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ON THE DEGREE OF APPROXIMATION OF PERIODIC FUNCTIONS FROM LIPSCHITZ AND THOSE FROM GENERALIZED LIPSCHITZ CLASSES

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ABSTRACT. In this paper we have introduced some new trigonometric polynomials. Using these polynomials, we have proved some theorems which determine the degree of approximation of periodic functions by a product of two special means of their Fourier series and the conjugate Fourier series. Many results proved previously by others are special case of ours.

Key words and phrases: Fourier series, Trigonometric polynomials, Degree of approximation, Lipschitz class.

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1. INTRODUCTION

Let $\sum_{n=0}^{\infty} w_n$ and infinite numerical series with sequence (s_n) of its nth partial sums. This series is said to be $(C,1) := C^1$-summable if the limit

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

exists.

Moreover, this series is said to be $(E,1)$-summable if the limit

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k$$

exists.

It was realized in [9], that the infinite series

$$1 - 4\sum_{n=1}^{\infty}(-3)^{n-1}$$

is not $(E,1)$ summable nor $(C,1)$ summable. However, it is showed that the above series is $(C,1)(E,1)$ summable. Therefore the product summability $(C,1)(E,1)$ is more powerful than the individual methods $(C,1)$ and $(E,1)$. Thus, $(C,1)(E,1)$ mean can be used in approximation for a wider class of 2π-periodic functions f by their Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

than the individual methods $(C,1)$ and $(E,1)$.

This fact can be enough to consider the finding of the degree of approximation of periodic functions by more general mean of their Fourier series. To recall the results of [7], dealing with this topic, we need first some preliminaries.

The L^p-norm of f is defined by

$$\|f\|_p = \begin{cases} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right]^{\frac{1}{p}}, & 1 \le p < \infty \\ \sup_{x\in[0,2\pi]} |f(x)|, & p = \infty, \end{cases}$$

while the best approximation $E_n(f)$ of the function f, is defined by

$$E_n(f) = \min_{T_n} \|f(x) - T_n(x)\|_p,$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

For a given function $f \in L^p := L^p[0,2\pi]$, $p \ge 1$, i.e. $\|f\|_p < +\infty$, let

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \tag{1.1}$$

denote the partial sums of the Fourier series of f at x.

Let $p := (p_n)$ be a sequence of non-negative and non-increasing real numbers such that

$$P_n = p_0 + p_1 + \cdots + p_n \neq 0 \quad \text{for} \quad n \ge 0,$$

and $P_n \to \infty$ as $n \to \infty$.

The product of $(C,1)$ summability with a N_p summability defines C^1N_p summability. Whence, C^1N_p mean is defined by

$$t_n^{CN}=\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}s_k. \tag{1.2}$$

If $t_n^{CN}\to w$ as $n\to\infty$, then the series $\sum_{n=0}^{\infty}w_n$ or the sequence (s_n) is said to be summable to the sum w by C^1N_p method.

Remark 1.1. Note that, see [7], the C^1N_p method is regular whenever C^1 and N_p are regular methods.

We write $u=\mathcal{O}(v)$ if there exists a positive constant K, such that $u\le Kv$, and we assume that all transformations under consideration are regular transformations even if they are not written explicitly.

Now we recall four well-known function classes:

1^0 A function $f\in \text{Lip}(\alpha)$, $(0<\alpha\le 1)$, if

$$|f(x+t)-f(x)|=\mathcal{O}(|t|^{\alpha}),$$

2^0 A function $f\in \text{Lip}(\alpha,p)$, $(0<\alpha\le 1)$, if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}}=\mathcal{O}(|t|^{\alpha}),$$

3^0 A function $f\in \text{Lip}(\xi(t),p)$, if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}}=\mathcal{O}(\xi(t)),$$

and

4^0 A function $f\in \text{W}(L^p,\xi(t))$, if

$$\left(\int_0^{2\pi}\left|[f(x+t)-f(x)]\sin^{\beta}\frac{x}{2}\right|^p dx\right)^{\frac{1}{p}}=\mathcal{O}(\xi(t)),$$

where $\beta\ge 0$, $p\ge 1$, and $\xi(t)$ is a positive increasing function of t.

We also denote

$$\phi_x(t):=\phi(t):=f(x+t)+f(x-t)-2f(x),$$

and τ instead of the integral part of π/t.

Theorem 1.1 ([7]). *Let N_p be a regular Nörlund method defined by a sequence (p_n) such that*

$$P_\tau\sum_{v=\tau}^{n}P_v^{-1}=\mathcal{O}(n+1). \tag{1.3}$$

Let $f\in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\le 1)$, then the degree of approximation of f by C^1N_p means of its Fourier series is given by

$$\|t_n^{CN}(f)-f\|_\infty=\begin{cases}\mathcal{O}\left((n+1)^{-\alpha}\right), & 0<\alpha<1;\\ \mathcal{O}\left(\frac{\log(\pi e(n+1))}{n+1}\right), & \alpha=1.\end{cases}$$

Theorem 1.2 ([7]). *If Let $f \in L^1[0,2\pi]$ be a 2π-periodic function and $f \in W(L^p, \xi(t))$, then the degree of approximation of f by C^1N_p means of its Fourier series is given by*

$$\|t_n^{CN}(f) - f\|_p = \mathcal{O}\left((n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),$$

provided $\xi(t)$ satisfies the following conditions:

(i) $\xi(t)/t$ *is a decreasing function,*

(ii) $\left[\int_0^{\frac{\pi}{n+1}} \left(\frac{t|\phi(t)\sin^\beta t|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = \mathcal{O}\left((n+1)^{-1}\right)$,

(iii) $\left[\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = \mathcal{O}\left((n+1)^{\delta}\right)$,

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $p^{-1}+s^{-1}=1$, $p \geq 1$, and these conditions hold true uniformly in x.

