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SOLVING NON-AUTONOMOUS NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS USING MULTI-STAGE DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT. Differential equations are basic tools to describe a wide variety of phenomena in nature such as, electrostatics, physics, chemistry, economics, etc. In this paper, a technique is developed to solve nonlinear and linear systems of ordinary differential equations based on the standard Differential Transform Method (DTM) and Multi-stage Differential Transform Method (MsDTM). Comparative numerical results that we are obtained by MsDTM and Runge-Kutta method are proposed. The numerical results showed that the MsDTM gives more accurate approximation as compared to the Runge-Kutta numerical method for the solutions of nonlinear and linear systems of ordinary differential equations.

Key words and phrases: Non-autonomous System, Linear systems, Nonlinear systems, Ordinary differential equations, Differential Transform Method, Multi-stage Differential Transform Method.

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1. INTRODUCTION

The DTM was first presented by [1] during his researches on electrical circuits. The DTM has been successfully implemented to solve linear and nonlinear problems in physics, chemistry, economics, mathematical science, engineering etc. The DTM has also been applied on linear ordinary differential equations and nonlinear differential equations [2], [3], [4], [5]. Furthermore, it is applied to partial, fractional, and algebraic differential equations [6], [7], [8]. The extended application of the DTM is due to its distinct features. One of them, the DTM is applied directly without linearization, discretization or perturbation transform [4]. However, the DTM is still suffering from some drawbacks such as, it converges over small time intervals [6], [9], [10]. To overcome this issue, the MsDTM is utilized to enhance the convergence range where it is implemented to solve linear and nonlinear systems of one or two ordinary differential equations (ODEs) [11], [12], [9]. In this paper, the new technique is developed to find a general technique to deal with the linear and nonlinear system of three ordinary differential equations or more based on DTM and MsDTM. The new technique is applied to solve two nonlinear systems of nonlinear ODEs. The numerical results show that the new presented technique is an effective tool to find an approximate analytical solution of linear and nonlinear systems and are accurate as compared to other semi analytical and numerical method such as DTM, Adomian Decomposition Method (ADM) and Runge-Kutta Method (RK4).

2. DIFFERENTIAL TRANSFORM METHOD (DTM)

Definition 2.1. [1], [14] If a function $u\left(t\right)$ is analytical with respect to t in the domain of interest, then

$$U\left(k\right)=\frac{1}{k!}\left[\frac{d^{k}u\left(t\right)}{dt^{k}}\right]_{t=t_0} , \tag{2.1}$$

is the transformed function of $u(t)$.

Definition 2.2. [1], [14] The differential inverse transforms of the set $\left\{U(k)\right\}_{k=0}^{n}$ is defined by

$$u\left(t\right)=\sum_{k=0}^{\infty}U\left(k\right)\left(t-t_0\right)^{k} . \tag{2.2}$$

Substituting (2.1) into (2.2), we deduce that

$$u\left(t\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^{k}u\left(t\right)}{dt^{k}}\right]_{t=t_0}\left(t-t_0\right)^{\mathrm{k}}. \tag{2.3}$$

From the above definitions (2.1) and (2.2), it is easy to see that the concept of the DTM is obtained from the power series expansion. To illustrate the application of the proposed DTM to solve systems of ordinary differential equations, we consider the nonlinear system

$$\frac{du\left(t\right)}{dt}=f\left(u\left(t\right),t\right),\ \ t\geq t_0, \tag{2.4}$$

where $f(u\left(t\right),t)$ is a nonlinear smooth function.
System (2.4) is supplied with some initial conditions

$$u(t_0)=u_0. \tag{2.5}$$

The DTM establishes the solution of (2.4), which can be written as

$$u\left(t\right)=\sum_{k=0}^{\infty}U\left(k\right)\left(t-t_0\right)^{k}, \tag{2.6}$$

where $U(0), U(1), U(2), \ldots$ are unknowns which are to be determined by the DTM. Applying the DTM to the initial conditions (2.5) and (2.4) respectively, the transformed initial conditions are obtained

$$U(0) = u_0, \tag{2.7}$$

with the recursion system

$$(1+k)\, U(k+1) = F(U(0), \ldots, U(k), k), \quad k = 0, 1, 2, \ldots, \tag{2.8}$$

where $F(U(0), \ldots, U(k), k)$ is the differential of $f(u(t), t)$.
Using (2.7) and (2.8), the unknown $U(k), k = 0, 1, 2, \ldots$ can be determined. Then, the differential inverse transformation of the set of values $\{U(k)\}_{k=0}^{m}$ gives the approximate solution

$$u(t) = \sum_{k=0}^{m} U(k)\,(t - t_0)^k, \tag{2.9}$$

where m is the approximation order of the solution. Equation (2.6) gives the exact solution of problem (2.4)-(2.5).
If $U(k)$ and $V(k)$ are the differential transforms of $u(t)$ and $v(t)$ respectively, then the main operations of the DTM are shown in the Table (2.1).

TABLE 2.1. Main Operation of DTM.

Function	Differential transform
$\alpha u(t) \pm \beta v(t)$	$\alpha U(k) \pm \beta V(k)$
$u(t)\, v(t)$	$\sum_{r=0}^{k} U(r)\, V(k-r)$
$u(t)\, v(t)\, w(t)$	$\sum_{r=0}^{k} \sum_{l=0}^{r} U(l)\, V(r-l)\, W(k-r)$
$\frac{d^n}{dt^n}[u(t)]$	$(k+1)\ldots(k+n)\, U(k+n)$
$e^{\lambda t}$	$\frac{\lambda e^{\lambda t_0}}{k!}$
$\sin(\omega t)$	$\frac{w^k}{k!} \sin\left(\omega t_0 + \frac{\pi k}{2}\right)$
$\cos(\omega t)$	$\frac{w^k}{k!} \cos\left(\omega t_0 + \frac{\pi k}{2}\right)$

Applying the DTM to the initial conditions (2.5) and (2.4) to obtain a recursion system for unknowns $U(0), U(1), U(2), \ldots$, the solution series are finally obtained from DTM, but it have limited regions of convergence. Therefore, to improve the limitation of DTM, the multi-stage version of this method is applied.

