Aust. J. Math. Anal. Appl.
Vol. **17** (2020), No. 2, Art. 20, 18 pp.
AJMAA

$\psi(m,q)$-ISOMETRIC MAPPINGS ON METRIC SPACES

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Received 6 January, 2020; accepted 30 September, 2020; published 21 December, 2020.

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ABSTRACT. The concept of (m,p)-isometric operators on Banach space was extended to (m,q)-isometric mappings on general metric spaces in [6]. This paper is devoted to define the concept of $\psi(m,q)$-isometric, which is the extension of $A(m,p)$-isometric operators on Banach spaces introduced in [10]. Let $T,\psi : (\mathbb{E},\ d) \to (\mathbb{E},\ d)$ be two mappings.
For some positive integer m and $q \in (0,\infty)$. T is said to be an $\psi(m,q)$-isometry, if for all $y,z \in \mathbb{E}$,

$$\sum_{0\leq r\leq m}(-1)^{m-r}\binom{m}{r}d\big(\psi\circ T^r y,\ \psi\circ T^r z\big)^q = 0.$$

Key words and phrases: Metric space; Mapping; m-isometry; (m,p)-isometry.

2010 *Mathematics Subject Classification.* Primary 47B99, 54E40.

ISSN (electronic): 1449-5910

1. INTRODUCTION

A few years ago, the class of m-isometric operators in both Hilbert and Banach spaces attracted much attention. They have been the object of some intensive studies by many authors in the papers [1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 17, 22, 23].Also,the theory m-isometry is developed by J. Agler and T. Stankus (see [1, 2, 3]) with rich connections to Toeplitz operators. Let $Q(y,z)$ be a polynomial in two variables y and z of the form

$$Q(y,z) = \sum_{0\leq r\leq m}\sum_{0\leq l\leq m} \beta_{rl} z^l y^r, \quad \beta_{rl} \in \mathbb{C}.$$

For an operator $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space of complex infinite dimensional $\mathcal{H}$ into itself,

$$Q(T,T^*) = \sum_{0\leq r\leq m}\sum_{0\leq l\leq m} \beta_{rl} T^{*l} T^r, \quad \beta_{rl} \in \mathbb{C}.$$

T is an m-isometry for some integer $m \geq 1$ if

$$\Lambda_m(T) = \big(zy-1\big)^m(T) = \sum_{0\leq r\leq m} (-1)^{m-r}\binom{m}{r} T^{*r}T^r = 0,$$

or equivalently

$$\langle \Lambda_m(T)y \mid y\rangle = \sum_{0\leq r\leq m} (-1)^{m-r}\binom{m}{r} \|T^r y\|^2 = 0$$

for all $y \in \mathcal{H}$ ([1]). If $\Lambda_{m-1}(T) \neq 0$, then T is said to be a strict m-isometry for $m \geq 2$.

m-isometric operators are important in the study of some classes of operators as Dirichlet operators,they are also a natural extension of an isometry ($m = 1$).

In [4, 8, 14, 17] a generalizations of m-isometries to Banach spaces are studied.
For some integer $m \geq 1$ and $p \in (0,\infty)$, if

$$\beta_m^{(p)}(T,y) := \sum_{0\leq r\leq m} (-1)^{m-r}\binom{m}{r} \|T^r y\|^p = 0 \quad (\forall\ y \in \mathcal{X}),$$

$T \in \mathcal{B}(\mathcal{X})$, is called an (m,p)-isometry, (see [4, 14]). In [10], the author introduced the concepts of $A(m,p)$-isometries, where, for an operator $A \in \mathcal{B}(\mathcal{X}), T \in \mathcal{B}(\mathcal{X})$ is $A(m,p)$-isometric if

$$\beta_m^{(p)}(T,A;\ y) := \sum_{0\leq r\leq m} (-1)^{m-r}\binom{m}{r} \|AT^r y\|^p = 0 \quad (\forall\ y \in \mathcal{X}). \tag{1.1}$$

Evidently, an $I(m,p)$-isometry is an (m,p)-isometry.

If $\beta_m^{(p)}(T,A,y) \leq 0$ $\big($resp. $\beta_m^{(p)}(T,A,y) \geq 0\big), \forall\ y \in \mathcal{X}$, T is said to be $A(m,p)$-expansive $\big($resp. $A(m,p)$-contractive$\big)$. We refer the interested reader to [11, 15, 20, 21] for complete details.

Let $\mathbb{E}$ and $\mathbb{F}$ be metric spaces. A mapping $T : \mathbb{E} \longrightarrow \mathbb{F}$ is said to be an isometry if it satisfies $d_{\mathbb{F}}(Ty,Tz) = d_{\mathbb{E}}(y,z)$, for all $y,z \in \mathbb{E}$, where $d_{\mathbb{E}}(.,\ .))$ and $d_{\mathbb{F}}(.,\ .)$ denote the metrics in the spaces $\mathbb{E}$ and $\mathbb{F}$, respectively.

For an map $T : \mathbb{E} \to \mathbb{E}$, a positive integer m and $q \in (0,\infty)$ define

(1.2) $$\Theta_m^q(T;\ y,z) : \sum_{0\le r\le m}(-1)^r\binom{m}{r}d_{\mathbb{E}}\big(T^{m-r}y,T^{m-r}z\big)^q,\quad y,z\in\mathbb{E}.$$

The map T is said to be (m,q)-contractive $\big($respectively, (m,q)-expansive and (m,q)-isometric $\big)$ if $(-1)^m\Theta_m^q(T;\ y,z)\ge 0$ $\big($respectively, $(-1)^m\Theta_m^q(T;\ y,z)\le 0$ and $\Theta_m^q(T;\ y,z)=0$ $\big)$ for some positive integer m and $q\in(0,\infty)$.

Clearly, T is an (m,q)-contractive mapping if

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}d_{\mathbb{E}}\big(T^{m-r}y,T^{m-r}z\big)^q\ge 0,\ \ \forall\, y,z\in\mathbb{E}.$$

T is an (m,q)-expansive mapping if

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}d_{\mathbb{E}}\big(T^{m-r}y,T^{m-r}z\big)^q\le 0,\ \ \forall\, y,z\in\mathbb{E},$$

and T is an (m,q)-isometric mapping if and only if

$$\sum_{0\le r\le m}(-1)^{r}\binom{m}{r}d_{\mathbb{E}}\big(T^{m-r}y,T^{m-r}z\big)^q=0.\ \ \forall\, y,z\in\mathbb{E}.$$

It is well known that the concept of (m,q)-isometric mappings was introduced and studied by the authors T. Bermúdez et al. in the paper [6]. However the third named author has introduced and studied the concepts of (m,q)-expansive and (m,q) contractive mappings in the papers [18, 19].