Remark 1.2. As is pointed out in [8]-[12], the factor $\sin^\beta t$ in definition of the class $\mathrm{W}(L^p, \xi(t))$ as well as in some conditions have to be replaced by $\sin^\beta \frac{t}{2}$.

For our further investigation let $a := (a_n)$ and $b := (b_n)$ be sequences of non-negative integers with conditions

$$a_n < b_n, \quad n = 1, 2, \dots \tag{1.4}$$

and

$$\lim_{n\to\infty} b_n = +\infty. \tag{1.5}$$

The deferred Cesàro mean (see [1]) determined by a and b is defined as

$$D_a^b := \frac{S_{a_n+1} + S_{a_n+2} + \cdots + S_{b_n}}{b_n - a_n}$$

where (S_m) is a sequence of real or complex numbers.

Since each D_a^b with conditions (1.4) and (1.5) satisfies the Silverman-Toeplitz conditions, then each transformation D_a^b is regular.

Let us suppose that $\mathbb{F}$ is a subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of a strictly increasing sequence of positive integers, say $\mathbb{F} = (\lambda(n))_1^\infty$.

Let $\{p_k\}$ and $\{q_k\}$, $k = 0, 1, \dots, \lambda(n)$ be two sequences of non-negative numbers with $P_{\lambda(n)} := \sum_{k=0}^{\lambda(n)} p_k \neq 0$, $Q_{\lambda(n)} := \sum_{k=0}^{\lambda(n)} q_k \neq 0$, and we define their convolution by

$$W_{\lambda(n)} := \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} q_k.$$

Now we define the generalized Woronoi-Nörlund polynomial as follows

$$W_n^\lambda(f;x) := \frac{1}{W_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} q_k s_k(f;x). \tag{1.6}$$

Note that for $q_k = 1$, $k = 0, 1, \dots, \lambda(n)$, we obtain the polynomials

$$N_n^\lambda(f;x) = \frac{1}{N_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} s_k(f;x),$$

and for $p_k = 1$, $k = 0, 1, \dots, \lambda(n)$, the polynomials

$$R_n^\lambda(f;x) = \frac{1}{R_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} q_k s_k(f;x),$$

which has been introduced in [5].

Moreover, for $p_k = q_k = 1$, $k = 0, 1, \dots, \lambda(n)$, we obtain the polynomials (see [2], page 195)

$$C_n^\lambda(f;x) = \frac{1}{\lambda(n)+1} \sum_{k=0}^{\lambda(n)} s_k(f;x),$$

which for $\lambda(n) = n$, as particular case, they reduce to ordinary Cesàro mean.

If $W_n^\lambda \to w_1$ as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} w_n$ or the sequence (s_n) is said to be summable to the sum w_1 by W^λ method.

Since the sequence

$$\left(\frac{p_{\lambda(n)-k}q_k}{W_{\lambda(n)}}\right), \quad k = 0, 1, \dots, \lambda(n),$$

is a sub-sequence of the sequence

$$\left(\frac{p_{n-k}q_k}{W_n}\right), \quad k = 0, 1, \dots, n,$$

introduced in [3], page 353, then W^λ method is regular if and only if:

(a) $\sum_{k=0}^{\lambda(n)} |p_{\lambda(n)-k}q_k| < K|W_{\lambda(n)}|$, where K is a positive number independent of n;
(b) For all $k \geq 0$, $\frac{p_{\lambda(n)-k}q_k}{W_{\lambda(n)}} \to 0$ as $\lambda(n) \to \infty$.

The product of D_a^b summability with a W^λ summability defines $D_a^b W^\lambda$ summability. Whence, $D_a^b W^\lambda$ mean is defined by

$$t_n^{DW} = \frac{1}{b_n - a_n} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v s_v. \tag{1.7}$$

If $t_n^{DW} \to w_2$ as $n \to \infty$, then we say the series $\sum_{n=0}^{\infty} w_n$ or the sequence (s_n) is said to be summable to the sum w_2 by $D_a^b W^\lambda$ method.

Under conditions (1.4), (1.5), (a) and (b) we conclude that:

$$\begin{aligned} s_n \to w_2 \implies W^\lambda(s_n) &= \frac{1}{W_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} p_{\lambda(n)-k} q_k s_k \to w_2,\ n \to \infty, \\ \implies D_a^b(W^\lambda(s_n)) &:= t_n^{DW}(s_n) \to w_2,\ n \to \infty, \end{aligned}$$

which means that the method $D_a^b W^\lambda$ is a regular one.