3. MULTI-STAGE DIFFERENTIAL TRANSFORM METHOD (MSDTM)

The MsDTM was first introduced by [11]. The MsDTM is utilized to enhance the convergence over the interest interval. Due to the fact that the DTM failed to provide convergent approximate analytical solutions over large time intervals, the MsDTM has been introduced in [11], [12], [15] and [16]. The concept of MsDTM depends on dividing the main interval into equally sub-intervals. Suppose that the main interval is $[0, T]$. This interval is divided into equally sub-intervals $[t_{i-1} - t_i], i = 1, 2, ..., N$. The step size is $h = \frac{T}{N}$ and $t_i = ih$. The essential idea of the MsDTM is shown in the first step. By applying the DTM to Eq (2.2) over the sub-interval $[0, t_1]$, the approximate solutions are obtained as follows:
$u_1(t) = \sum_{l=0}^{K} U_l (t - t_1)^l$, where K is the order of the approximation for the power series.
The next step is applying the DTM to Eq (2.2) over the sub-intervals $[t_{i-1}, t_i]$, where

$i = 2, ..., N-1$ by using the initial conditions
$u_0^{(i)}(t) = \sum_{k=0}^{K} U_k^{(i)}(t - t_{i-1})^k$.
The approximate solutions are obtained as the follows:
$u_i(t) = \sum_{l=0}^{K} U_l(t - t_i)^l$,$i = 2, 3..., N-1$.
The second step is repeated until $i = N$ then, the approximate solution over $[0, T]$ is obtained as follows:

$$u(t) = \begin{cases} u_1(t) & 0 \le t < t_1, \\ u_2(t) & t_1 \le t < t_2, \\ . \\ . \\ . \\ u_n(t) & t_{N-1} \le t \le T. \end{cases} \tag{3.1}$$

4. SOLVING NONLINEAR AND LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Assume a system of nonlinear ODES that has the following form:

$$\begin{cases} \frac{du_1(t)}{dt} = f_1(t, u_1(t), u_2(t), ..., u_n(t)), \\ \frac{du_2(t)}{dt} = f_2(t, u_1(t), u_2(t), ..., u_n(t)), \\ . \\ . \\ . \\ \frac{du_n(t)}{dt} = f_n(t, u_1(t), u_2(t), ..., u_n(t)), \end{cases} \tag{4.1}$$

subject to the initial conditions

$$u_1(t_0) = u_1(0), u_2(t_0) = u_2(0), ..., u_n(t_0) = u_n(0). \tag{4.2}$$

Based on the definitions of DTM which are presented previously, by applying DTM to both sides of the system given in Eq (4.1) and Eq (4.2), the following is obtained:

$$\begin{cases} (k+1)U_1(k+1) = F_1(k), \\ (k+1)U_2(k+1) = F_2(k), \\ . \\ . \\ . \\ (k+1)U_n(k+1) = F_n(k). \end{cases} \tag{4.3}$$

Therefore, according to DTM the n^{th} term approximations for (4.3) can be presented as

$$\begin{cases} u_1(t) = \sum_{k=1}^{n} U_1(k)t^k \\ u_2(t) = \sum_{k=1}^{n} U_2(k)t^k \\ . \\ . \\ . \\ u_n(t) = \sum_{k=1}^{n} U_n(k)t^k. \end{cases} \tag{4.4}$$

The DTM is not valid over large intervals, hence, the MsDTM is applied. The main range $[0, T]$ is divided into N equal sub-intervals, then, DTM is applied in every sub-intervals to obtain the approximate solutions over $[0, T]$ as follows:

$$u(t) = \begin{cases} u_1(t), & 0 \leq t \leq t_1 \\ u_2(t), & t_1 \leq t \leq t_2 \\ . \\ . \\ . \\ u_n(t), & t_{N-1} \leq t \leq T. \end{cases} \tag{4.5}$$

5. NUMERICAL EXAMPLES

Example 1: Genesio nonlinear differential equation system

Consider Genesio nonlinear differential equation system [17]:

$$\begin{cases} \frac{du_1}{dt} = u_2, \\ \frac{du_2}{dt} = u_3, \\ \frac{du_3}{dt} = -a.u_3 - b.u_2 - c.u_1 + u_1^2, \end{cases} \tag{5.1}$$

where, u_1, u_2, u_3 are the state variable. The initial conditions are :

$$\begin{cases} u_1(0) = 0.2, \\ u_2(0) = 0.3, \\ u_3(0) = 0.1. \end{cases} \tag{5.2}$$

When $a = 1.2, \quad b = 2.92, \quad c = 6,$ system (5.1) gets chaotic.
By applying the differential transform on the both sides of system (5.1) the following is obtained:

$$\begin{cases} (k+1)U_1(k+1) = DT[u_2] = F_1(k), \qquad k \geq 0, \\ (k+1)U_2(k+1) = DT[u_3] = F_2(k), \\ (k+1)U_3(k+1) = DT[-1.2u_3 - 2.92u_2 - 6u_1 + u_1^2] = F_3(k). \end{cases} \tag{5.3}$$

Then, the following is obtained:

$$F(0) = \begin{bmatrix} U_{02} \\ U_{03} \\ -1.2U_{03} - 2.92U_{02} - 6U_{01} + {U_{01}}^2 \end{bmatrix},$$

$$F(1) = \begin{bmatrix} U_{12} \\ U_{13} \\ -1.2U_{13} - 2.92U_{12} - 6U_{11} + 2U_{01}U_{11} \end{bmatrix},$$

$$F(2) = \begin{bmatrix} U_{22} \\ U_{23} \\ -1.2U_{23} - 2.92U_{22} - 6U_{21} + {U_{11}}^2 + 2U_{01}U_{21} \end{bmatrix},$$

$$F(3) = \begin{bmatrix} U_{32} \\ U_{33} \\ -1.2U_{33} - 2.92U_{32} - 6U_{31} + 2U_{11}U_{21} + 2U_{01}U_{31} \end{bmatrix},$$