Following [16], a mapping T (not necessarily linear) on a normed space $\mathcal{X}$ is an (m,p)-isometry ($m\ge 1$ integer and $p>0$ real) if, for all $y,z\in\mathcal{X}$,

(1.3) $$\beta_m^{(p)}(T;\ y,z) := \sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}\|T^ry-T^rz\|^p=0.$$

When $m=1$, (1.3) is equivalent to $\|Ty-Tz\|=\|y-z\|,\ \forall\, y,z\in\mathcal{X}$, and when $m=2$, (1.3) is equivalent to

$$\|T^2y-T^2z\|^p-2\|Ty-Tz\|^p+\|y-z\|^p=0,\ \forall\, y,z\in\mathcal{X}.$$

In this paper, we extend the concept of $A(m,p)$-isometries on Banach spaces to general metric spaces. Let $T,\psi:(\mathbb{E},\ d)\to(\mathbb{E},\ d)$ be two mappings. T is said to be $\psi(m,q)$-isometric mapping, if for all $y,z\in\mathbb{E}$,

(1.4) $$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}d\big(\psi\circ T^ry,\ \psi\circ T^rz\big)^q=0,$$

for some positive integer m and $q\in(0,\infty)$.

Observe that if $T,\psi\in\mathcal{B}(\mathcal{X})$, we can write (1.4) as

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}\big\|\psi\circ T^ry\big\|^q=0.$$

The contents of the paper is as follows. In Section one we set up notation and terminology. Furthermore, we collect some facts about m-isometries and (m,p)-isometries. In Section two, we introduce and study the concept of $\psi(m,q)$-isometric mappings on general metric spaces. Several properties for members from this class of mappings are investigated. We prove under suitable conditions that $\psi(m,q)$-isometry must be $\psi(m-1,q)$-isometry for $m \geq 2$ (Proposition 2.4, Theorem 2.8). Recall that if T is an m-isometric (resp. (m,q)-isometry or $A(m,p)$-isometry), then so are all its power T^n; for $n \geq 1$ (cf [1, 4, 6]). It turns out that the same assertion remains true for $\psi(m,q)$-isometry (Theorem 2.11). Moreover, we prove that if T is an $\psi(m,q)$ and T is an $\psi(n,q)$-isometry such $TS = ST$, Then TS is an $\psi(m+n-1,q)$-isometry (Theorem 2.13). In section three, we prove that a map $M : (\mathbb{E},\ d) \to (\mathbb{E},\ d)$ is an $\psi(m,q)$-isometry if and only if $T : \big(\mathbb{E},\ \widetilde{\rho_{T,\,\psi}}\big) \to \big(\mathbb{E},\ \widetilde{\rho_{T,\,\psi}}'\big)$ is an isometry for some distances $\widetilde{\rho_{T,\,\psi}}$ and $\widetilde{\rho_{T,\,\psi}}'$ on $\mathbb{E}$ associated to T and ψ.

2. MAIN RESULTS

From now in this paper, $\psi : \mathbb{E} \to \mathbb{E}$ is a self mapping on a metric space $(\mathbb{E},\ d)$.

Definition 2.1. Let T be self mappings on $(\mathbb{E},\ d)$. T is said to be $\psi(m,q)$-isometry if it satisfies for all $y, z \in \mathbb{E}$

$$\sum_{0\leq r\leq m} (-1)^{m-r}\binom{m}{r} d\big(\psi \circ T^r y,\ \psi \circ T^r z\big)^q = 0, \tag{2.1}$$

for some positive integer m and real $q \in (0, \infty)$.

Remark 2.1. (1) When $m = 1$, Equation (2.1) is equivalent to

$$d\big(\psi \circ Ty,\ \psi \circ Tz\big) = d\big(\psi y,\ \psi z\big);\ \ ;\forall\ \ y, z \in \mathbb{E}.$$

(2) When $m = 2$, Equation (2.1) is equivalent to

$$d\big(\psi \circ T^2 y,\ \psi \circ T^2 z\big)^q - 2d\big(\psi \circ Ty,\ \psi \circ Tz)^q + d\big(\psi y,\ \psi z)^q = 0,\ \ \forall\ y, z \in \mathbb{E}.$$

(3) When $m = 3$, Equation (2.1) is equivalent to

$$d(\psi\circ T^3 y,\ \psi\circ T^3 z\big)^q - 3d\big(\psi\circ T^2 y,\ \psi\circ T^2 z\big)^q + 3d\big(\psi\circ Ty,\ \psi\circ Tz\big)^q - d\big(\psi y,\ \psi z\big)^q = 0\ \ \forall\ y, z \in \mathbb{E}.$$

Remark 2.2. If $\psi \equiv I_{\mathbb{E}}$ (the identity map), then Definition 2.1 coincides with [6, Definition1.1].

Remark 2.3. (1) It is will known that every (m,q)-isometry is injective map. Moreover, in general an $\psi(m,q)$-isometry is not necessary injective map.

(2) Let T be a self map on a metric spaces $\mathbb{E}$ such is $\psi(m,q)$-isometric. If $T\circ\psi$ or ψ is injective, then T is injective.

Let $y, z \in \mathbb{E}$ such that $Ty = Tz$. It is obvious that $T^r y = T^r z$ for all $r \in \mathbb{N}$. Under the assumption that T is a $\psi(m,q)$-isometric mapping we get

$$\sum_{0\leq r\leq m} (-1)^r\binom{m}{r} d(\psi \circ T^{m-r} y, \psi \circ T^{m-r} z)^q = 0,$$

which means that $d(\psi y, \psi z) = 0$. So that $\psi y = \psi z$. Thus, $T \circ \psi y = T \circ \psi z$. Consequently $y = z$ under one of the above conditions.

Remark 2.4. The following example shows that there exists a map that is $\psi(m,q)$-isometry but is not (m,q)-isometry for some positive integer m and real q.

Example 2.1. *Consider the metric space* $(\mathbb{E}, d_0)$ *where* $E = \mathbb{R}^2$ *and*

$$d_0\big((y,z),\ (u,v)\big) = |y-u| + |z-v|.$$

Define $T, \psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *as follows:*

$$T(y,z) = \big(\frac{y+z-1}{2},\ \frac{y+z+1}{2}\big)\ \ \textit{and}\ \psi(y,z) = \big(\frac{y+z}{2}, \frac{y+z}{2}\big).$$

A simple computation shows that $d_0\big(T(y,z),\ T(u,v)\big) \neq d_0\big((y,z),\ (u,v)\big)$ *and*

$$d_0\big(\psi \circ T(y,z),\ \psi \circ T(u,v)\big) = d_0\big(\psi(y,z),\ \psi(u,v)\big).$$

This means that T *is a* ψ*-isometry but* T *is not isometry.*

The following theorem extended [6, Proposition 1.4].