It is the purpose of this paper to determine the degree of approximation of the functions f and $\widetilde{f}$ by $D_a^b W^\lambda$ means of their Fourier series and conjugate series of the Fourier series (see section 3.1.), respectively. Later on, we will see that our results cover a lot of results obtained previously by others and they are expressed in terms of b_n (which give better degree of approximation) instead of n. More results on this topic, the interested reader, can find in [4] and the references therein.

To achieve this purpose we need first to prove some helpful lemmas given in next section.

2. AUXILIARY LEMMAS

At first, we denote

$$\mathbb{F}_n^{DW}(t) := \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}.$$

Lemma 2.1. *For* $0 < t \leq \frac{\pi}{b_n+1}$*, the inequality*

$$|\mathbb{F}_n^{DW}(t)| = \mathcal{O}\left(b_n + 1\right).$$

Proof. Applying the inequalities $\sin(\beta m) \leq \beta m$ and $\sin\frac{t}{2} \geq \frac{t}{\pi}$ for $0 < t \leq \frac{\pi}{b_n+1}$, we obtain

$$\begin{aligned}
|\mathbb{F}_n^{DW}(t)| &= \frac{1}{2\pi(b_n - a_n)} \left| \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right| \\
&\leq \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \left| \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right| \\
&\leq \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \left| \frac{\left(v+\frac{1}{2}\right)t}{\frac{t}{\pi}} \right| \\
&= \frac{1}{2(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \left(v+\frac{1}{2}\right) \\
&\leq \frac{1}{4(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \left(2\lambda(k)+1\right) \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \\
&= \frac{1}{4(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \left(2\lambda(k)+1\right) \\
&\leq \frac{1}{4(b_n - a_n)} \left(2b_n+1\right)\left(b_n - a_n\right) \\
&= \mathcal{O}\left(b_n+1\right).
\end{aligned}$$

The proof is completed. ∎

Lemma 2.2. *For* $\frac{\pi}{b_n+1} < t \leq \pi$*, the inequality*

$$|\mathbb{F}_n^{DW}(t)| = \mathcal{O}\left(\frac{1}{t^2(b_n - a_n)} + \frac{1}{t}\right).$$

Proof. We divide the kernel $\mathbb{F}_n^{DW}(t)$ as follows

$$\mathbb{F}_n^{DW}(t) = \frac{1}{2\pi(b_n - a_n)} \left[\sum_{\lambda(k)=a_n+1}^{\tau} + \sum_{\lambda(k)=\tau+1}^{b_n} \right] \times \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}} := \mathbb{I}_1 + \mathbb{I}_2, \tag{2.1}$$

where τ denote the integer part of π/t

Based on well-known inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$ for $0 < t \leq \pi$, we have

$$|\mathbb{I}_1| = \mathcal{O}\left(\frac{1}{t(b_n - a_n)}\right) \sum_{\lambda(k)=a_n+1}^{\tau} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v = \mathcal{O}\left(\frac{\tau}{t(b_n - a_n)}\right) = \mathcal{O}\left(\frac{1}{t^2(b_n - a_n)}\right). \tag{2.2}$$

Now we need to estimate $|\mathbb{I}_1|$. Indeed, once again, using the inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$ for $0 < t \leq \pi$, we obtain

$$|\mathbb{I}_2| = \mathcal{O}\left(\frac{1}{t(b_n - a_n)}\right) \sum_{\lambda(k)=\tau+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v = \mathcal{O}\left(\frac{b_n - \tau}{t(b_n - a_n)}\right) = \mathcal{O}\left(\frac{1}{t}\right). \tag{2.3}$$

Relation (2.1) along with (2.2) and (2.3) imply the estimation of $|\mathbb{F}_n^{DW}(t)|$ as required.
The proof is completed. ∎

In the sequel, we denote

$$\psi_x(t) := \psi(t) := f(x+t) + f(x-t),$$

and

$$\widetilde{\mathbb{F}}_n^{DW}(t) := \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}.$$

Lemma 2.3. *For $0 < t \leq \frac{\pi}{b_n+1}$, the inequality*

$$|\widetilde{\mathbb{F}}_n^{DW}(t)| = \mathcal{O}\left(\frac{1}{t}\right).$$

Proof. Using the inequalities $|\cos\beta| \leq 1$ and $\sin\frac{t}{2} \geq \frac{t}{\pi}$ for $0 < t \leq \frac{\pi}{b_n+1}$, we obtain

$$\begin{aligned}
|\widetilde{\mathbb{F}}_n^{DW}(t)| &\leq \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \left|\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \\
&\leq \frac{1}{2\pi(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} \frac{1}{W_{\lambda(k)}} \sum_{v=0}^{\lambda(k)} p_{\lambda(n)-v} q_v \left|\frac{\pi}{t}\right| \\
&= \frac{1}{2t(b_n - a_n)} \sum_{\lambda(k)=a_n+1}^{b_n} 1 = \mathcal{O}\left(\frac{1}{t}\right).
\end{aligned}$$

The proof is completed. ∎

Lemma 2.4. *For $\frac{\pi}{b_n+1} < t \leq \pi$, the inequality*

$$\widetilde{\mathbb{F}}_n^{DW}(t)| = \mathcal{O}\left(\frac{1}{t^2(b_n - a_n)} + \frac{1}{t}\right).$$

Proof. The proof can be done by same arguments as the proof of Lemma 2.2. That is why we omit the details.
The proof is completed. ∎

3. MAIN RESULTS

This section is separated into two subsections. The first one deals with Fourier series, while the second one, with conjugate series of the Fourier series.