$$F(4) = \begin{bmatrix} U_{42} \\ U_{43} \\ -1.2U_{43} - 2.92U_{42} - 6U_{41} + {U_{21}}^2 + 2U_{11}U_{31} + 2U_{01}U_{41} \end{bmatrix}.$$

$$\vdots$$

Therefore, in a sequential pattern the following is obtained:

$$U(0) = \begin{bmatrix} 0.2 \\ -0.3 \\ 0.1 \end{bmatrix}, \qquad F(0) = \begin{bmatrix} -0.3 \\ 0.1 \\ -0.404 \end{bmatrix}, \qquad U(1) = \tfrac{F(0)}{0+1} = \begin{bmatrix} -0.3 \\ 0.1 \\ -0.404 \end{bmatrix},$$

$$F(1) = \begin{bmatrix} 0.1 \\ 2 \\ -0.404 \end{bmatrix}, \qquad U(2) = \tfrac{F(1)}{1+1} = \begin{bmatrix} 0.05 \\ -0.202 \\ 0.9364 \end{bmatrix}, \qquad F(2) = \begin{bmatrix} -0.20199999999 \\ 0.9363999999 \\ -0.72384 \end{bmatrix},$$

$$U(3) = \tfrac{F(2)}{2+1} = \begin{bmatrix} -0.06733333333 \\ 0.3121333333 \\ -0.24128 \end{bmatrix}, \qquad F(3) = \begin{bmatrix} 0.3121333333 \\ -0.24128 \\ -0.2748266665 \end{bmatrix},$$

$$U(4)=\tfrac{F(3)}{3+1}=\begin{bmatrix}0.07803333332\\-0.06032\\-0.06870666662\end{bmatrix},\qquad U(5)=\tfrac{F(4)}{4+1}=\begin{bmatrix}-0.012064\\-0.01374133332\\-0.02710085334\end{bmatrix}.$$

$$\vdots$$

Then, the approximate solution is obtained:

$$u(t)=\sum_{k=0}^{\infty}U(k)t^k=\begin{bmatrix}0.2-0.3t+0.05t^2-0.0673333333333t^3+0.07803333332t^4-0.012064t^5+\ldots\\-0.3-0.1t-0.202t^2-0.3121333333t^3+0.06032t^4-0.1374133332t^5+\ldots\\0.1-0.404t+0.9364t^2-0.24128t^3-0.068706666662t^4-0.0270085334^5+\ldots\end{bmatrix}$$

By applying the MsDTM to the system (5.1), the main interval $[0,7]$ is divided into 300 equal sub-interval. Then, applying DTM over every sub-intervals, the following approximate solutions over every equal sub-interval are obtained as follows:

$$u_0(t)=\begin{bmatrix}0.2+0.07803333332\,t^4-0.06733333333\,t^3+0.05\,t^2-0.3\,t\\-0.3-0.06032\,t^4+0.3121333333\,t^3-0.202\,t^2+0.1\,t\\0.1-0.06870666662\,t^4-0.24128\,t^3+0.9364\,t^2-0.404\,t\end{bmatrix},$$

$0\le t<0.02333333333,$

$$u_1(t)=\begin{bmatrix}0.199999999996307+0.0766068816850832\,t^4-0.0672664682657978\,t^3+0.0499984364295275\,t^2-0.299999981641494\,t\\-0.300000000007394-0.0619591151766338\,t^4+0.312210377489326\,t^3-0.202001804573596\,t^2+0.100000021300018\,t\\0.100000000032243-0.0717487413160875\,t^4-0.241139911517990\,t^3+0.936396753246393\,t^2-0.403999963248190\,t\end{bmatrix},$$

$0.02333333333\le t<0.046666666667,$

$$u_2(t)=\begin{bmatrix}0.199999997370445+0.0751413780003067\,t^4-0.0670609880639438\,t^3+0.0499872335944311\,t^2-0.299999701125753\,t\\-0.300000003077215-0.0636663605869019\,t^4+0.312449899312904\,t^3-0.202014871296198\,t^2+0.100000348744206\,t\\0.0999999951994727-0.0745447805935609\,t^4-0.240750416634325\,t^3+0.936375646132342\,t^2-0.403999438764385\,t\end{bmatrix},$$

$0.046666666667\le t<0.07,$

$$u_3(t)=\begin{bmatrix}0.199999978547386+0.0736353004110879\,t^4-0.0667092489688893\,t^3+0.0499560184208996\,t^2-0.299998454099613\,t\\-0.300000025215231-0.0654358857993379\,t^4+0.312863249668166\,t^3-0.202051561949169\,t^2+0.100001814877116\,t\\0.0999999640530241-0.0770855515088917\,t^4-0.240159588774862\,t^3+0.936323433790137\,t^2-0.403997362521667\,t\end{bmatrix},$$

$0.07\le t<0.09333333333,$

$$\vdots$$

$$u_{299}(t)=\begin{bmatrix}183.556022510499+0.130441579544786\,t^4-3.30203799243174\,t^3+30.7037722821109\,t^2-124.077801920598\,t\\-639.271069198469-0.217459449481631\,t^4+6.59033468910472\,t^3-73.4136819556440\,t^2+356.788299809543\,t\\-848.685075146366-0.508821821063691\,t^4+13.3296831569203\,t^3-128.826982675661\,t^2+544.318383715572\,t\end{bmatrix},$$

$6.976666667\le t\le 7.$

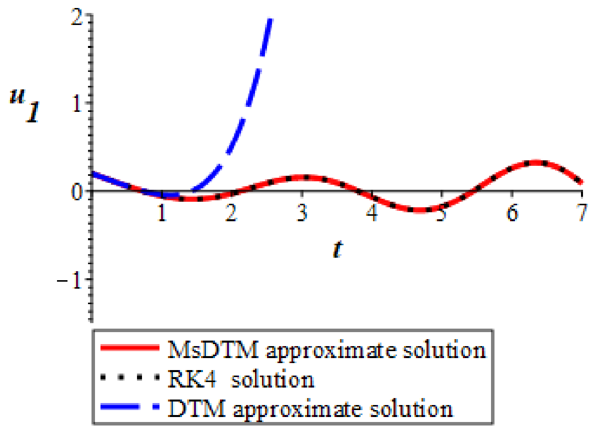

FIGURE 1. Comparison between MsDTM, DTM and RK4 Solution of Component u_1 for the System (5.1)

