_h, v)_K - R(\mathbf{W}) - (f - f_h, v)_K \\ &= -(A : D^2 (u - u_h), v)_K - (f - A : D^2 u_h, v)_K - (f - f_h, v)_K \\ &= -(A : D^2 e_u, v)_K - (f_h - A : D^2 u_h, v)_K - 2(f - f_h, v)_K\end{aligned}$$

Using Cauchy Schwarz inequality, we obtain

$$|(f_h - \det\sigma_h - A : D^2u_h, v)_K| \lesssim \big(|e_u|_{2,K} + \| f_h - A : D^2u_h \|_K + \| f - f_h \|_K\big) \| v \|_K$$

Hence, from (3.27), we have,

$$\begin{aligned}\sup_{v\in H^1(K)} \frac{|(f_h - \det\sigma_h - A : D^2u_h, v)_K|}{\| v \|_K} &\lesssim |e_u|_{2,K} + \| e_u \|_K + \zeta_K \\ &\lesssim \| e_u \|_{2,K} + \zeta_K.\end{aligned}$$

Thus,

$$\| f_h - \det\sigma_h - A : D^2u_h \|_K \lesssim \| e_u \|_{2,K} + \zeta_K. \tag{3.29}$$

(4) **Residual element** $(g_h - \lambda_h)$: For each $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h(K \cap \partial\Omega)$, we have

$$\begin{aligned} g_h - \lambda_h &= (g_h - \lambda) + (\lambda - \lambda_h) \\ &= (g_h - g) + e_\lambda \end{aligned}$$

Thus,

$$\| g_h - \lambda_h \|_E \leq \| g_h - g \|_E + \| e_\lambda \|_E$$

We sum up over all (boundary) faces $E \in \mathcal{E}_h(K \cap \partial\Omega)$ and obtain,

$$\sum_{E\in\mathcal{E}_h(K\cap\partial\Omega)} \| g_h - \lambda_h \|_E \lesssim \|(e_\sigma, e_u, e_\lambda)\|_{h,\tilde{w}_K} + \zeta_K. \tag{3.30}$$

The estimates (3.27), (3.28), (3.29) and (3.30) provide the desired local lower error bound. ∎

4. SUMMARY

In this paper, we have proposed and rigorously analyzed a new a posteriori residual type error estimators for the Monge-Ampère equation on isotropic meshes. Our investigations cover conforming discretization in $\mathbb{R}^2$. The residual type a posteriori error estimator is provided. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods. Also, we intend use the technical used by [17] where the adaptative technical was used for a nonlinear problem. Indeed [17] apply the Brezzi-Rappaz-Raviart theorem for implicitly obtain an inf-sup condition. In the same way, we will like to use this approach for Monge-Ampère problem in order to build our error indicator.

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