Theorem 2.1. *Let* T *be a self map on a metric space* $(\mathbb{E},\ d)$*. If* T *is a bijective* $\psi(m,q)$*-isometry, then* T^{-1} *is also an* $\psi(m,q)$*-isometry.*

Proof. As the proof is similar to [6, Proposition 1.4], we omit it. ∎

Set

$$\Theta_{m,\,q}(T,\psi;y,z) := \sum_{0\le r\le m} (-1)^{m-r}\binom{m}{r} d(\psi\circ T^r y, \psi \circ T^r z)^q, \quad \forall\ y,z\in\mathbb{E}. \tag{2.2}$$

Proposition 2.2. *For a self map* T *on a metric space* $\mathbb{E}$*,* $m\in\mathbb{N}$*,* $q\in(0,\infty)$ *and* $y,z\in\mathbb{E}$*, the following identity holds.*

$$\Theta_{m,\,q}(T,\psi;\ y,z) = \Theta_{m-1,\,q}(T,\psi;\ Ty,Tz) - \Theta_{m-1,\,q}(T;\psi,\ y,z). \tag{2.3}$$

Proof. In view of the identity $\binom{m}{r} = \binom{m-1}{r}+\binom{m-1}{r-1}$ for $j=1,\cdots,m-1$, we have the equalities

$$\begin{aligned}
\Theta_{m,\,q}(T,\psi;\ y,z) &= \sum_{0\le r\le m} (-1)^{m-r}\binom{m}{r} d\big(\psi\circ T^r y, \circ T^r z\big)^q \\
&= (-1)^m d\big(\psi y,\psi z\big)^q + \sum_{1\le r\le m-1}(-1)^{m-r}\binom{m}{r} d\big(\psi\circ T^r y,\psi\circ T^r z\big)^q \\
&\quad + d\big(\psi\circ T^m y,\psi\circ T^m z\big)^q \\
&= (-1)^m d\big(\psi y,\psi z\big)^q \\
&\quad + \sum_{1\le r\le m-1}(-1)^{m-r}\Big(\binom{m-1}{r}+\binom{m-1}{r-1}\Big) d\big(\psi\circ T^r y,\psi\circ T^r z\big)^q \\
&\quad + d\big(\psi\circ T^m y,\psi\circ T^m z\big)^q \\
&= -\Theta_{m-1,\,q}(T,\psi;\ y,z) + \Theta_{m-1,\,q}(T,\psi;\ Ty,Tz).
\end{aligned}$$

∎

Corollary 2.3. *If* T *is a self map on a metric space* $(\mathbb{E},\ d)$ *such is an* $\psi(m,q)$*-isometry, then* T *is an* $\psi(m+1,q)$*-isometry.*

The converse of Corollary 2.3 is not in general true (see [6]).

Proposition 2.4. *Let* T *be a self mapping on a metric space* $(\mathbb{E},\ d)$ *such is an* $\psi(m,q)$*-isometry. If* T *satisfies*

$$d\big(\psi\circ Ty,\ \psi\circ Tz\big) \le d\big(\psi y,\psi z\big),\quad \forall\ y,z\in\mathbb{E},$$

then T *is an* $\psi(m-1,q)$*-isometry.*

Proof. Since T satisfies the condition $d\big(\psi\circ Ty,\ \psi\circ Tz\big)\le d\big(\psi y,\psi z\big),\ \ \forall\ y,z\in\mathbb{E}$, it follows that,

$$d\big(\psi\circ T^{n+1}y,\ \psi\circ T^{n+1}z\big)^q \le d\big(\psi\circ T^n y,\psi\circ T^n z\big)^q;\forall\ y,z\in\mathbb{E},\ \text{and}\ \ n\in\mathbb{N}.$$

This means that $\Big(d\big(\psi\circ T^n y,\psi\circ T^n z\big)^q\Big)_{n\in\mathbb{N}}$ is deceasing sequence, so convergent.

Under the assumption that T is an $\psi(m,q)$-isometry and together (2.3), we get

$$\Theta_{m-1,\,q}(T,\psi;\ y,z)=\Theta_{m-1,\,q}(T,\psi;\ Ty,Tz)=...=\Theta_{m-1,\,q}(T,\psi;\ T^ny,T^nz).$$

However

$$\Theta_{m-1,\,q}(T,\psi;\ T^ny,T^nz)=\Theta_{m-2,\,q}(T,\psi;\ T^{n+1}y,T^{n+1}z)-\Theta_{m-2,\,q}(T,\psi;\ T^ny,T^nz),$$

so that

$$\begin{aligned}&\Theta_{m-1,\,q}(T,\psi;\ T^ny,T^nz)\\ =\ &\sum_{0\le j\le m-2}(-1)^{m-j}\binom{m-2}{j}\Big[d\big(\psi\circ T^{n+j+1}y,\psi\circ T^{n+j+1}z\big)^q\\ &-d\big(\psi\circ T^{n+j}y,\ \psi\circ T^{n+j}z\big)^q\Big].\end{aligned}$$

By taking the limit $n\to\infty$ in the preceding equality leads to

$$\Theta_{m-1,\,q}(T,\psi;\ T^ny,T^nz)\to 0.$$

Consequently, $\Theta_{m-1,\,q}(T,\psi;\ y,z)=0$. Therefore, T is an $\psi(m-1,q)$-isometry. ∎

Corollary 2.5. *Let T be a self mapping on a metric space $(\mathbb{E},\ d)$. If T satisfies*

$$d\big(\psi\circ Ty,\ \psi\circ Tz\big)\le d\big(\psi y,\psi z\big),\ \ \forall\ y,z\in\mathbb{E},$$

then T is an $\psi(m,q)$-isometry if and only if T is an ψ-isometry.

Proof. We can derive the result from Proposition 2.4. ∎

Proposition 2.6. *Let T be a self mapping on a metric space $(\mathbb{E},\ d)$. Then the following identities hold for $n\ge m\ge 1$.*

$$\Theta_{m,\,q}(T,\psi;y,z)=d\big(\psi\circ T^my,\ \psi\circ T^mz\big)^q-\sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r,\,q}(T,\psi;\ y,z)\tag{2.4}$$

$$\begin{aligned}&\sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ Ty,Tz)=\\ &\sum_{0\le r\le m-1}\binom{n+1}{r}\Theta_{r,\,q}(T,\psi,\ y,z)+\binom{n}{m-1}\Theta_{m,\,q}(T,\psi;\ y,z),\end{aligned}\tag{2.5}$$

where $\Theta_{0,\,q}(T,\psi;y,z)=d\big(\psi y,\ \psi z\big)^q$.

Proof. We will prove (2.4) by induction on $m\ge 1$. One may let $m=1$ in (2.4) to see that

$$\Theta_{1,\,q}(T,\psi;\ y,z)=d\big(\psi\circ Ty,\ \psi\circ Tz\big)^q-d\big(\psi y,\psi z\big)^q,$$

which is obviously true. Suppose that the induction hypothesis holds for m. By the induction hypothesis and (2.3), we obtain

$$\begin{aligned}&\Theta_{m+1,\,q}(T,\psi;\ y,z)=\Theta_{m,\,q}(T,\psi;\ Ty,Tz)-\Theta_{m,\,q}(T,\psi;\ y,z)\\ =\ &d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q-\sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r,\,q}(T,\psi;\ Ty,Tz)\\ &-d\big(\psi\circ T^my,\ \psi\circ T^mz\big)^q+\sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r,\,q}(T,\psi;\ y,z)\Big)\end{aligned}$$