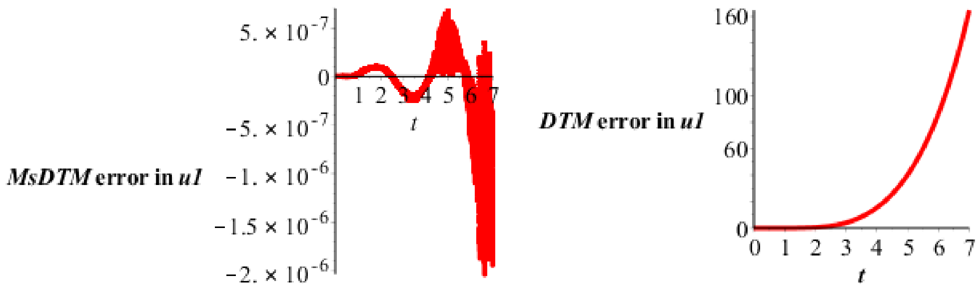

FIGURE 2. Error of Component u_1 using MsDTM and DTM for the System (5.1)

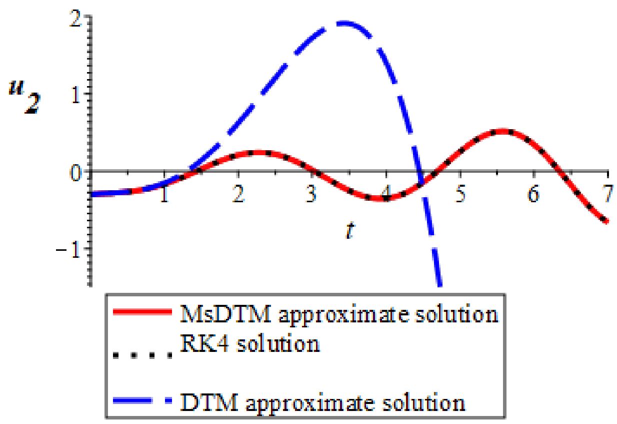

FIGURE 3. Comparison between MsDTM, DTM and RK4 Solution of Component u_2 for the System (5.1)

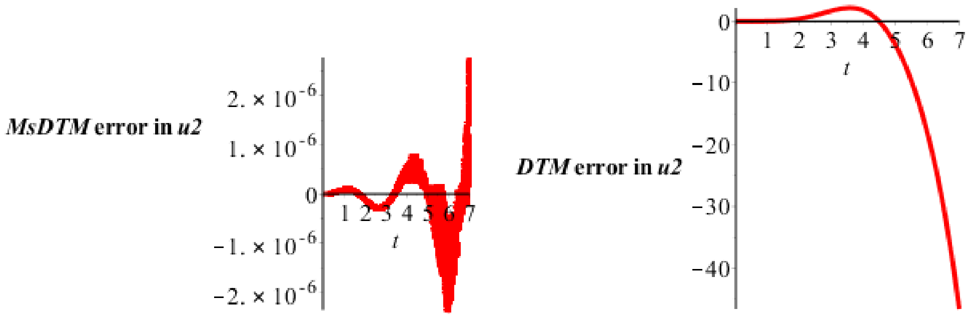

FIGURE 4. Error of Component u_2 using MsDTM and DTM for the System (5.1)

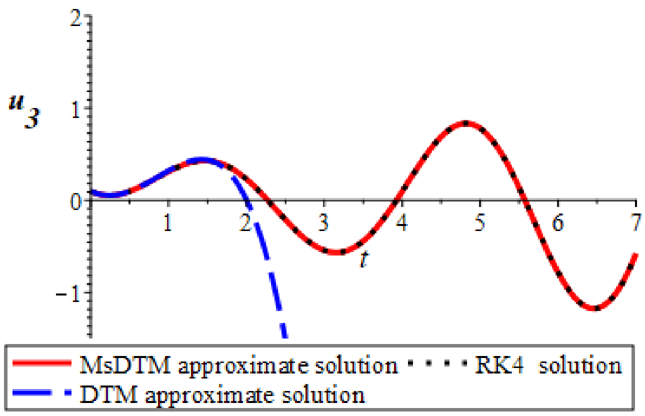

FIGURE 5. Comparison between MsDTM, DTM and RK4 Solution of Component u_3 for the System (5.1)

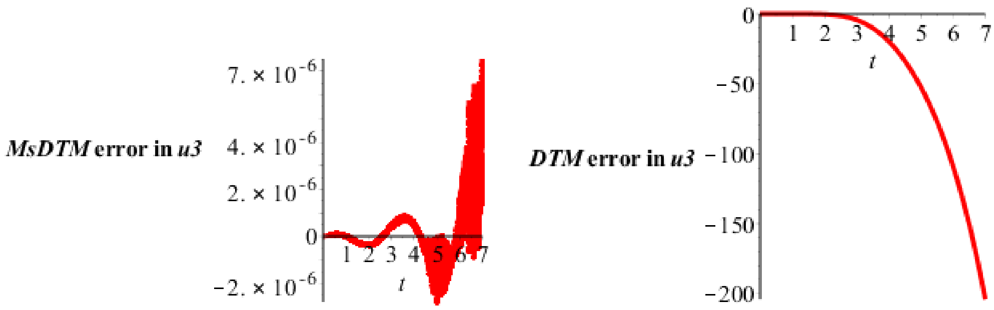

FIGURE 6. Error of Component u_3 using MsDTM and DTM for the System (5.1)

The results in Figures 1, 3 and 5 show that the approximate solution of the MsDTM is in an excellent agreement with the RK4 solution for the three components u_1, u_2 and u_3 respectively along the interest interval. Unfortunately, the approximate solution of the DTM diverges along the interest interval, certainly for $t > 1$, for all the components. It can be observed that the MsDTM rigorously converged throughout the interest interval.