3.1. **The approximation of functions by Fourier series.** In this section we are going to prove the analogues of Theorem 1.1-1.2 using $D_a^bW^\lambda$ means of partial sums of the Fourier series of the function f. We begin with the first theorem.

Theorem 3.1. *Let $f \in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\leq 1)$, then the degree of approximation of f by $D_a^bW^\lambda$ means of its Fourier series is given by*

$$\|t_n^{DW}(f)-f\|_\infty = \begin{cases} \mathcal{O}\Big((b_n+1)^{-\alpha}\Big), & 0<\alpha<1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha=1. \end{cases}$$

Proof. It has been showed in [14] that

$$s_v(f;x)=\frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt,$$

then denoting $D_a^bW^\lambda$ mean of $s_v(f;x)$ by $t_n^{DW}(f;x):=t_n^{DW}(x)$, we have

$$t_n^{DW}(x)-f(x)=\frac{1}{2\pi(b_n-a_n)}\int_0^\pi \phi(t)\sum_{\lambda(k)=a_n+1}^{b_n}\frac{1}{W_{\lambda(k)}}\sum_{v=0}^{\lambda(k)}p_{\lambda(n)-v}q_v\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$=\int_0^{\frac{\pi}{b_n+1}}\phi(t)\mathbb{F}_n^{DW}(t)dt+\int_{\frac{\pi}{b_n+1}}^\pi \phi(t)\mathbb{F}_n^{DW}(t)dt:=\mathbb{J}_1+\mathbb{J}_2. \tag{3.1}$$

Since $f\in$ Lip(α), $(0<\alpha\leq 1)$, implies $\phi\in$ Lip(α), then using Lemma 2.1 we get

$$|\mathbb{J}_1|\leq \int_0^{\frac{\pi}{b_n+1}}|\phi(t)||\mathbb{F}_n^{DW}(t)|dt$$

$$=\mathcal{O}\left(b_n+1\right)\int_0^{\frac{\pi}{b_n+1}}t^\alpha dt=\mathcal{O}\left((b_n+1)^{-\alpha}\right). \tag{3.2}$$

The use of Lemma 2.2, implies

$$|\mathbb{J}_2|\leq \int_{\frac{\pi}{b_n+1}}^\pi |\phi(t)||\mathbb{F}_n^{DW}(t)|dt$$

$$=\mathcal{O}\left(1\right)\int_{\frac{\pi}{b_n+1}}^\pi t^\alpha\left(\frac{1}{t^2(b_n-a_n)}+\frac{1}{t}\right)dt$$

$$=\mathcal{O}\left(\frac{1}{b_n-a_n}\int_{\frac{\pi}{b_n+1}}^\pi t^{\alpha-2}dt+\int_{\frac{\pi}{b_n+1}}^\pi t^{\alpha-1}dt\right):=\mathcal{O}\left(\mathbb{J}_{21}+\mathbb{J}_{22}\right). \tag{3.3}$$

For $\mathbb{J}_{21}$ we have

$$\mathbb{J}_{21}=\frac{1}{b_n-a_n}\int_{\frac{\pi}{b_n+1}}^\pi t^{\alpha-2}dt=\begin{cases} \mathcal{O}\Big((b_n+1)^{-\alpha}\Big), & 0<\alpha<1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha=1, \end{cases} \tag{3.4}$$

while for $\mathbb{J}_{22}$ we get

$$\mathbb{J}_{22} = \int_{\frac{\pi}{b_n+1}}^{\pi} t^{\alpha-1} dt = \mathcal{O}\left((b_n+1)^{-\alpha}\right). \tag{3.5}$$

Whence, from (3.3), (3.4), and (3.5), we find that

$$\mathbb{J}_2 = \begin{cases} \mathcal{O}\Big((b_n+1)^{-\alpha} \Big), & 0 < \alpha < 1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1} \Big), & \alpha = 1, \end{cases} \tag{3.6}$$

Finally, using 3.2), (3.7), and (3.1) we obtain

$$\sup_{x\in[0,2\pi]} |t_n^{DW}(x) - f(x)| = \begin{cases} \mathcal{O}\Big((b_n+1)^{-\alpha} \Big), & 0 < \alpha < 1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1} \Big), & \alpha = 1. \end{cases}$$

The proof is completed. ∎

Theorem 3.2. *If Let $f \in L^1[0,2\pi]$ be a 2π-periodic function and $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of f by $D_a^b W^\lambda$ means of its Fourier series is given by*

$$\|D_a^b W^\lambda(f) - f\|_p = \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{b_n+1}\right)\right),$$

provided $\xi(t)$ satisfies the following conditions:

(i) *$\xi(t)/t$ is a decreasing function,*

(ii) $\left[\int_0^{\frac{\pi}{b_n+1}} \left(\frac{|\phi(t)|\sin^\beta \frac{t}{2}}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = \mathcal{O}(1)$,

(iii) $\left[\int_{\frac{\pi}{b_n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = \mathcal{O}\left((b_n+1)^\delta\right)$,

where δ is an arbitrary number such that $q(\beta-\delta)-1 > 0$, $p^{-1}+q^{-1} = 1$, $p \geq 1$, and conditions (ii) and (iii) hold true uniformly in x.

