Aust. J. Math. Anal. Appl.
Vol. **17** (2020), No. 2, Art. 18, 10 pp.
AJMAA

ESTIMATES OF NORMS ON KREIN SPACES

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Received 6 May, 2020; accepted 9 October, 2020; published 21 December, 2020.

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ABSTRACT. Various norms can be defined on a Krein space by choosing different underlying fundamental decompositions. Some estimates of norms on Krein spaces are discussed and a few results in Bognar's paper are generalized.

Key words and phrases: Krein space ; Fundamental decomposition; J-norm.

2010 *Mathematics Subject Classification.* 46C05, 46C20.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $\mathcal{K}$ be a complex vector space with a Hermitian sesquilinear form defined on it. Then we call $(\mathcal{K}, [.,.])$ an inner product space. An element $x \in \mathcal{K}$ is called neutral, positive, or negative if $[x, x] = 0$, $[x, x] > 0$, or $[x, x] < 0$ respectively. If $\mathcal{K}$ contains positive as well as negative elements, then it is called an indefinite inner product space, otherwise it is called a semi-definite inner product space. We refer [3, 4] for basics on indefinite inner product spaces. The concept of indefinite inner product was first found in a paper on quantum field theory by Dirac in 1942 [6]. Pontrjagin gave the mathematical interpretation of indefinite inner product. Bognar [2], Hansen [7], Langer [1] et al. have investigated the notion of norm in indefinite inner product spaces.

An indefinite inner product space $(\mathcal{K}, [.,.])$ is decomposable if it can be written as an orthogonal direct sum of a neutral subspace $\mathcal{K}^0$, a positive definite subspace $\mathcal{K}^+$ and a negative definite subspace $\mathcal{K}^-$:

$$\mathcal{K} = \mathcal{K}^0[\dot{+}]\mathcal{K}^+[\dot{+}]\mathcal{K}^-. \tag{1.1}$$

Then (1.1) is known as a fundamental decomposition of $\mathcal{K}$.

An indefinite inner product space $(\mathcal{K}, [.,.])$ is a Krein space if it can be written as an orthogonal direct sum of a positive definite subspace $\mathcal{K}^+$ and a negative definite subspace $\mathcal{K}^-$ such that $(\mathcal{K}^+, [.,.])$ and $(\mathcal{K}^-, -[.,.])$ are Hilbert spaces. Let a fundamental decomposition of a Krein space $\mathcal{K}$ be given by

$$\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^- \tag{1.2}$$

and $P^{\pm}$ be the orthogonal projections onto $\mathcal{K}^{\pm}$. The linear map

$$J = P^+ - P^-,$$

is called the fundamental symmetry corresponding to (1.2). Then

$$(x, y)_J = [Jx, y]$$

is a positive definite inner product on $\mathcal{K}$, called J-inner product corresponding to the fundamental decomposition (1.2). We can write

$$(x, x)_J = [Jx, x] = [(2P^+ - I)x, x] = 2[P^+x, P^+x] - [x, x]. \tag{1.3}$$

The corresponding norm (called J-norm) is denoted by

$$\|x\|_J = (x, x)_J^{\frac{1}{2}} = [Jx, x]^{\frac{1}{2}}.$$

Different fundamental decompositions induce different J-norms. Hence various norms can be defined on a Krein space by choosing different underlying fundamental decompositions.

A different fundamental decomposition of $\mathcal{K}$ say, $\mathcal{K} = \mathcal{K}_1^{+'}[\dot{+}]\mathcal{K}_2^{-'}$ makes the norm of an element x larger than $|[x, x]|$. Roughly speaking, if the spaces $\mathcal{K}_1^{+'}$ and $\mathcal{K}_2^{-'}$ "come closer" to a neutral set $\mathcal{K}^0$, these norms in general be unbounded. It is interesting to observe that how the norm of a single element actually depends upon the choice of fundamental decomposition [1]. We end the section with some examples. In the second section, some preliminary results are given which will be used in the sequel. The third section contains our main results concerning estimates of norms on Krein spaces.