Figures 2, 4 and 6 show a comparison between the MsDTM error and the DTM error for the components u_1, u_2 and u_3 respectively for the system (5.1). In this case, the exact solution is not available, hence, the MsDTM error or the DTM error is the difference between the MsDTM approximate solution or the DTM solution and the RK4 solution. It can be seen clearly that the

MsDTM error is very small compared to the DTM error for all components over the interest interval. The results in these figures indicate that the proposed method expands the domain of convergence to contain the entire interval, unlike the DTM method which is only valid over the interval $[0, 1]$. System (5.1) was also solved by the Modified DTM (MDTM) in [17], which is obtained from DTM by applying the Laplace transform and Pade' approximant. The results obtained by MDTM is not in good agreement with the MsDTM results where, the MDTM absolute error for the three components u_1, u_2 and u_3 in this system were $2\times10^{-6}, 1\times10^{-5}, 7\times10^{-5}$ respectively over the interval $[0, 0.5]$. But, the results obtained by the proposed method show that the MsDTM error for the three components u_1, u_2 and u_3 were $5\times10^{-7}, 1\times10^{-6}, 2\times10^{-6}$ respectively, over the interval $[0, 7]$. This comparison confirms that the MsDTM is a more accurate and a more reliable method than the MDTM and DTM for solving a system of nonlinear ODEs.

Example 2: A Novel Four-Scroll Chaotic System

Consider a chaotic nonlinear differential equation system [18]:

$$\begin{cases} \frac{du_1}{dt} = a(u_2 - u_1) + bu_2u_3, \\ \frac{du_2}{dt} = -10u_2^3 - u_2 + 4u_1u_3, \\ \frac{du_3}{dt} = -cu_3 - u_1u_2, \end{cases} \tag{5.4}$$

where, u_1, u_2, u_3 are state variable, as $a = 3, \quad b = 14, \quad c = 3.9,$ the system (5.4) gets chaotic. The initial conditions are :

$$\begin{cases} u_1(0) = 0.2, \\ u_2(0) = 0.4, \\ u_3(0) = 0.2. \end{cases} \tag{5.5}$$

By applying the DTM on both sides of system (5.4) the following is obtained:

$$\begin{cases} (k+1)U_1(k+1) = DT[3(u_2 - u_1) + 14u_2u_3] = F_1(k), \qquad k \geq 0, \\ (k+1)U_2(k+1) = DT[-10u_2^3 - u_2 + 4u_1u_3] = F_2(k), \\ (k+1)U_3(k+1) = DT[-3.9u_3 - u_1u_2] = F_3(k), \end{cases} \tag{5.6}$$

Subsequently, the following is obtained:

$$F(0) = \begin{bmatrix} 14U_{02}U_{03} - 3U_{01} + 3U_{02} \\ -10\,{U_{02}}^3 + 4U_{01}U_{03} - U_{02} \\ 3.9U_{03} - U_{01}U_{02} \end{bmatrix},$$

$$F(1) = \begin{bmatrix} 14U_{02}U_{13} + 14U_{12}U_{03} - 3U_{11} + 3\,U_{12} \\ -30{U_{02}}^2U_{12} + 4U_{01}U_{13} + 4U_{11}U_{03} - U_{12} \\ 3.9U_{13} - U_{11}U_{02} - U_{01}U_{12} \end{bmatrix},$$

$$F(2) = \begin{bmatrix} 14U_{02}U_{23} + 14U_{22}U_{03} + 14U_{12}U_{13} - 3U_{21} + 3U_{22} \\ -30{U_{02}}^2U_{22} - 30U_{02}{U_{12}}^2 + 4U_{01}U_{23} + 4U_{21}U_{23} + 4U_{11}U_{13} - U_{22} \\ 3.9U_{23} - U_{21}U_{02} - U_{11}U_{12} - U_{01}U_{22} \end{bmatrix},$$

$$F(3) = \begin{bmatrix} 14U_{02}U_{33} + 14U_{32}U_3 + 14U_{12}U_{23} + 14U_{22}U_{13} - 3U_{31} + 3U_{32} \\ -30{U_{02}}^2U_{32} - 60U_{02}U_{12}U_{22} - 10{U_{12}}^3 + 4U_{01}U_{33} + 4U_{31}U_{03} + 4U_{11}U_{23} + 4U_{21}U_{13} - U_{32} \\ 3.9U_{33} - U_{31}U_2 - U_{21}U_{12} - U_{11}U_{22} - U_1U_{32} \end{bmatrix},$$

$$F(4)=\begin{bmatrix} 14U_{02}U_{43}+14U_{42}U_{03}+14U_{12}U_{33}+14U_{32}U_{13}+14U_{22}U_{23}-3U_{41}+3U_{42} \\ -30{U_{02}}^2U_{42}-60U_{02}U_{12}U_{32}-30U_{02}{U_{22}}^2-30{U_{12}}^2U_{22}+4U_{01}U_{43}+4U_{41}U_{03}+4\,U_{11}U_{33}+4U_{31}U_{13}+4U_{21}U_{23}-U_{42} \\ 3.9U_{43}-U_{41}U_2-U_{31}U_{12}-U_{21}U_{22}-U_{11}U_{32}-U_{01}U_{42} \end{bmatrix}.$$