$$\begin{aligned}
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q - d\big(\psi\circ T^{m}y,\ \psi\circ T^{m}z\big)^q \\
&\quad - \sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r+1,\,q}(T,\psi;\ y,z) \\
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q - \Theta_m(T,\psi,\ y,z) - \sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r,\,q}(T,\psi;\ y,z) \\
&\quad - \sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r+1,\,q}(T,\psi;\ y,z) \\
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q - \Theta_{m,\,q}(T,\psi;\ y,z) - \sum_{0\le r\le m-1}\binom{m}{r}\Theta_{r,\,q}(T,\psi;\ y,z) \\
&\quad - \sum_{1\le r\le m}\binom{m}{r-1}\Theta_{r,\,q}(T,\psi;\ y,z) \\
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q - \Theta_{m,\,q}(T,\psi;\ y,z) \\
&\quad -\Theta_{0,\,q}(T,\psi;\ y,z) - \sum_{1\le r\le m-1}\left(\binom{m}{r}+\binom{m}{r-1}\right)\Theta_{r,\,q}(T,\psi,\ y,z) \\
&\quad -\binom{m}{m-1}\Theta_{m,\,q}(T,\psi;\ y,z) \\
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}z\big)^q - \Theta_{0,\,q}(T,\psi,\ y,z) \\
&\quad - \sum_{1\le r\le m-1}\binom{m+1}{r}\Theta_{r,\,q}(T,\psi;\ y,z) - \binom{m+1}{m}\Theta_{m,\,q}(T,\psi,\ y,z) \\
&= d\big(\psi\circ T^{m+1}y,\ \psi\circ T^{m+1}\big)^q - \sum_{0\le r\le m}\binom{m+1}{r}\Theta_{r,\,q}(T,\psi;\ y,z).
\end{aligned}$$

Hence, the desired conclusion follows.

To prove (2.5), we have by (2.3) that

$$\begin{aligned}
&\sum_{1\le r\le m}\binom{n}{r-1}\Theta_{r,\,q}(T,\psi;\ y,z) \\
&= \sum_{1\le r\le m}\binom{n}{r-1}\Big(\Theta_{r-1,\,q}(T,\psi;\ Ty,Tz) - \Theta_{r-1,\,q}(T,\psi;\ y,z)\Big) \\
&= \sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ Ty,Tz) - \sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ y,z).
\end{aligned}$$

From this, we deduce that

$$\begin{aligned}
&\sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ Ty,Tz) = \\
&\sum_{1\le r\le m}\binom{n}{r-1}\Theta_{r,\,q}(T,\psi;\ y,z) + \sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ y,z)
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n}{m-1}\Theta_{m,\,q}(T,\psi;\ y,z) + \sum_{1\le r\le m-1}\left(\binom{n}{r-1}+\binom{n}{r}\right)\Theta_{r,\,q}(T,\psi;\ y,z)\\
&= +\Theta_{0,\,q}(T,\psi;\ y,z)\\
&= \sum_{0\le r\le m-1}\binom{n+1}{r}\Theta_{r,\,q}(T,\psi;\ y,z) + \binom{n}{m-1}\Theta_{m,\,q}(T,\psi;\ y,z).
\end{aligned}
$$

The proof is so completed. ∎

Theorem 2.7. *Let T is a self mapping on a metric space $(\mathbb{E},\ d)$. The following properties hold.*

(1)

$$d\big(\psi\circ T^n y,\ \psi\circ T^n z)^q = \sum_{0\le r\le m}\binom{n}{r}\Theta_{r,\,q}(T,\psi;\ y,z),\quad \forall\, y,z\in\mathbb{E}. \tag{2.6}$$

(2) *T is an $\psi(m,q)$-isometry if and only if*

$$d\big(\psi\circ T^n y,\ \psi\circ T^n z)^q = \sum_{0\le r\le m-1}\binom{n}{r}\Theta_{r,\,q}(T,\psi;y,z),\quad \forall\, y,z\in\mathbb{E}. \tag{2.7}$$

(3) *If T is an $\psi(m,q)$-isometry, then*

$$\Theta_{m-1,\,q}(T,\psi;y,z) = \lim_{n\to\infty}\frac{1}{\binom{n}{m-1}}d(\psi\circ T^n y,\ \psi\circ T^n z)^q,\quad \forall\, y,z\in\mathbb{E}. \tag{2.8}$$

In particular $\Theta_{m-1,\,q}(T,\psi,\ y,z)\ge 0,\quad y,z\in\mathbb{E}.$

Proof. We proceed by the induction to prove (2.6). It is easy to see that (2.6) is true for $n=1$. Now assume that (2.6) holds for n and prove it for $n+1$. By (2.2) and the induction hypothesis,

$$
\begin{aligned}
&d\big(\psi\circ T^{n+1}y,\ \psi\circ T^{n+1}z\big)^q =\\
&\Theta_{n+1,\,q}(T,\psi;\ y,z) - \sum_{0\le j\le n}(-1)^{n+1-j}\binom{n+1}{j}d\big(\psi\circ T^j y,\ \psi\circ T^j z\big)^q\\
=\ &\Theta_{n+1,\,q}(T,\psi;\ y,z) - \sum_{0\le j\le n}(-1)^{n+1-j}\binom{n+1}{j}\sum_{0\le r\le j}\binom{j}{r}\Theta_{r,\,q}(T,\psi;\ y,z)\\
=\ &\Theta_{n+1,\,q}(T,\psi;\ y,z) - \sum_{0\le r\le n}\Theta_{r,\,q}(T,\ \psi;\ y,z)\sum_{r\le j\le n}(-1)^{n+1-j}\binom{n+1}{j}\binom{j}{r}\\
=\ &\Theta_{n+1,\,q}(T,\ \psi;\ y,z) - \sum_{0\le r\le n}\binom{n+1}{r}\Theta_{r,\,q}(T,\psi;\ y,z)\underbrace{\left(\sum_{r\le j\le n}(-1)^{n+1-j}\binom{n+1-j}{j-r}\right)}_{=-1}\\
=\ &\sum_{0\le r\le n+1}\binom{n+1}{r}\Theta_{r,\,q}(T,\psi,\ y,z).
\end{aligned}
$$

Thus (2.6) holds for $(n+1)$.

(2) If T is an $\psi(m,q)$-isometric mapping, then $\Theta_{r,\,q}(T,\psi;\ y,z)=0$ for all $r\ge m$. Hence we drive (2.7) from (2.6). On the other hand, if (2.7) holds for all $n\ge 1$. Then $\Theta_{r,\,q}(T,\psi;\ y,z)=0$ for $r\ge m$ by (2.6), so T is an $\psi(m,q)$-isometry.

(3) One first has to observe that, by (2.7) if T is an $\psi(m,q)$-isometry, then

$$d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)^q=\sum_{0\le j\le m-2}\binom{n}{j}\Theta_{j,\,q}(T,\psi;\ y,z)+\binom{n}{m-1}\Theta_{m-1,\,q}(T,\psi;\ y,z).$$

Dividing both sides by $\binom{n}{m-1}$, we get

$$\frac{1}{\binom{n}{m-1}}d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)^q=\sum_{0\le j\le m-2}\frac{\binom{n}{j}}{\binom{n}{m-1}}\Theta_{j,\,q}(T,\psi,\ y,z)+\Theta_{m-1,\,q}(T,\psi,\ y,z).$$

Since $\dfrac{\binom{n}{j}}{\binom{n}{m-1}}\longrightarrow 0$ for $0\le j\le m-2$, by taking $n\to\infty$ we get the statement (3). ∎

It was observed that for an even integer m, every invertible m-isometric operator is also an $(m-1)$-isometric operator. See [1, Proposition 1.23] and [9, Proposition A]. The following theorem shows that this property is also satisfied by the class of $\psi(m,q)$-isometry.