Example 1.1. [8] *Minkowski space M^{n+1} is defined as the set of $(n+1)$-dimensional column vectors $x = (x_1, x_2, \ldots, x_n, t_1)^t$ (t indicates the transpose of a matrix) with complex entries and the indefinite inner product is defined by*

$$[x, y] = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n} - t_1\overline{t_2}$$

where $x = (x_1, x_2, \ldots, x_n, t_1)^t$, $y = (y_1, y_2, \ldots, y_n, t_2)^t \in M^{n+1}$. *Then* M^{n+1} *is a Krein space. A fundamental symmetry for the space is given by the matrix*

$$\begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

where I_n *denotes the identity matrix of order* n.

Example 1.2. *Consider* $\mathcal{K} = \ell_2$, *the linear space of square-summable sequences, with*

$$[x, y] = \sum_{i=1}^{\infty} (-1)^i x_i \overline{y_i} \quad \textit{for} \;\; x = (x_i)_{i=1}^{\infty},\; y = (y_i)_{i=1}^{\infty} \in \mathcal{K}.$$

Let $\mathcal{K}^+ = \{(x_i)_{i=1}^{\infty} : x_i = 0 \text{ if } i \text{ is odd}\}$ *and* $\mathcal{K}^- = \{(x_i)_{i=1}^{\infty} : x_i = 0 \text{ if } i \text{ is even}\}$. *Then* $\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$, *where* $\mathcal{K}^+$ *and* $\mathcal{K}^-$ *are complete with respect to the induced norm and hence* $\mathcal{K}$ *is a Krein space.*

Example 1.3. *Consider* $\mathcal{K} = C[-1, 1]$ *the linear space of all complex-valued continuous functions defined on the interval* $[-1, 1]$ *with*

$$[x, y] = \int_{-1}^{1} x(t)\overline{y(-t)}dt \quad \textit{for} \;\; x, y \in \mathcal{K}.$$

Then $\mathcal{K}$ *admits a fundamental decomposition* $\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$ *where* $\mathcal{K}^+$ *and* $\mathcal{K}^-$ *the spaces of all continuous even and odd functions on* $[-1, 1]$ *respectively, are complete with respect to the induced norm and hence* $\mathcal{K}$ *is a Krein space.*

Example 1.4. [5] *Let* Ω *be a set and* Σ *be a* σ*-algebra on* Ω. *Let* μ_+ *and* μ_- *be two mutually singular positive measures defined on* Σ. *Set* $\mu = \mu_+ + \mu_-$. *Define*

$$[f, g] = \int_{\Omega} f\overline{g}d\mu_+ - \int_{\Omega} f\overline{g}d\mu_- \quad \textit{for} \;\; f, g \in L^2(\mu).$$

Then $L^2(\mu) = L^2(\mu_+)[\dot{+}]L^2(\mu_-)$ *forms a fundamental decomposition, since* $(L^2(\mu_+), [., .])$ *and* $(L^2(\mu_-), -[., .])$ *are Hilbert spaces. Thus* $L^2(\mu)$ *is a Krein space.*

2. PRELIMINARIES

Theorem 2.1. [4] *Let* $\mathcal{K}$ *be a Krein space. Then* $\mathcal{K}$ *has several fundamental decompositions with non-zero components. All norms induced by different fundamental decompositions are equivalent and hence they induce the same topology.*

Theorem 2.2. [3] *Let* $(\mathcal{K}, [., .])$ *be a Krein space. Then the following are equivalent:*

(1) *There exists a fundamental decomposition of* $\mathcal{K}$.
(2) *There exists a maximal uniformly positive ortho-complemented subspace.*
(3) *There exists a maximal uniformly negative ortho-complemented subspace.*

Example 2.1. *Let* $\mathcal{K}$ *be a two-dimensional vector space with basis* $\{e_1, e_2\}$ *and an indefinite inner product defined by* $[e_1, e_1] = 1, [e_2, e_2] = -1$ *and* $[e_1, e_2] = 0$. *If we take* $Y = \text{span}\{e_1\}$, *then it is a maximal uniformly positive definite subspace and hence there exists a fundamental decomposition of* $\mathcal{K}$ *with* $\mathcal{K}^+ = Y$ *and* $\mathcal{K}^- = \text{span}\{e_2\}$. *Choosing* $\mathcal{K}_n^+ = \text{span}\{(n, 1)\}$ *and* $\mathcal{K}_n^- = \text{span}\{(1, n)\}$ *where* $n > 1$, *we get several fundamental decompositions. The corresponding fundamental symmetries* J_n *are given by*

$$J_n = \begin{pmatrix} \frac{n^2+1}{n^2-1} & \frac{-2n}{n^2-1} \\ \frac{2n}{n^2-1} & \frac{-(n^2+1)}{n^2-1} \end{pmatrix}.$$