Proof. To start the proof we use (3.1)

$$t_n^{DW}(x) - f(x) = \int_0^{\frac{\pi}{b_n+1}} \phi(t)\mathbb{F}_n^{DW}(t)dt + \int_{\frac{\pi}{b_n+1}}^{\pi} \phi(t)\mathbb{F}_n^{DW}(t)dt := \mathbb{J}_3 + \mathbb{J}_4. \tag{3.7}$$

once again (labeled now as (3.7)).

Using Hölder's inequality, condition (ii) of theorem, Lemma 2.1, the well-known inequality $\sin\frac{u}{2} \geq \frac{u}{\pi}$ for $0 < u \leq \frac{\pi}{b_n+1}$, and implication $f \in \mathrm{W}(L^p, \xi(t)) \Longrightarrow \phi \in \mathrm{W}(L^p, \xi(t))$, we have

$$\begin{aligned} |\mathbb{J}_3| &= \left| \int_0^{\frac{\pi}{b_n+1}} \phi(t)\mathbb{F}_n^{DW}(t)dt \right| \\ &\leq \int_0^{\frac{\pi}{b_n+1}} \left| \frac{\phi(t)\xi(t)\widetilde{\mathbb{F}}_n^{DW}(t)\sin^\beta\frac{t}{2}}{\xi(t)\sin^\beta\frac{t}{2}} \right| dt \\ &\leq \left(\int_0^{\frac{\pi}{b_n+1}} \left| \frac{\phi(t)\sin^\beta\frac{t}{2}}{\xi(t)} \right|^p dt \right)^{1/p} \left(\lim_{\varepsilon\to 0} \int_\varepsilon^{\frac{\pi}{b_n+1}} \left| \frac{\xi(t)\widetilde{\mathbb{F}}_n^{DW}(t)}{\sin^\beta\frac{t}{2}} \right|^q dt \right)^{1/q} \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&= \mathcal{O}(1)\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\frac{\pi}{b_n+1}}\left|\frac{\pi^{\beta}\xi(t)\mathcal{O}(b_n+1)}{t^{\beta}}\right|^q dt\right)^{1/q}\\
&= \mathcal{O}\left(\xi\left(\frac{\pi}{b_n+1}\right)(b_n+1)\right)\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\frac{\pi}{b_n+1}} t^{-q\beta}dt\right)^{1/q}\\
&= \mathcal{O}\left(\xi\left(\frac{\pi}{b_n+1}\right)(b_n+1)^{1+\beta-\frac{1}{q}}\right)\\
&= \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right),
\end{aligned}
$$

because of $1/p + 1/q = 1$ and condition (i) of the function $\xi(t)/t$.

Now we are going to estimate $|\mathbb{J}_4|$. In this case, we use Lemma 2.2 to obtain

$$
\begin{aligned}
|\mathbb{J}_4| &\leq \int_{\frac{\pi}{b_n+1}}^{\pi}\left|\phi(t)\mathbb{F}_n^{DW}(t)\right|dt\\
&= \mathcal{O}\left(\int_{\frac{\pi}{b_n+1}}^{\pi}\frac{|\phi(t)|}{t^2(b_n-a_n)}dt + \int_{\frac{\pi}{b_n+1}}^{\pi}\frac{|\phi(t)|}{t}dt\right) := \mathbb{J}_{41}+\mathbb{J}_{42}. \qquad (3.9)
\end{aligned}
$$

By Hölder's inequality, and condition (iii), we get

$$
\begin{aligned}
|\mathbb{J}_{41}| &= \mathcal{O}\left(\frac{1}{b_n-a_n}\right)\int_{\frac{\pi}{b_n+1}}^{\pi}\left|\frac{\phi(t)\xi(t)t^{\delta-2-\delta}\sin^{\beta}\frac{t}{2}}{\xi(t)\sin^{\beta}\frac{t}{2}}\right|dt\\
&= \mathcal{O}\left(\frac{1}{b_n-a_n}\right)\left(\int_0^{\frac{\pi}{b_n+1}}\left|\frac{\phi(t)t^{-\delta}\sin^{\beta}\frac{t}{2}}{\xi(t)}\right|^p dt\right)^{1/p}\left(\int_{\frac{\pi}{b_n+1}}^{\pi}\left|\frac{\xi(t)t^{\delta-2}}{\sin^{\beta}\frac{t}{2}}\right|^q dt\right)^{1/q}\\
&= \mathcal{O}\left((b_n-a_n)^{\delta-1}\right)\left(\int_{\frac{\pi}{b_n+1}}^{\pi}\left(\xi(t)t^{\delta-\beta-2}\right)^q dt\right)^{1/q}\\
&= \mathcal{O}\left((b_n-a_n)^{\delta-1}\right)\left(\int_{\frac{1}{\pi}}^{\frac{b_n+1}{\pi}}\left(\frac{\xi\left(\frac{1}{t}\right)}{\frac{1}{t}}t^{-\delta+\beta+1}\right)^q\frac{dt}{t^2}\right)^{1/q}\\
&= \mathcal{O}\left((b_n-a_n)^{\delta}\,\xi\left(\frac{\pi}{b_n+1}\right)\right)\left(\int_{\frac{1}{\pi}}^{\frac{b_n+1}{\pi}} t^{-q\delta+q\beta+q-2}dt\right)^{1/q}\\
&= \mathcal{O}\left((b_n-a_n)^{\delta}\,\xi\left(\frac{\pi}{b_n+1}\right)(b_n-a_n)^{1-\delta+\beta-\frac{1}{q}}\right)\\
&= \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right), \qquad (3.10)
\end{aligned}
$$

since by (i) $\frac{\xi(t)}{t}$ is a decreasing function and $1/p + 1/q = 1$.