Theorem 2.8. *Let $T:(\mathbb{E},\ d)\to(\mathbb{E},\ d)$ be a map such is an invertible $\psi(m,q)$-isometry. If m is even, then T is an $\psi(m-1,q)$-isometry.*

Proof. Since T and T^{-1} are an $\psi(m,q)$-isometries, it follows in view of the statement (3) of Theorem 2.7 that

$$\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^r y,\ \psi\circ T^r z\big)^q\ge 0,\ \ \forall\ \ y,z\in\mathbb{E}$$

and

$$\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^{-r} y,\ \psi\circ T^{-r} z\big)^q\ge 0,\ \ \forall\ \ y,z\in\mathbb{E}.$$

Then one has

$$\begin{aligned}
&\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^{-r} y,\ \psi\circ T^{-r} z\big)^q\ge 0,\ \ \forall\ \ y,z\\
\Longrightarrow\ &\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{m-1-r}d\big(\psi\circ T^{m-1-r} y,\ \psi\circ T^{m-1-r} z\big)^q\ge 0\\
\Longrightarrow\ &\sum_{0\le r\le m-1}(-1)^{r}\binom{m-1}{r}d\big(\psi\circ T^{r} y,\ \psi\circ T^{r} z\big)^q\ge 0,\ \ \forall\ \ y,z\\
\Longrightarrow\ &-\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^{r} y,\ \psi\circ T^{r} z\big)^q\ge 0\ \ \text{(since m is even integer)}\\
\Longrightarrow\ &\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^{r} y.\ \psi\circ T^{r} z\big)^q\le 0,\ \ \forall\ \ y,z.
\end{aligned}$$

Hence we have

$$\sum_{0\le r\le m-1}(-1)^{m-1-r}\binom{m-1}{r}d\big(\psi\circ T^{r} y.\ \psi\circ T^{r} z\big)^q=0,\ \ \forall\ \ y,z\in\mathbb{E}.$$

Consequently, T is an $\psi(m-1,q)$-isometry. So the proof is complete.

∎

Lemma 2.9. *([12]) Let m be a non negative integer. Then the following identities hold.*

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}r^j=0 \tag{2.9}$$

for $j=0,1,\cdots,m-1$ and

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}r^m=m!. \tag{2.10}$$

For $n,r\in\mathbb{N}$ we set $n^{(r)}:=\binom{n}{r}r!$.

Proposition 2.10. *Let $T:(\mathbb{E},\ d)\to(\mathbb{E},\ d)$ be a map. Then T is an $\psi(m,q)$-isometry if and only if*

$$d\big(\psi\circ T^ny,\ \psi\circ T^nz\big)^q=\sum_{0\le j\le m-1}\Bigg(\sum_{j\le r\le m-1}(-1)^{r-j}\frac{n^{(r)}}{r!}\binom{r}{j}\Bigg)d\big(\psi\circ T^jy,\ \psi\circ T^jz\big)^q, \tag{2.11}$$

for all $n\in\mathbb{N}$ and $y,z\in\mathbb{E}$.

Proof. Firstly, assume that T is an $\psi(m,q)$-isometric mapping. From (2.6), it follows that

$$d\big(\psi\circ T^ny,\ \psi\circ T^nz\big)^q=\sum_{0\le j\le m-1}\frac{n^{(j)}}{j!}\Theta j,\ q(T,\psi;\ y,z).$$

In view of (2.2), we have

$$\begin{aligned}
&d\big(\psi\circ T^ny,\ \psi\circ T^nz\big)^q=\\
&\sum_{0\le j\le m-1}\frac{n^{(j)}}{j!}\sum_{0\le r\le j}(-1)^{j-r}\binom{j}{r}d\big(\psi\circ T^ry,\ \psi\circ T^rz\big)^q\\
=&\sum_{0\le j\le m-1}\frac{n^{(j)}}{j!}(-1)^j\binom{j}{0}d\big(\psi y,\ \psi z\big)^q+\sum_{1\le j\le m-1}\frac{n^{(j)}}{j!}(-1)^{j-1}\binom{j}{1}d\big(\psi\circ Ty,\ \psi\circ Tz\big)^q+\\
&\cdots+\sum_{m-1\le j\le m-1}\frac{n^{(j)}}{j!}(-1)^{j-m+1}\binom{j}{m-1}d\big(\psi\circ T^{m-1}y,\ \psi\circ T^{m-1}z\big)^q\\
=&\sum_{0\le j\le m-1}\Bigg(\sum_{j\le r\le m-1}(-1)^{r-j}\frac{n^{(r)}}{r!}\binom{r}{j}\Bigg)d\big(\psi\circ T^jy,\ \psi\circ T^jz\big)^q.
\end{aligned}$$

Conversely, assume that (2.11) holds, then we obtain that $n\longmapsto d\big(\psi\circ T^ny,\ \psi\circ T^nz\big)^q$ is a polynomial in n of degree $\le m-1$;

$$d\big(\psi\circ T^jy,\ \psi\circ T^jz\big)^q=p_0+p_1n+\cdots+p_{m-1}n^{m-1}$$

where $p_r=\sum_{0\le j\le r}\beta_jd\big(\psi\circ T^jy,\ \psi\circ T^jz\big)^q$ for $\beta_j\in\mathbb{R}$. Applying (2.9) of Lemma 2.9, we obtain that

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}d\big(\psi\circ T^ry,\ \psi\circ T^rz\big)^q=0.$$

Hence T is an $\psi(m,q)$-isometric mapping. ∎

The following result shows that a power of an $\psi(m,q)$-isometry is again an $\psi(m,q)$-isometry.

Theorem 2.11. *Let $T:(\mathbb{E},\ d)\to(\mathbb{E},\ d)$ be a map such is an $\psi(m,q)$-isometry. Then T^n is an $\psi(m,q)$-isometry for each positive integer n.*

Proof. Assume that T is an $\psi(m,q)$ isometry. From (2.6), it follows that

$$d\big(\psi\circ T^{nr}y,\ \psi\circ T^{nr}z\big)^q=\sum_{0\le j\le m-1}\frac{(nr)^{(j)}}{j!}\Theta_{j,\,q}(T,\psi;\ y,z).$$

By (2.2), it holds

$$\begin{aligned}\Theta_{m,\,q}(T^n,\psi,\ y,z) &= \sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}d\big(\psi\circ T^{nr}y,\ \psi\circ T^{nr}z\big)^q\\ &= \sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}\Bigg(\sum_{0\le j\le m-1}\frac{(nr)^{(j)}}{j!}\Theta_{j,\,q}(T,\psi,\ y,z)\Bigg)\\ &= \sum_{0\le j\le m-1}\frac{1}{j!}\underbrace{\Bigg(\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}(nr)^{(j)}\Bigg)}_{=0\ \text{(by Lemma 2.9)}}\Theta_{j,\,q}(T,\psi,\ y,z)\\ &= 0.\end{aligned}$$

Hence T^n is an $\psi(m,q)$-isometry as desired. ∎

Lemma 2.12. *Let T be a self map on a metric space $(\mathbb{E},\ d)$ is an $\psi(m,q)$-isometry. Then the following identities hold for $n\ge m$ and $y,z\in\mathbb{E}$,*

$$\sum_{0\le r\le n}(-1)^r\binom{n}{r}r^i d\big(\psi\circ T^{n-r}y,\ \psi\circ T^{n-r}z\big)^q=0 \tag{2.12}$$

for $i=0,1,\cdots,n-m$.