Here we can see that the fundamental symmetries J_n satisfy $J_n^2 = I$, $[J_n x, y] = [x, J_n y]$ and $[J_n x, J_n y] = [x, y]$ for all $x, y \in \mathcal{K}$.

3. MAIN RESULTS

Theorem 3.1. [1] *Assume that $\mathcal{K}$ is a Krein space such that $[.,.]$ is indefinite and let $x \in \mathcal{K}$, $x \neq 0$. Then the following holds.*

(i) *If $[x,x] \neq 0$, then*

$$\left\{ \|x\|_J : J \text{ is a fundamental symmetry} \right\} = [|[x,x]|^{\frac{1}{2}}, \infty). \tag{3.1}$$

Moreover,

$$\|x\|_J = |[x,x]|^{\frac{1}{2}} \quad \text{if and only if} \quad x \in \mathcal{K}_J^+ \cup \mathcal{K}_J^-,$$

where $\mathcal{K} = \mathcal{K}_J^+ [\dot{+}] \mathcal{K}_J^-$ is the fundamental decomposition associated with J.

(ii) *If $[x,x] = 0$, then*

$$\left\{ \|x\|_J : J \text{ is a fundamental symmetry} \right\} = (0, \infty).$$

Theorem 3.2. *Assume that $(\mathcal{K}, [.,.])$ is a Krein space and let $0 \neq x \in \mathcal{K}$.*

(a) *If $[x,x] \neq 0$, then for each real number $a > |[x,x]|^{\frac{1}{2}}$ there exists a fundamental symmetry J_a such that $\|x\|_{J_a} = a$.*

(b) *If $[x,x] = 0$, then for each positive real number a there exists a fundamental symmetry J_a such that $\|x\|_{J_a} = a$.*

Proof. (a) Let $[x,x] < 0$. Let $\mathcal{K} = \mathcal{M}^+ [\dot{+}] \mathcal{M}^-$ be a fundamental decomposition such that $x \in \mathcal{M}^-$. Choose $0 \neq y \in \mathcal{M}^+$ and let $\mathcal{L}^+$ and $\mathcal{L}^-$ be subspaces such that

$$\mathcal{M}^+ = \mathcal{L}^+ [\dot{+}] \text{span}\{y\} \qquad\qquad \mathcal{M}^- = \mathcal{L}^- [\dot{+}] \text{span}\{x\}.$$

Consider $u(s) = sy + (1-s)x$, $s \in [0,1]$. We have $[u(0), u(0)] < 0$, $[u(1), u(1)] > 0$ and $[.,.]$ is continuous. Hence there exists $s_0 \in (0,1)$ such that $[u(s_0), u(s_0)] = 0$. Let $z = u(s_0)$, then

$$[z,z] = 0, \qquad [y,z] = s_0[y,y] > 0, \qquad [z,x] = (1-s_0)[x,x] < 0.$$

Let $v(t) = ty + (1-t)z$, $t \in (0,1]$, which is a positive element for $t \in (0,1]$. Now set

$$\mathcal{K}_t^+ = \mathcal{L}^+ [\dot{+}] \text{span}\{v(t)\}$$

since y and z are orthogonal to $\mathcal{L}^+$. Thus the orthogonal projection P_t^+ onto $\mathcal{K}_t^+$ can be written as $P_t^+ = P_{\mathcal{L}_+} + P_{v(t)}$, where $P_{\mathcal{L}_+}$ is the orthogonal projection onto $\mathcal{L}_+$ and $P_{v(t)}$ is the orthogonal projection onto span$\{v(t)\}$. We also have