Finally, we estimate $|\mathbb{J}_{42}|$. With similar reasoning, as in (3.10), we have

$$
|\mathbb{J}_{42}| = \mathcal{O}(1)\left(\int_0^{\frac{\pi}{b_n+1}}\left|\frac{\phi(t)t^{-\delta}\sin^{\beta}\frac{t}{2}}{\xi(t)}\right|^p dt\right)^{1/p}\left(\int_{\frac{\pi}{b_n+1}}^{\pi}\left|\frac{\xi(t)t^{\delta-1}}{\sin^{\beta}\frac{t}{2}}\right|^q dt\right)^{1/q} \qquad (3.11)
$$

$$= \mathcal{O}\left((b_n - a_n)^{\delta}\right)\left(\int_{\frac{\pi}{b_n+1}}^{\pi} \left(\xi(t)t^{\delta-\beta-1}\right)^q dt\right)^{1/q}$$

$$= \mathcal{O}\left((b_n - a_n)^{\delta}\right)\left(\int_{\frac{1}{\pi}}^{\frac{b_n+1}{\pi}} \left(\frac{\xi\left(\frac{1}{t}\right)}{\frac{1}{t}} t^{-\delta+\beta}\right)^q \frac{dt}{t^2}\right)^{1/q}$$

$$= \mathcal{O}\left((b_n - a_n)^{\delta+1}\,\xi\left(\frac{\pi}{b_n+1}\right)\right)\left(\int_{\frac{1}{\pi}}^{\frac{b_n+1}{\pi}} t^{-q\delta+q\beta-2} dt\right)^{1/q}$$

$$= \mathcal{O}\left((b_n - a_n)^{\delta+1}\,\xi\left(\frac{\pi}{b_n+1}\right)(b_n - a_n)^{-\delta+\beta-\frac{1}{q}}\right)$$

$$= \mathcal{O}\left((b_n + 1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right),$$

because of (i) and $1/p + 1/q = 1$.

Therefore, from (3.9), (3.10), and (3.11), we obtain

$$|\mathbb{J}_4| = \mathcal{O}\left((b_n + 1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right). \tag{3.12}$$

Whence, using (3.7), (3.8), and (3.12) the conclusion follows.

The proof is completed. ∎

3.2. **The approximation of functions by conjugate series of the Fourier series.** The conjugate series of a Fourier series is of the form

$$\sum_{k=1}^{\infty}(b_k \cos kx - a_k \sin kx).$$

We already know (see [15], Th. (3.1) Ch. IV) that if $f \in L^1[0, 2\pi]$, then

$$\widetilde{f}(x) := -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\frac{t}{2}dt = -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \psi(t)\cot\frac{t}{2}dt$$

exists for almost all x (it is called conjugate function of the function f).

In this section we are going to prove the analogues of Theorem 1.1-1.2 using $D_a^bW^{\lambda}$ means of partial sums of conjugate series of the Fourier series of the function f.

Theorem 3.3. *Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0 < \alpha \le 1)$, then the degree of approximation of conjugate function $\widetilde{f}$ by $D_a^bW^{\lambda}$ means of its conjugate series of the Fourier series is given by*

$$\|t_n^{DW}(\widetilde{f}) - \widetilde{f}\|_{\infty} = \begin{cases} \mathcal{O}\Big((b_n+1)^{-\alpha}\Big), & 0 < \alpha < 1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha = 1. \end{cases}$$

Proof. Since

$$s_v(\widetilde{f}; x) - \widetilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt,$$

then we have:

$$t_n^{DW}(\widetilde{f};x)-\widetilde{f}(x)=\frac{1}{2\pi(b_n-a_n)}\int_0^\pi\psi(t)\sum_{\lambda(k)=a_n+1}^{b_n}\frac{1}{W_{\lambda(k)}}\sum_{v=0}^{\lambda(k)}p_{\lambda(n)-v}q_v\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$(3.13)\qquad =\int_0^{\frac{\pi}{b_n+1}}\psi(t)\widetilde{\mathbb{F}}_n^{DW}(t)dt+\int_{\frac{\pi}{b_n+1}}^{\pi}\psi(t)\widetilde{\mathbb{F}}_n^{DW}(t)dt:=\mathbb{J}_5+\mathbb{J}_6.$$

Because of the similarity in reasoning we will shorten the proof. So, using Lemma 2.3, we have obtained

$$(3.14)\qquad |\mathbb{J}_5|=\mathcal{O}\left((b_n+1)^{-\alpha}\right),$$

and using Lemma 2.4, we get

$$(3.15)\qquad |\mathbb{J}_6|=\begin{cases}\mathcal{O}\Big((b_n+1)^{-\alpha}\Big), & 0<\alpha<1;\\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha=1,\end{cases}$$

Finally, using 3.13), (3.14), and (3.15) we obtain the conclusion.
The proof is completed. ∎

Next theorem is given without proof.