Proof. Since T is an $\psi(m,q)$-isometry, it is known that T is an $\psi(n,q)$-isometry for each $n\ge m$. Thus, for $i=0$, (2.12) is immediate. Assume that $i\ge 1$ and prove (2.12) by induction on n. The result is true for $n=m$ (by Proposition 2.10). Suppose that (2.12) is true for $i\in\{1,2,\cdots,n-m\}$ and prove it for $i\in\{1,2,\cdots,n-m+1\}$. By the induction hypothesis, we obtain

$$\begin{aligned}
&\sum_{0\le r\le n+1}(-1)^r\binom{n+1}{r}r^i d\big(\psi\circ T^{n-r+1}y,\ \psi\circ T^{n-r+1}z\big)^q\\
=&\sum_{1\le r\le n+1}(-1)^r\binom{n+1}{r}r^i d\big(\psi\circ T^{n-r+1}y,\psi\circ T^{n-r+1}z\big)^q\\
=&\sum_{0\le r\le n}(-1)^{r+1}\binom{n+1}{r+1}(r+1)^i d\big(\psi\circ T^{n-r}y,\ \psi\circ T^{n-r}z\big)^q\\
=&-(n+1)\sum_{0\le r\le n}(-1)^r\frac{n!}{r!(n-r)!}(r+1)^{i-1}d\big(\psi\circ T^{n-r}y,\ \psi\circ T^{n-r}z\big)^q\\
=&-(n+1)\sum_{0\le r\le n}(-1)^r\binom{n}{r}\Big(\sum_{0\le j\le i-1}\binom{i-1}{j}r^j\Big)d\big(\psi\circ T^{n-r}y,\ \psi\circ T^{n-r}z\big)^q\\
=&-(n+1)\sum_{0\le j\le i-1}\binom{i-1}{j}\Big(\sum_{0\le r\le n}(-1)^r\binom{n}{r}r^j d\big(\psi\circ T^{n-r}y,\ \psi\circ T^{n-r}z\big)^q\Big)\\
=&\ 0.
\end{aligned}$$

∎

Theorem 2.13. *Let T and S be self mappings on a metric space $(\mathbb{E},\ d)$ such that $T\circ S = S\circ T$. Assume that T is an $\psi(m,q)$ isometry and S is an $\psi(n,q)$-isometry, then $T\circ S$ is an $\psi(m+n-1,q)$-isometry..*

Proof. We need to prove that $\Theta_{m+n-1,\,q}(T\circ S,\psi;\ y,z)=0\quad$ for $y,z\in\mathbb{E}$.
In fact, under the assumption that $T\circ S=S\circ T$, we have

$$\begin{aligned}
&\Theta_{m+n-1,\,q}(T\circ S,\psi;\ y,z)=\\
&\sum_{0\le r\le m+n-1}(-1)^r\binom{m+n-1}{r}d\big(\psi\circ(T\circ S)^{m+n-1-r}y,\ \psi\circ(T\circ S)^{m+n-1-r}z\big)^q=\\
&\sum_{0\le r\le m+n-1}(-1)^r\binom{m+n-1}{r}d\big(\psi\circ T^{m+n-1-r}\circ S^{m+n-1-r}y,\ \psi\circ T^{m+n-1-r}S^{m+n-1-r}z\big)^q.
\end{aligned}$$

On the other hand since T is an $\psi(m,q)$-isometry, it follows by Proposition 2.10 that

$$\begin{aligned}
&d\big(\psi\circ T^{m+n-1-r}S^{m+n-1-r}y,\ \psi\circ T^{m+n-1-r}S^{m+n-1-r}z\big)^q=\\
&\sum_{0\le l\le m-1}\Big(\sum_{l\le p\le m-1}(-1)^{p-l}\frac{1}{p!}(m+n-1-r^{(p)}\binom{p}{l}\Big)d\big(\psi\circ T^lS^{m+n-r}y,\ \psi\circ T^lS^{m+n-1-r}z\big)^q.
\end{aligned}$$

By observing that $(m+n-1-r)^{(p)}=\sum_{0\le\alpha\le p}b_\alpha r^\alpha$, we obtain that

$$\begin{aligned}
&\Theta_{m+n-1,\,q}(T\circ S,\psi;\ y,z)=\\
&\sum_{0\le r\le m+n-1}\ \sum_{l\le p\le m-1}\ \sum_{0\le\alpha\le p}b_\alpha(-1)^r\binom{m+n-1}{r}r^\alpha d\big(\psi\circ S^{m+n-1-r}T^ly,\ \psi\circ S^{m+n-1-r}T^lz\big)^q.
\end{aligned}$$

In order to prove that $\Theta_{m+n-1,\,q}(TS,\psi,\ y,z)=0$ it suffices to prove that for $l\in\{0,1,\cdots,m-1\}$ we have

$$\sum_{0\le r\le m+n-1}\sum_{0\le\alpha\le p} b_\alpha(-1)^{r+p-l}\binom{m+n-1}{r}r^\alpha d\big(\psi\circ S^{m+n-1-r}T^l y,\ \psi\circ S^{m+n-1-r}T^l z\big)^q=0.$$

In view of the fact that S is an $\psi(n,q)$-isometry, it follows by Lemma 2.12 that

$$\sum_{0\le r\le m+n-1}(-1)^r\binom{m+n-1}{r}r^\alpha d\big(\psi\circ S^{m+n-1-r}T^l y,\ \psi\circ S^{m+n-1-r}T^l z\big)^q=0$$

for $\alpha\in\{0,1,\cdots,m-1\}$. Therefore $T\circ S$ is an $\psi(m+n-1,q)$-isometry. ∎

Corollary 2.14. *Let T and W be self mappings on a metric space $(\mathbb{E},d)$ such that $T\circ W=W\circ T$. If T is an $\psi(m,q)$-isometry and W-is an $\psi(n,q)$-isometry, then $T^p\circ W^v$ is an $\psi(m+n-1,q)$-isometry for all positive integers p and v.*

Proof. The proof is an consequence of Theorem 2.11 and Theorem 2.13. ∎

Lemma 2.15. *(*[13, Lemma 3.15]*) If $(a_j)_{j\ge0}$ is a sequence of complex numbers and v,u,m,l are positive integers satisfying*

$$\sum_{0\le r\le m}(-1)^r\binom{m}{r}a_{vr+j}=0 \tag{2.13}$$

and

$$\sum_{0\le r\le l}(-1)^r\binom{l}{r}a_{ur+j}=0 \tag{2.14}$$

for all $j\ge0$, then

$$\sum_{0\le r\le p}(-1)^r\binom{p}{r}a_{hr}=0, \tag{2.15}$$

where $h=gcd(v,u)$ and $p=\min(m,l)$.