$$P_{v(t)} u = \frac{[u, v(t)]}{[v(t), v(t)]} v(t) \tag{3.2}$$

for any $u \in \mathcal{K}$. Let $w(t)$ be a non-zero element in span$\{y, x\}$ which is orthogonal to $v(t)$ and hence negative. With $\mathcal{K}_t^- = \mathcal{L}^- [\dot{+}] \text{span}\{w(t)\}$, we have a fundamental decompositon $\mathcal{K} = \mathcal{K}_t^+ [\dot{+}] \mathcal{K}_t^-$ and a corresponding fundamental symmetry $J_t = 2P_t^+ - I$. Now we get

$$[P_t^+ x, P_t^+ x] = [P_t^+ x, x] = \frac{|[x, v(t)]|^2}{[v(t), v(t)]} = \frac{(1-t)^2 |[x,z]|^2}{t^2 [y,y] + 2t(1-t)[y,z]}. \tag{3.3}$$

From equation (1.3) we get,

$$(x,x)_{J_t} = \frac{2(1-t)^2 |[x,z]|^2}{t^2[y,y] + 2t(1-t)[y,z]} - [x,x]. \tag{3.4}$$

The construction of $(x,x)_{J_t}$ in equation (3.4) is taken from the proof of the Theorem (3.1). The details are given for the sake of completeness of the proof.

As t varies in (0,1], $(x,x)_{J_t}$ takes all values in $[|[x,x]|,\infty)$. Thus $\|x\|_{J_t}$ takes all values in $[|[x,x]|^{\frac{1}{2}},\infty)$. Let $a \in [|[x,x]|^{\frac{1}{2}},\infty)$ be such that $a^2 = b > |[x,x]|$. Now let us try to find $t \in (0,1]$ for which $(x,x)_J = b$ so that $\|x\|_J = a$.
From (3.4) and $(x,x)_J = b$ we get,

$$\text{(3.5)} \qquad \frac{2(1-t)^2|[x,z]|^2}{t^2[y,y]+2t(1-t)[y,z]} - [x,x] = b.$$

We have $[x,x] < 0$, $[y,y] > 0$ and $[y,z] > 0$. So let $h = [y,y]$. Replacing y by $\frac{y}{\sqrt{h}}$ we get $[y,y] = 1$. Now set $A = |[x,z]|^2$, $B = [y,y] = 1$, $C = [y,z]$, $D = [x,x]$. Thus from (3.4) we get $\frac{2(1-t)^2A}{t^2+2t(1-t)C} - D = b$ which implies $2A(1-2t+t^2) = (b+D)(t^2-2Ct^2+2Ct)$ so that

$$t^2[(b+D)(1-2C)-2A] + t[(b+D)2C+4A] - 2A = 0,$$

which is a quadratic equation in t whose discriminant is $4C^2(b+D)^2 + 8A(b+D)$, which is positive as $b+D$ and A are positive. Thus there exists $t \in (0,1]$ such that it satisfies equation (3.5). Let us denote it by t_b. Then the subspaces

$$\mathcal{K}_t^+ = \mathcal{L}^+[\dot{+}]\text{span}\{v(t_b)\},\ \mathcal{K}_t^- = \mathcal{L}^-[\dot{+}]\text{span}\{w(t_b)\}$$

give a fundamental symmetry corresponding to t_b. We denote it by J_a and hence we see that $(x,x)_{J_a} = b$ and $\|x\|_{J_a} = a$.
Let $[x,x] > 0$. Then choose a fundamental decomposition $\mathcal{K} = \mathcal{M}^+[\dot{+}]\mathcal{M}^-$ such that $x \in \mathcal{M}^+$ and continue the proof as discussed above.

(b) Let $[x,x] = 0$. Let y be another neutral element that satisfies $[x,y] = 1$. Define $u = \frac{1}{\sqrt{2}}(x+y)$, $v = \frac{1}{\sqrt{2}}(x-y)$. Then $x = \frac{1}{\sqrt{2}}(u+v)$, $[u,u] = 1$, $[v,v] = -1$ and $[u,v] = 0$. Let $\mathcal{K} = \mathcal{M}^+[\dot{+}]\mathcal{M}^-$ be a fundamental decomposition such that $\mathcal{M}^+ = \mathcal{L}^+[\dot{+}]\text{span}\{u\}$, $\mathcal{M}^- = \mathcal{L}^-[\dot{+}]\text{span}\{v\}$ with some subspaces $\mathcal{L}^\pm$. Set