Theorem 3.4. *If Let $f\in L^1[0,2\pi]$ be a 2π-periodic function and $f\in W(L^p,\xi(t))$ with $0\le\beta\le 1-1/p$, then the degree of approximation of $\widetilde{f}$ by $D_a^bW^\lambda$ means of conjugate series of its Fourier series is given by*

$$\|D_a^bW^\lambda(\widetilde{f})-\widetilde{f}\|_p=\mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{b_n+1}\right)\right),$$

provided $\xi(t)$ satisfies the following conditions:

(i) *$\xi(t)/t$ is a decreasing function,*

(ii') $\left[\int_0^{\frac{\pi}{b_n+1}}\left(\frac{|\psi(t)|\sin^\beta\frac{t}{2}}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}=\mathcal{O}(1)$,

(iii') $\left[\int_{\frac{\pi}{b_n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}=\mathcal{O}\left((b_n+1)^\delta\right)$,

where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p\ge 1$, and conditions (ii') and (iii') hold true uniformly in x.

Proof. The proof is very similar to the proof of Theorem 3.2. Therefore, we omit it. ∎

4. COROLLARIES AND REMARKS

In this section we shall write corollaries of the main results. Indeed, if $\lambda(n)=n$, then the mean in (1.6) becomes the generalized Nörlund mean (N,p,q) (see [3]) Therefore, Theorems 3.1–3.4 imply:

Corollary 4.1. *Let $f\in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\le 1)$, then the degree of approximation of f by $D_a^b(N,p,q)$ means of its Fourier series is given by*

$$\|[D_a^b(N,p,q)]_n(f)-f\|_\infty=\begin{cases}\mathcal{O}\Big((b_n+1)^{-\alpha}\Big), & 0<\alpha<1;\\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha=1.\end{cases}$$

Corollary 4.2. *If Let $f \in L^1[0,2\pi]$ be a 2π-periodic function and $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of f by $D_a^b(N,p,q)$ means of its Fourier series is given by*

$$\|[D_a^b(N,p,q)]_n(f) - f\|_p = \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right),$$

provided $\xi(t)$ satisfies conditions (i), (ii), and (iii), where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p \geq 1$, and conditions (ii) and (iii) hold true uniformly in x.

Corollary 4.3. *Let $f \in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\leq 1)$, then the degree of approximation of conjugate function $\widetilde{f}$ by $D_a^b(N,p,q)$ means of its conjugate series of the Fourier series is given by*

$$\|[D_a^b(N,p,q)]_n(\widetilde{f}) - \widetilde{f}\|_\infty = \begin{cases} \mathcal{O}\Big(\,(b_n+1)^{-\alpha}\Big), & 0<\alpha<1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha = 1. \end{cases}$$

Corollary 4.4. *If Let $f \in L^1[0,2\pi]$ be a 2π-periodic function and $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of $\widetilde{f}$ by $D_a^b(N,p,q)$ means of conjugate series of its Fourier series is given by*

$$\|[D_a^b(N,p,q)]_n(\widetilde{f}) - \widetilde{f}\|_p = \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right),$$

provided $\xi(t)$ satisfies conditions (i), (ii'), and (iii'), where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p \geq 1$, and conditions (ii') and (iii') hold true uniformly in x.

If we take $\lambda(n)=n$ and $q_k = 1$ for all $k=0,1,\ldots,n$, then $D_a^b W^\lambda$ mean reduces to $D_a^b N_p$ mean, and from Theorems 3.3-3.4 we derive the following.

Corollary 4.5 ([6]). *Let $f \in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\leq 1)$, then the degree of approximation of conjugate function $\widetilde{f}$ by $D_a^b N_p$ means of its conjugate series of the Fourier series is given by*

$$\|(D_a^b N_p)_n(\widetilde{f}) - \widetilde{f}\|_\infty = \begin{cases} \mathcal{O}\Big(\,(b_n+1)^{-\alpha}\Big), & 0<\alpha<1; \\ \mathcal{O}\Big(\frac{\log(b_n+1)}{b_n+1}\Big), & \alpha = 1. \end{cases}$$

Corollary 4.6 ([6]). *If Let $f \in L^1[0,2\pi]$ be a 2π-periodic function and $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of $\widetilde{f}$ by $D_a^b N_p$ means of conjugate series of its Fourier series is given by*

$$\|(D_a^b N_p)_n(\widetilde{f}) - \widetilde{f}\|_p = \mathcal{O}\left((b_n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{b_n+1}\right)\right),$$

provided $\xi(t)$ satisfies conditions (i), (ii'), and (iii'), where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p \geq 1$, and conditions (ii') and (iii') hold true uniformly in x.