Therefore, in a sequential pattern the following is obtained:

$$U(0)=\begin{bmatrix} 0.2 \\ -0.3 \\ 0.1 \end{bmatrix},\qquad F(0)=\begin{bmatrix} 0.2 \\ -0.3 \\ 0.1 \end{bmatrix},\qquad U(1)=\frac{F(0)}{0+1}=\begin{bmatrix} -1.92 \\ 0.650 \\ 0.45 \end{bmatrix},$$

$$F(1)=\begin{bmatrix} 6.73000000000000042 \\ -2.81300000000000017 \\ 1.04899999999999993 \end{bmatrix},\qquad U(2)=\frac{F(1)}{1+1}=\begin{bmatrix} 3.365 \\ -1.4065 \\ 0.5245 \end{bmatrix},$$

$$F(2)=\begin{bmatrix} -14.3915000010000007 \\ 7.31615000100000046 \\ 4.58435000099999978 \end{bmatrix},\qquad U(3)=\frac{F(2)}{2+1}=\begin{bmatrix} -4.797166667 \\ 2.438716667 \\ 1.528116667 \end{bmatrix},$$

$$F(3)=\begin{bmatrix} 14.6157633279999999 \\ -26.8930850079999999 \\ -0.854968332400000030 \end{bmatrix},\qquad U(4)=\frac{F(3)}{3+1}=\begin{bmatrix} 3.653940832 \\ -6.723271252 \\ -0.2137420831 \end{bmatrix},$$

$$F(4)=\begin{bmatrix} -20.7046520899999997 \\ 77.0201730499999968 \\ 14.1406092099999992 \end{bmatrix},\qquad U(5)=\frac{F(4)}{4+1}=\begin{bmatrix} -4.140930418 \\ 15.40403461 \\ 2.828121842 \end{bmatrix}.$$

Then, the approximate solution is obtained:

$$u(t)=\sum_{k=0}^{\infty}U(k)t^k=\begin{bmatrix} 0.2-1.92t+3.365t^2-4.797166667t^3+3.653940832t^4-4.140930418t^5+... \\ -0.3-0.65t-1.4065t^2+2.438716667t^3-6.723271252t^4+15.40403461t^5+... \\ 0.1+0.45t+0.5245t^2+1.528116667t^3-0.2137420831t^4+2.828121842^5+... \end{bmatrix}.$$

By applying the MsDTM to the system (5.4), the main interval $[0,3]$ is divided into 300 equal sub-intervals. Then, applying DTM over every sub-interval, the following approximate solutions over every equal sub-intervals are obtained:

$$u_0(t)=\begin{bmatrix} 0.2-11.51223334\,t^4+9.1728\,t^3-3.172\,t^2+1.72\,t \\ 0.4+35.06703332\,t^4-8.847733333\,t^3+3.52\,t^2-0.88\,t \\ 0.2-0.6051958332\,t^4+2.1345\,t^3+1.109\,t^2+0.7\,t \end{bmatrix},$$

$0\leq t<0.01,$

$$u_1(t)=\begin{bmatrix} 0.199999998944319-9.85563570242039\,t^4+9.14034348484780\,t^3-3.17167900725179\,t^2+1.71999848469451\,t \\ 0.400000002323328+29.6536368018110\,t^4-8.74347566115410\,t^3+3.51897731056318\,t^2-0.879995017809122\,t \\ 0.199999999973078-0.331094876221447\,t^4+2.12916461095784\,t^3+1.10905260997900\,t^2+0.699999736030373\,t \end{bmatrix},$$

$0\leq t<0.01,$

$$u_2(t)=\begin{bmatrix} 0.200000084141058-8.37973113102058\,t^4+9.05232396181441\,t^3-3.16963696155721\,t^2+1.71997678451631\,t \\ 0.399999754336612+25.3044758359338\,t^4-8.48564798392580\,t^3+3.51302885326647\,t^2-0.879931987110542\,t \\ 0.200000013626711-0.0968173633937595\,t^4+2.11522786356339\,t^3+1.10937519384165\,t^2+0.699996302150300\,t \end{bmatrix},$$

$0.02 \leq t < 0.03,$

$$u_3(t) = \begin{bmatrix} 0.200000635490251 - 7.04669761585337\,t^4 + 8.91944253265202\,t^3 - 3.16460320561409\,t^2 + 1.71989099491291\,t \\ 0.399998318456393 + 21.7879465152591\,t^4 - 8.13645093460079\,t^3 + 3.49985014298699\,t^2 - 0.879708094180930\,t \\ 0.200000096714031 + 0.104687243838486\,t^4 + 2.09517677404955\,t^3 + 1.11013345088378\,t^2 + 0.699983390217885\,t \end{bmatrix},$$

$0.03 \leq t < 0.04,$

$$\vdots$$

$$u_{299}(t) = \begin{bmatrix} 515.558374219336 + 3.91683354980570\,t^4 - 51.9557323424376\,t^3 + 262.150658420283\,t^2 - 596.228835868634\,t \\ -96.4882700648167 - 0.841385314803319\,t^4 + 10.7724485134852\,t^3 - 52.3394069006767\,t^2 + 114.889350009676\,t \\ -17.7593398054183 + 0.728853480642598\,t^4 - 5.54176231131966\,t^3 + 11.9051373771224\,t^2 + 0.476251600743609\,t \end{bmatrix},$$

$2.99 \leq t \leq 3.$

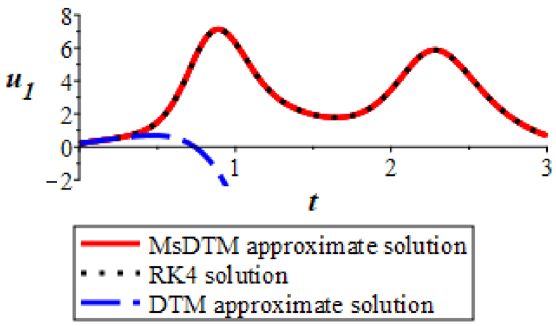

FIGURE 7. Comparison between MsDTM, DTM and RK4 Solution of Component u_1 for the System (5.4)

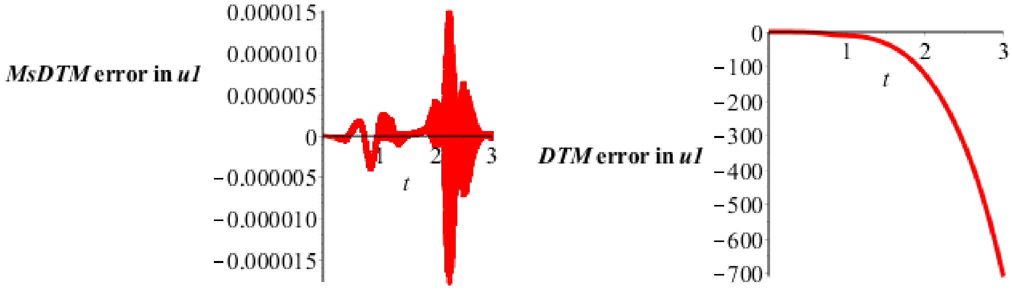

FIGURE 8. Error of Component u_1 using MsDTM and DTM for the System (5.4)