$$w(t) = u + tv, t \in (-1,1).$$

Then $[w(t),w(t)] = (1-t^2) > 0$, $t \in (-1,1)$. Hence $\mathcal{K}_{t,+} = \mathcal{L}^+[\dot{+}]\text{span}\{w(t)\}$ is a maximal uniformly positive subspace. Now the projection $P_{t,+}$ onto $\mathcal{K}_{t,+}$ can be written as $P_{t,+} = P_{\mathcal{L}_+} + P_{w(t)}$ and we get

$$[P_{t,+}x, P_{t,+}x] = [P_{t,+}x, x] = \frac{|[x,w(t)]|^2}{[w(t),w(t)]} = \frac{1-t}{2(1+t)},$$

which takes all the values in $(0,\infty)$ if t varies in $(-1,1)$. Let $a \in (0,\infty)$ be such that $a^2 = b$. Thus from (1.3) and solving $(x,x)_{J_t} = b$ we get $2[P_{t,+}x, P_{t,+}x] = b$. That is, $\frac{2(1-t)}{2(1+t)} = b$ and hence $t = \frac{1-b}{1+b}$. Thus for $t_b = \frac{1-b}{1+b}$,

$$\mathcal{K}_{t_b}^+ = \mathcal{L}^+[\dot{+}]\text{span}\{w(t_b)\}$$

is a maximal uniformly positive subspace and hence there exists a fundamental decomposition. We denote the corresponding fundamental symmetry by J_a. Hence we get that $(x,x)_{J_a} = b$ and $\|x\|_{J_a} = a$. ∎

Corollary 3.3. *Let $0 \neq x \in \mathcal{K}$, $[x,x] \neq 0$ and J be a given fundamental symmetry. Then there exists a fundamental symmetry K such that*

$$\|x\|_J < \|x\|_K.$$

Proof. Choose a positive real number $k > \|x\|_J$. Then by Theorem 3.2(a) there exists a fundamental symmetry K such that $\|x\|_K = k$, which is greater than $\|x\|_J$ and that implies $\|x\|_J < \|x\|_K$. ∎

Corollary 3.4. *Let $0 \neq x \in \mathcal{K}$ be a neutral element and J be a given fundamental symmetry. Then there exist fundamental symmetries K_1 and K_2 such that*

$$\|x\|_{K_1} < \|x\|_J < \|x\|_{K_2}.$$

Proof. Choose positive real numbers k_1, k_2 such that $0 < k_1 < \|x\|_J < k_2$. Then by Theorem 3.2 (b) there exist fundamental symmetries K_1 and K_2 such that $\|x\|_{K_1} < \|x\|_J < \|x\|_{K_2}$. ∎

Corollary 3.5. *Let x be any arbitrary non-zero (neutral or non-neutral) element in $\mathcal{K}$. If $\|x\|_{J_1} < \|x\|_{J_2}$ for some fundamental symmetries J_1 and J_2, then there exists a fundamental symmetry J such that $\|x\|_{J_1} < \|x\|_J < \|x\|_{J_2}$.*

Proof. Choose $a > |[x,x]|^{\frac{1}{2}}$ such that $\|x\|_{J_1} < a < \|x\|_{J_2}$. Then by Theorem 3.2 the result follows. ∎

Corollary 3.6. *Let $\mathcal{K}$ be a Krein space with a fundamental decomposition and a corresponding fundamental symmetry J. Let (x_n) be a sequence of non-zero neutral or non-neutral vectors such that (x_n) converges to some x in $\mathcal{K}$. Then there exists a sequence of fundamental symmetries (J_n) such that $\|x_n\|_{J_n} \to \infty$.*

Proof. Let $J_1 = J$. Now choose a real number $a_2 > \max\{|[x_2, x_2]|^{\frac{1}{2}}, \|x_1\|_{J_1}\}$. Then by Theorem 3.2 (a) there exists a fundamental symmetry J_2 such that $\|x_2\|_{J_2} = a_2 > \|x_1\|_{J_1}$. In a similar way, we can find J_k by choosing

$$a_k > \max\left\{|[x_k, x_k]|^{\frac{1}{2}}, \|x_{k-1}\|_{J_{k-1}}\right\}$$

and by using Theorem 3.2 (a) we get J_k such that $\|x_k\|_{J_k} = a_k > \|x_{k-1}\|_{J_{k-1}}$. Continuing the process we see that $\|x_n\|_{J_n} \to \infty$ as $n \to \infty$. ∎