Moreover, if we take $b_n = n$, $a_n = 0$, then D_a^b mean becomes ordinary Cesàro mean, and in addition, if we assume $\lambda(n)=n$ and $q_k=1$ for all $k=0,1,\ldots,n$, then $D_a^b W^\lambda$ mean reduces to N_p mean. Consequently, from our theorems we obtain:

Corollary 4.7 ([7]). *Let $f \in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\leq 1)$, then the degree of approximation of f by $(C^1 N_p)$ means of its Fourier series is given by*

$$\|(C^1N_p)_n(f)-f\|_\infty = \begin{cases}\mathcal{O}\Big((n+1)^{-\alpha}\Big), & 0<\alpha<1;\\ \mathcal{O}\Big(\frac{\log(n+1)}{n+1}\Big), & \alpha=1.\end{cases}$$

Corollary 4.8 ([7]). *If Let $f\in L^1[0,2\pi]$ be a 2π-periodic function and $f\in W(L^p,\xi(t))$ with $0\le\beta\le 1-1/p$, then the degree of approximation of f by (C^1N_p) means of its Fourier series is given by*

$$\|(C^1N_p)_n(f)-f\|_p=\mathcal{O}\left((n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),$$

provided $\xi(t)$ satisfies conditions (i), (ii), and (iii), where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p\ge 1$, and conditions (ii) and (iii) hold true uniformly in x.

Corollary 4.9. *Let $f\in L^1[0,2\pi]$ be a 2π-periodic function belonging to Lip(α), $(0<\alpha\le 1)$, then the degree of approximation of conjugate function $\widetilde{f}$ by (C^1N_p) means of its conjugate series of the Fourier series is given by*

$$\|(C^1N_p)_n(\widetilde{f})-\widetilde{f}\|_\infty = \begin{cases}\mathcal{O}\Big((n+1)^{-\alpha}\Big), & 0<\alpha<1;\\ \mathcal{O}\Big(\frac{\log(n+1)}{n+1}\Big), & \alpha=1.\end{cases}$$

Corollary 4.10 ([10]). *If Let $f\in L^1[0,2\pi]$ be a 2π-periodic function and $f\in W(L^p,\xi(t))$ with $0\le\beta\le 1-1/p$, then the degree of approximation of $\widetilde{f}$ by (C^1N_p) means of conjugate series of its Fourier series is given by*

$$\|(C^1N_p)_n(\widetilde{f})-\widetilde{f}\|_p=\mathcal{O}\left((n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right),$$

provided $\xi(t)$ satisfies conditions (i), (ii'), and (iii'), where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p\ge 1$, and conditions (ii') and (iii') hold true uniformly in x.

Remark 4.1. If we take $\lambda(n)=n$, $b_n=n$, $a_n=0$, $p_k=\frac{1}{2^n}\binom{n}{k}$, and $q_k=1$ for all $k=0,1,\dots,n$, then $D_a^bW^\lambda$ mean reduces to $(C,1)(E,1)$ mean, and from Theorems 3.3-3.4 the results presented in [11] are consequences of ours.

Taking $\xi(t)=t^\alpha$ and $\beta=0$, then W$(L^p,\xi(t))\equiv$ Lip(α,p), and from Theorem 3.2 we obtain the following.

Corollary 4.11. *Let $f\in L^1[0,2\pi]$ be a 2π-periodic function and $f\in$ Lip(α,p), then the degree of approximation of f by $D_a^bW^\lambda$ means of its Fourier series is given by*

$$\|(D_a^bW^\lambda)_n(f)-f\|_p=\mathcal{O}\left(\frac{1}{(b_n+1)^{\alpha-\frac{1}{p}}}\right),$$

provided that $\frac{1}{p}<\alpha<1$.

We have to note here that if we take $p=\infty$ in Corollary 4.11, then we get:

Corollary 4.12. *Let $f\in L^1[0,2\pi]$ be a 2π-periodic function and $f\in$ Lip(α), $0<\alpha<1$, then the degree of approximation of f by $D_a^bW^\lambda$ means of its Fourier series is given by*

$$\|(D_a^bW^\lambda)_n(f)-f\|_\infty=\mathcal{O}\left(\frac{1}{(b_n+1)^{\alpha}}\right),$$

Remark 4.2. The same degree of approximation, as in Corollary 4.11 and Corollary 4.12, can be obtain for the function $\widetilde{f}$ using Theorem 3.4.

If we choose $a_n = 2n - 1$ and $b_n = 2n$, then from Corollary 4.11 (or Theorem 3.2 with conditions as in Corollary 4.11) we obtain:

Corollary 4.13. *Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function and $f \in Lip(\alpha, p)$, then the degree of approximation of f by $D_a^b W^\lambda$ means of its Fourier series is given by*

$$\|W_{2n}^\lambda(f) - f\|_p = \mathcal{O}\left(\frac{1}{(2n+1)^{\alpha - \frac{1}{p}}}\right),$$

provided that $\frac{1}{p} < \alpha < 1$.

The latest corollary, for $p = \infty$, takes the form:

Corollary 4.14. *Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function and $f \in Lip(\alpha, p)$, $0 < \alpha < 1$, then the degree of approximation of f by $D_a^b W^\lambda$ means of its Fourier series is given by*

$$\|W_{2n}^\lambda(f) - f\|_\infty = \mathcal{O}\left(\frac{1}{(2n+1)^{\alpha}}\right).$$

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