Theorem 2.16. *Let T be a self map on a metric space $(\mathbb{E},d)$ such that T^r is an $\psi(m,q)$-isometry and T^m is an $\psi(l,q)$-isometry, then T^h is a $\psi(p,q)$-isometry, where h is the greatest common divisor of r and m, and p is the minimum of m and l.*

Proof. Fix $t,z\in\mathbb{E}$ and denote $a_j=d\big(\psi\circ T^j y,\ \psi\circ T^j z\big)^q$ for $j=1,2,\cdots$. As T^r is an $\psi(m,q)$-isometry the sequence $(a_j)_{j\ge0}$ verifies the recursive equation

$$\sum_{0\le r\le m}(-1)^{m-r}\binom{m}{r}a_{rr+j}=0,\ \text{for all } j\ge0.$$

Analogously, as T^s is an $\psi(l,q)$-isometry the sequence $(a_j)_{j\ge0}$ verifies the recursive equation

$$\sum_{0\le r\le l}(-1)^{l-r}\binom{l}{r}a_{rm+j}=0,\quad \text{for all }\ j\ge0.$$

Applying Lemma 2.15 we obtain that

$$\sum_{0\le r\le p}(-1)^{p-r}\binom{p}{r}a_{hr}=0,$$

where h is the greatest common divisor of r and m, and p is the minimum of m and l. Consequently, T^h is an $\psi(p,q)$-isometry. ∎

The following corollary is direct consequence of preceding theorem.

Corollary 2.17. *Let $T : (\mathbb{E}, d) \to (\mathbb{E}, d)$ be a map and let v, u, m, l be positive integers. The following properties hold.*

(1) *If T is an $\psi(m, q)$-isometry such that T^u is an $\psi(1, q)$-isometry , then T is an $\psi(1, q)$-isometry .*

(2) *If T^v and T^{v+1} are $\psi(m, q)$-isometries, then so is T.*

(3) *If T^v is an $\psi(m, q)$-isometry and T^{v+1} is an $\psi(l, q)$-isometry with $m < l$, then T is an $\psi(m, q)$-isometry.*

Lemma 2.18. *Let T be a self map on a metric space $\mathbb{E}$ such is $\psi(2,\ p)$-isometric, then for all integer $r \geq 2$ and $y, z \in \mathbb{E}$, the following identity holds.*

$$d\big(\psi \circ T^r y, \psi \circ T^r z\big)^q - d\big(\psi \circ T^{r-1} y, \psi \circ T^{r-1} z\big)^q = d\big(\psi \circ Ty, \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)^q.$$

Proof. By by induction on r. The identity is obviously true for $r = 2$, since T is an $\psi(2,\ p)$-isometric mapping. Now assume that the identity is true for $r \geq 2$ i.e.;

$$d\big(\psi \circ T^r y, \psi \circ T^r z\big)^q - d\big(\psi \circ T^{r-1} y, \psi \circ T^{r-1} z\big)^q = d\big(\psi \circ Ty, \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)^q,\ \forall\ \ y, z \in \mathbb{E}.$$

Consequently, we obtain the following equality

$$\begin{aligned} & d\big(\psi \circ T^{r+1} y, \psi \circ T^{r+1} z\big)^q - d\big(\psi \circ T^r y, \psi \circ T^r z\big)^q \\ = \ & d\big(\psi \circ T^2 y, \psi \circ T^2 z\big)^q - d\big(\psi \circ Ty, \psi \circ Tz\big)^q \\ = \ & d\big(\psi \circ Ty, \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)^q. \end{aligned}$$

∎

Lemma 2.19. *Let T be a self map on a metric space $(\mathbb{E},\ d)$ wish is $\psi(2,\ q)$-isometric map, then the following statements are true.*

(1) $d\big(\psi \circ T^n t, \psi \circ T^n z\big)^q = n.d\big(\psi \circ Ty,\ \psi \circ Tz\big)^q - (n-1) d\big(\psi y, \psi z\big)^q,\ y, z \in \mathbb{E},\ n = 0, 1, 2, \cdots$

(2) $d\big(\psi \circ Ty,\ \psi \circ Tz\big)^q \geq \dfrac{n-1}{n} d\big(\psi y, \psi z\big)^q,\ \ n \geq 1,\ y, z \in \mathbb{E}.$

(3) $d\big(\psi \circ Ty, \psi \circ Tz\big)^q \geq d\big(\psi y, \psi z\big)^q$ *for all* $y, z \in \mathbb{E}$.

(4) $d\big(\psi \circ Ty, \psi \circ Tz\big) \leq 2^{\frac{1}{q}} d\big(\psi y, \psi z\big)\ \ \forall\ y, z \in \mathcal{R}(T)$ *(the range of T).*

Proof.)(1) Since T is $\psi(2, q)$-isometric map it follows from Lemma 2.18 that

$$d\big(\psi \circ T^{r+1} y, \psi \circ T^{r+1} z\big)^q - d\big(\psi \circ T^r y, \psi \circ T^r z\big)^q = d\big(\psi \circ Ty, \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)^q.$$

This means that

$$\begin{aligned} & d\big(\psi \circ T^n y, \psi \circ T^n z\big)^q \\ = \ & d\big(\psi \circ Ty,\ \psi \circ Tz\big)^q + \sum_{1 \leq r \leq n-1} \Big(d\big(\psi \circ Ty,\ \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)\Big)^q \\ = \ & d\big(\psi \circ Ty,\ \psi \circ Tz\big)^q + (n-1)\big(d\big(\psi \circ Ty, \psi \circ Tz\big)^q - d\big(\psi y, \psi z\big)^q\big) + \\ = \ & n\ d\big(\psi \circ Ty, \psi \circ Tz\big)^q + (1-n) d\big(\psi y, \psi z\big)^q. \end{aligned}$$

(2) Since $d\big(\psi \circ T^n y, \psi \circ T^n z\big)^q \geq 0$ for all $y, z \in \mathbb{E}$, we get

$$d\big(\psi\circ Ty,\psi\circ Tz\big)^q \geq \frac{n-1}{n} d\big(\psi y,\psi z\big)^q.$$

(3) By taking $n \longrightarrow \infty$ in (2) yields (3).