Corollary 3.7. *Let (x_n) be a sequence of non-zero neutral elements in $\mathcal{K}$. Then there exists a sequence of fundamental symmetries (J_n) such that $\|x_n\|_{J_n} \to 0$ as $n \to \infty$.*

Proof. We have $x_1 \neq 0$, then we can find a fundamental symmetry J_1 such that $\|x_1\|_{J_1} > 0$. Choose a real number a_2 such that $\|x_1\|_{J_1} > a_2 > 0$. Then by Theorem 3.2 (b) there exists a fundamental symmetry J_2 such that $\|x_2\|_{J_2} = a_2$ and so we get $\|x_1\|_{J_1} > \|x_2\|_{J_2}$. Choose a real number a_k such that

$$\|x_{k-1}\|_{J_{k-1}} > a_k > 0.$$

Then by Theorem 3.2 (b) there exists a fundamental symmetry J_k such that $\|x_{k-1}\|_{J_{k-1}} > \|x_k\|_{J_k}$. Thus we see that $(\|x_n\|_{J_n})$ is a decreasing sequence which is bounded below by 0 and hence $\|x_n\|_{J_n} \to 0$ as $n \to \infty$. ∎

Remark 3.1. Corollaries 3.6 and 3.7 generalize the Lemma in [2] which says for a non-neutral element x there exists a sequence of fundamental norms (p_n) such that $p_n(x) \to \infty$ as $n \to \infty$ and for a neutral element x there exist sequences of fundamental norms (p_n) and (q_n) such that $p_n(x) \to \infty$ and $q_n(x) \to 0$ as $n \to \infty$.

Example 3.1. *Consider the fundamental symmetry J_n given in Example 2.1. Then for $x = (x_1, y_1) \in \mathcal{K}$*

$$\|x\|_{J_n}^2 = \frac{(n^2+1)((x_1^2 + y_1^2) - 4nx_1y_1}{n^2 - 1}.$$

We fix $x = (2,1)$. Then $[x,x] = 3$. Let $a = 2 > |[x,x]|^{\frac{1}{2}}$. By solving $\|x\|_{J_n}^2 = 4$, we see that n equals to the positive square root of the equation $5n^2 - 8n - 7 = 0$.

Theorem 3.8. *Let $\mathcal{K}$ be a Krein space. Then the following are true.*

(a) *Let $0 \neq x \in \mathcal{K}$ be a non-neutral element, $\alpha \in \mathbb{C}$. Then for every ε such that $\varepsilon > |[x,x]|^{\frac{1}{2}}|1-|\alpha||$, there exists a fundamental symmetry J such that $|\|x\|_J - \|\alpha x\|_J| < \varepsilon$.*

(b) *Let $0 \neq x \in \mathcal{K}$ be a neutral element, $\alpha \in \mathbb{C}$. Then for every $\varepsilon > 0$, there exists a fundamental symmetry J such that $|\|x\|_J - \|\alpha x\|_J| < \varepsilon$.*

Proof. Suppose $|\alpha| = 1$, the result is trivial. Next we asssume that $|\alpha| \neq 1$.

(a) We have $\varepsilon > |[x,x]|^{\frac{1}{2}}|1-|\alpha||$, which implies $|[x,x]|^{\frac{1}{2}} < \frac{\varepsilon}{|1-|\alpha||}$. Let $c \in \mathbb{R}$ be such that $|[x,x]|^{\frac{1}{2}} < c < \frac{\varepsilon}{|1-|\alpha||}$. Then by Theorem 3.2 there exists a fundamental symmetry J such that $\|x\|_J = c$ which implies $\|x\|_J < \frac{\varepsilon}{|1-|\alpha||}$ so that we get $|\|x\|_J - \|\alpha x\|_J| < \varepsilon$.