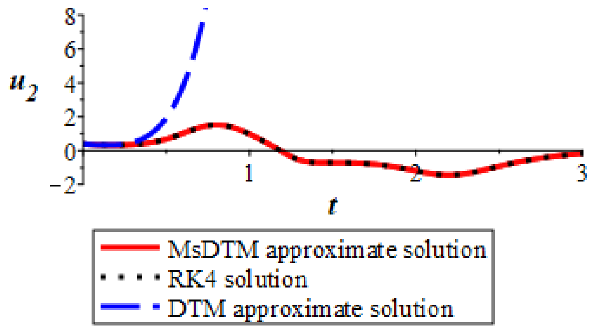

FIGURE 9. Comparison between MsDTM, DTM and RK4 Solution of Component u_2 for the System (5.4)

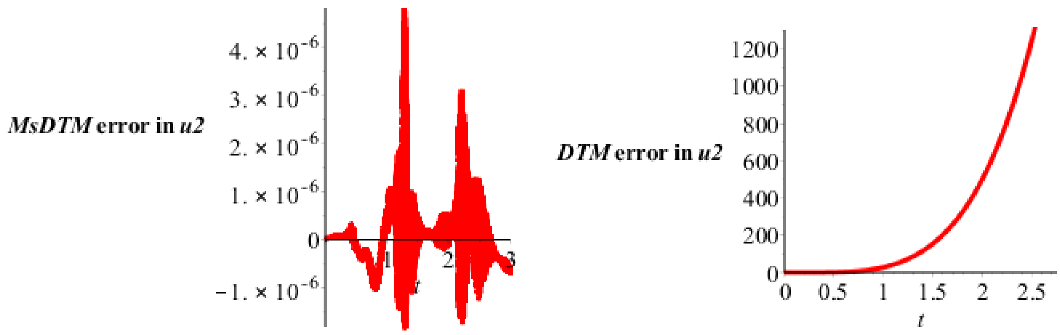

FIGURE 10. Error of Component u_2 using MsDTM and DTM for the System (5.4)

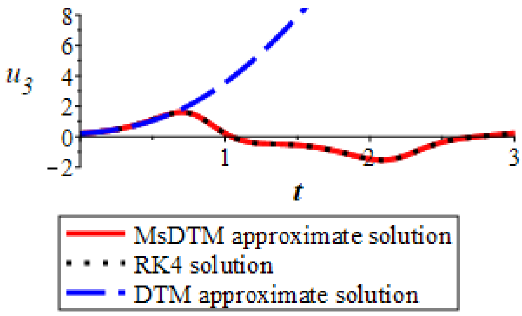

FIGURE 11. Comparison between MsDTM, DTM and RK4 Solution of Component u_3 for the System (5.4)

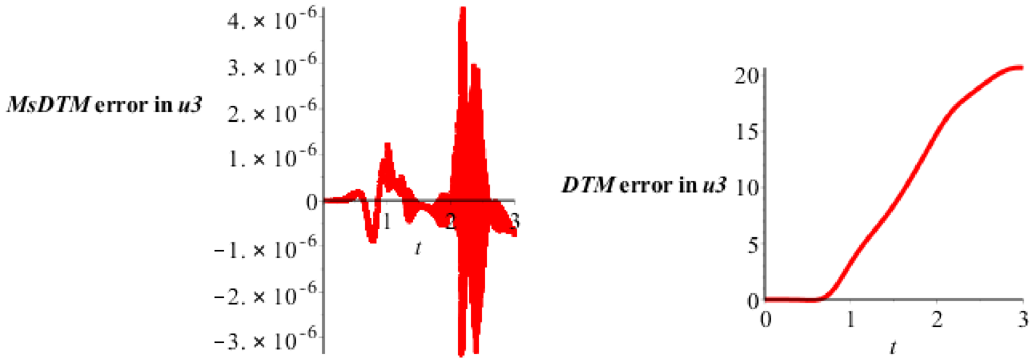

FIGURE 12. Error of Component u_3 using MsDTM and DTM for the System (5.4)

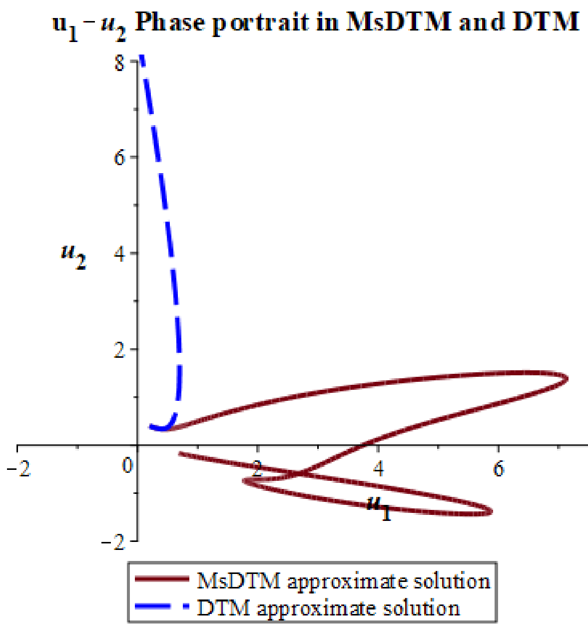

FIGURE 13. $u_1 - u_2$ Phase Portrait using MsDTM and DTM for the System (5.4)