(4) The fact that T is $\psi(2,p)$-isometric gives

$$d\big(\psi\circ T^2 y,\psi\circ T^2 z\big)^q = 2d\big(\psi\circ Ty,\psi\circ Tz\big)^q - d\big(\psi y,\psi z\big)^q \leq 2d\big(\psi\circ Ty,\psi\circ Tz\big)^q.$$

This means that,

$$d\big(\psi\circ T^2 y,\psi\circ T^2 z\big) \leq 2^{\frac{1}{q}} d\big(\psi\circ Ty,\psi\circ Tz\big).$$

∎

3. DISTANCES ASSOCIATED TO $\psi(m,q)$-ISOMETRIES

In this section we introduce some distances related to $\psi(m,q)$-isometries. Our inspiration cames from the papers [4, 6, 19].

let T be an $\psi(m,q)$-isometry, we set $\rho_{_{T,\psi}}(y,z) = \Big(\Theta_{m-1,\,q}(T,\psi;\ y,z)\Big)^{\frac{1}{q}}$ for $y,z\in\mathbb{E}$, $m\geq 1$ and $q\geq 1$.

Proposition 3.1. *If T is an $\psi(m,q)$-isometry, then*

$$\rho_{_{T,\,\psi}}(y,z) = \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)}{\sqrt[q]{n^{(m-1)}}}. \tag{3.1}$$

Moreover $\rho_{_{T,\psi}}$ is a semi-distance on $\mathbb{E}$.

Proof. Under the assumption that T is an $\psi(m,q)$-isometry, we have from the statement (2) of Theorem 2.7

$$d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)^q = \sum_{0\leq r\leq m-1}\binom{n}{r}\Theta_{m-1,\,q}(T,\psi;\ y,z).$$

Note that the map $n\mapsto\binom{n}{r}$ is polynomial in n of degree r and $\Theta_{r,\,q}(T,\psi;\ \ y,z) = 0$ for $r>m-1$. Therefore

$$\Theta_{m-1,\,q}(T,\psi;\ y,z) = \lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)^q}{\binom{n}{m-1}} = (m-1)!\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)^q}{n^{(m-1)}}.$$

This means that

$$\rho_{_{T,\,\psi}}(y,z) = \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)}{\sqrt[q]{n^{(m-1)}}}. \tag{3.2}$$

To show that $\rho_{_{T,\,\psi}}$ is a semi-metric, firstly, we observe that $\rho_{_{T,\,\psi}}(y,z)\geq 0$, by the statement (3) of Theorem 2.7. Clearly $\rho_{_{T,\,\psi}}(y,y)=0$ and $\rho_{_{T,\,\psi}}(y,z)=\rho_{_{T,\,\psi}}(z,y)\ \forall\ y,z\in\mathbb{E}$.

Next to prove the triangle inequality, we have for $y, z, z \in \mathbb{E}$,

$$\begin{aligned}
\rho_{_{T,\,\psi}}(y,z) &= \Theta_{m-1,\,q}(T,\psi;\ y,z)^{\frac{1}{q}} \\
&= \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)}{\sqrt[q]{n^{(m-1)}}} \\
&\leq \sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n y,\ \psi\circ T^n z\big)}{\sqrt[q]{n^{(m-1)}}} \\
&\quad +\sqrt[q]{(m-1)!}\lim_{n\longrightarrow\infty}\frac{d\big(\psi\circ T^n z,\ \psi\circ T^n z\big)}{\sqrt[q]{n^{(m-1)}}} \\
&= \rho_{_{T,\,\psi}}(y,z)+\rho_{_{T,\,\psi}}(z,z).
\end{aligned}$$

∎

Remark 3.1. In view of Proposition 2.2, if T is an $\psi(m,q)$-isometry, then

$$\Theta_{m-1,\,q}(T,\psi;\ y,z)=\Theta_{m-1,\,q}(T,\psi;\ Ty,Tz).$$

This means that $\rho_{_{T,\,\psi}}(y,z)=\rho_{_{T,\,\psi}}(Ty,Tz)$ and therefore

$$T:(\mathbb{E},\ \rho_{_{T,\,\psi}})\longrightarrow(\mathbb{E},\rho_{_{T,\,\psi}}),$$

is an isometry.

By observing that

$$\begin{aligned}
&\Theta_{m,\,q}(T,\psi;\ y,z)= \\
&\sum_{0\leq r\leq m}(-1)^r\binom{m}{r}d\big(\psi\circ T^{m-r}y,\ \psi\circ T^{m-r}z\big)^q \\
=&\sum_{\substack{0\leq r\leq m\\ r\ \text{(even)}}}\binom{m}{r}d\big(\psi\circ T^{m-r}y,\ \psi\circ T^{m-r}z\big)^q \\
&-\sum_{\substack{0\leq r\leq m\\ r\ \text{(odd)}}}\binom{m}{r}d\big(\psi\circ T^{m-r}y,\ \psi\circ T^{m-r}z\big)^q \\
=&\sum_{\substack{0\leq r\leq m\\ r\ \text{(even)}}}\binom{m}{r}d\big(\psi\circ T^{m-r}y,\ \psi\circ T^{m-r}z\big)^q \\
&-\sum_{\substack{0\leq r\leq m\\ r\ \text{(odd)}}}\binom{m}{r}d\big(\psi\circ T^{m-r-1}Ty,\ \psi\circ T^{m-r-1}Tz\big)^q \\
=&\ \widetilde{\rho_{_{T,\,\psi}}}(y,z)-\widetilde{\rho_{_{T,\,\psi}}}'(Ty,Tz).
\end{aligned}$$

Lemma 3.2. *If ψ is a injective self map on $\mathbb{E}$, then $(\mathbb{E},\widetilde{\rho_{_{T,\,\psi}}})$ and $(\mathbb{E},\widetilde{\rho_{_{T,\,\psi}}}')$ are both metric space.*

Theorem 3.3. *Let $T:(E,d)\to(E,d)$ be a map and $q\geq 1$. If ψ is injective, then following statements are equivalent.*

(1) $T:(\mathbb{E},\ d)\to(\mathbb{E},\ d)$ *is an* $\psi(m,q)$*-isometry*

(2) $T:\left(\mathbb{E},\ \widetilde{\rho_{T,\,\psi}}\right)\to\left(\mathbb{E},\ \widetilde{\rho_{T,\,\psi}}'\right)$ *is an isometry.*

Proof. In view of Proposition 2.2 it follows that,

$$\begin{aligned}
& T \text{ is an } \psi(m,q)\text{-isometry} \\
\Leftrightarrow\quad & \sum_{\substack{0\le r\le m \\ r\ (\text{even})}} \binom{m}{r} d\big(\psi\circ T^{m-r}y,\ \psi\circ T^{m-r}z\big)^{q} \\
& = \sum_{\substack{0\le r\le m \\ r\ (\text{odd})}} \binom{m}{r} d\big(\psi\circ T^{m-r-1}Ty,\ \psi\circ T^{m-r-1}Tz\big)^{q},\ \forall\, y,z\in\mathbb{E}, \\
\Leftrightarrow\quad & \widetilde{\rho_{T,\,\psi}}(y,z)=\widetilde{\rho_{T,\,\psi}}'(Ty,Tz),\ \forall\, y,z\in\mathbb{E} \\
\Leftrightarrow\quad & T \text{ is an isometry.}
\end{aligned}$$

∎

4. CONCLUSION

In this study, some properties of m-isometries of Hilbert and Banach spaces operators are characterized for m-isometries for mappings on general metric spaces.

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