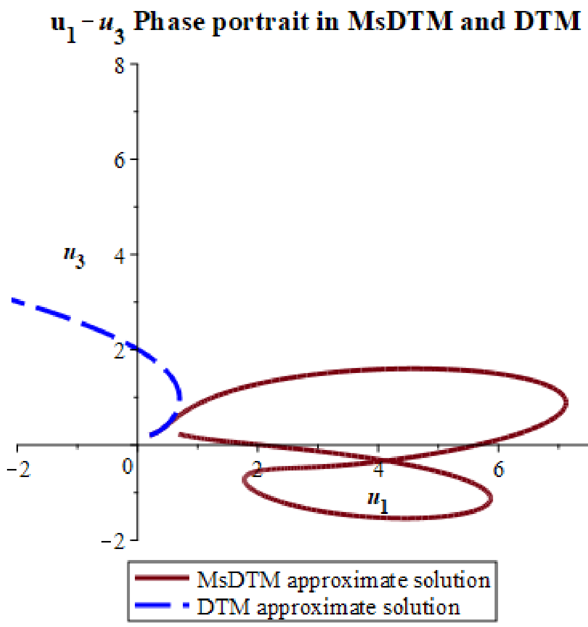

FIGURE 14. $u_1 - u_3$ Phase Portrait using MsDTM and DTM for the System (5.4)

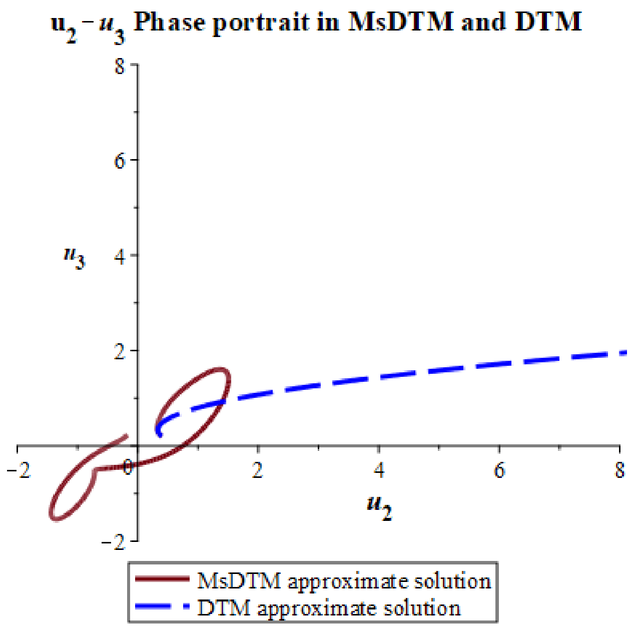

FIGURE 15. $u_2 - u_3$ Phase Portrait using MsDTM and DTM for the System (5.4)

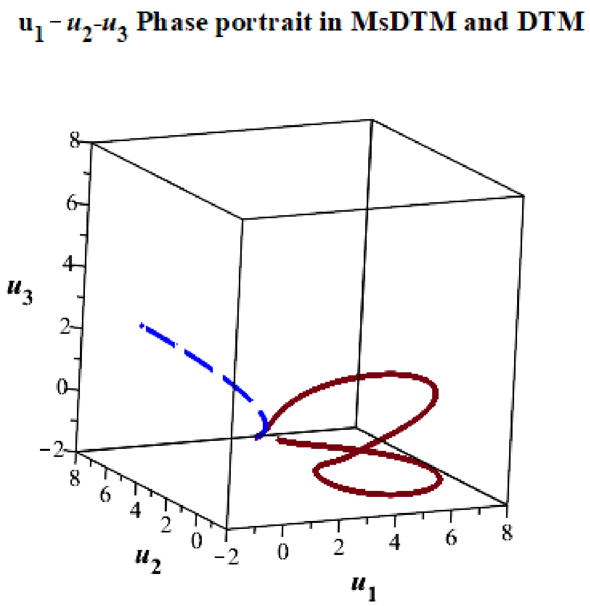

FIGURE 16. $u_1 - u_2 - u_3$ Phase Portrait using MsDTM and DTM for the System (5.4)

Figures 7, 9 and 11, show that the curves of the MsDTM approximate solution, the DTM approximate solution and the RK4 solution for the three components, u_1, u_2 and u_3 respectively for system (5.4), where $t \in [0, 3]$ is the time interval, $N = 300$ is the number of subintervals, $K = 4$ is the order of the approximation and $h = 0.01$ is the time step. The results in these figures show that the MsDTM approximate solution is in excellent agreement with the RK4 solution for the three components, u_1, u_2 and u_3 respectively along the interest interval. On the other hand, the DTM approximate solution diverges along the interest interval for $t > 0.4$, $t > 0.4$, and $t > 0.5$ respectively. It is clear that the MsDTM is an effective method to enlarge the domain of convergence for a system of nonlinear ODEs. The error of the MsDTM and the error of the DTM are plotted in Figures 8, 10 and 12 for the three components u_1, u_2 and u_3 respectively. It can be observed that the MsDTM error is very small compared to the DTM error for all components. The results in these figures indicate the MsDTM is more accurate and more reliable.

Figures 13, 14 and 15, show that the phase portrait of the MsDTM approximate solution and the phase portrait of the DTM approximate solution in 2-D views with respect to $u_1 - u_2$, $u_1 - u_3$ and $u_2 - u_3$ respectively. The results confirm that the MsDTM is a more accurate and a more powerful device for solving a system of several nonlinear ODEs. Figure 16 shows a comparison between the MsDTM approximate solution and the DTM approximate solution in 3-D views. It can be seen easily the clear diverge between the MsDTM approximate solution and the DTM approximate solution in 3-D views. This confirms the MsDTM is a more accurate and a more efficient. System (5.4) was also solved using the RK4 method [18]. The results obtained by RK4 were analyzed and its fundamental properties like dissipativity, symmetry and invariance, equilibria, Lyapunov exponents and Kaplan-Yorke dimension were examined. Similarly, the proposed method gives the same phase portraits in 2-D views and 3-D views as shown Figures 13, 14, 15 and 16. This indicates the proposed method is in good agreement with the RK4 method.

6. CONCLUSION

In this paper, a new proposed technique, the MsDTM, is applied to solve nonlinear systems of ODES. Comparison between DTM and MsDTM shows that MsDTM can solve nonlinear systems of ODEs more accurately without linearization, discretization or perturbation transform. The analytical approximate solutions obtained by MsDTM are valid over larger time intervals than the standard DTM.

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