(b) We have $|1-|\alpha|| > 0$, which implies $\frac{\varepsilon}{|1-|\alpha||} > 0$. Let $c \in \mathbb{R}$ be such that $0 < c < \frac{\varepsilon}{|1-|\alpha||}$. Then by Theorem 3.2 there exists a fundamental symmetry J such that $\|x\|_J = c$ which implies $\|x\|_J < \frac{\varepsilon}{|1-|\alpha||}$ so that we get $|\|x\|_J - \|\alpha x\|_J| < \varepsilon$.

∎

Theorem 3.9. *Let x and y be orthogonal non-neutral elements of a Krein space $\mathcal{K}$ with a fundamental decomposition $\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$. If x and y are linearly independent and if*

$$dim(\mathcal{K}^+) > 1, dim(\mathcal{K}^-) > 0, [y,y] > 0 \tag{3.6}$$

or

$$dim(\mathcal{K}^-) > 1, dim(\mathcal{K}^+) > 0, [y,y] < 0 \tag{3.7}$$

then there exists a sequence of fundamental symmetries (J_n) such that $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.

Proof. The case (3.7) can be reduced to (3.6) by passing to the inner product $[u,v]' = -[u,v]$ where $u, v \in \mathcal{K}$. Thus we consider only the case (3.6). From the hypothesis, we can find at least two positive elements x_1, x_2 and a negative element y_1 in $\mathcal{K}$ such that

$$\mathcal{K} = \mathcal{L}_1^+[\dot{+}]\text{span}\{x_1\}[\dot{+}]\text{span}\{x_2\}[\dot{+}]\mathcal{L}_2^-[\dot{+}]\text{span}\{y_1\},$$

where $\mathcal{L}_1^+$ and $\mathcal{L}_2^-$ are positive and negative subspaces respectively.

We now first discuss the case when $[x,x] > 0$. Choose $x_1 = y$ and and $x_2 = \frac{x}{\sqrt{[x,x]}}$ so that $[x_2,x_2] = 1$ and choose y_1 such that $[y_1,y_1] = -1$. We can find a neutral element $e_1 = s_0x_2 + (1-s_0)y_1$ for some $s_0 \in (0,1)$. Take $v(t_n) = t_nx_2 + (1-t_n)e_1$ where $t_n = \frac{1}{n}, n > 1$. Then $[v(t_n), v(t_n)] = {t_n}^2 + 2s_0t_n(1-t_n) > 0$ and $[v(t_n), x_1] = 0$. Set

$${\mathcal{K}_n}^+ = \mathcal{L}_1[\dot{+}]\text{span}\{x_1\}[\dot{+}]\text{span}\{v(t_n)\}.$$

Thus the orthogonal projection ${P_n}^+$ onto ${\mathcal{K}_n}^+$ can be written as

$${P_n}^+ = P_{\mathcal{L}_1} + P_{x_1} + P_{v(t_n)}$$

where $P_{\mathcal{L}_1}$ is the orthogonal projection onto $\mathcal{L}_1$, P_{x_1} is the orthogonal projection onto span$\{x_1\}$ and $P_{v(t_n)}$ is the orthogonal projection onto span$\{v(t_n)\}$, which has the form

$$P_{v(t_n)}z = \frac{[z, v(t_n)]}{[v(t_n), v(t_n)]}v(t_n).$$

Choosing a non-zero element $u(t_n)$ in the span$\{x_2, y_1\}$, which is orthogonal to $v(t_n)$, we get a fundamental decompositon with ${\mathcal{K}_n}^+$ and ${\mathcal{K}_n}^- = \mathcal{L}_2[\dot{+}]\text{span}\{u(t_n)\}$ and a corresponding fundamental symmetry $J_n = 2{P_n}^+ - I$. For a vector $z \in \mathcal{K}$ we have

$$\|z\|_{J_n}^2 = [J_nz, z] = [(2{P_n}^+ - I)z, z] = 2[{P_n}^+z, z] - [z,z].$$

Let us calculate $\|y\|_{J_n}^2$. Since $x_1 = y$ we have $P_n{}^+y = P_{x_1}y = y$. Thus $\|y\|_{J_n} = [y,y]^{\frac{1}{2}}$ for all $n > 2$. Let us find $\|x\|_{J_n}$.We have

$$P_n{}^+x_2 = P_{v(t_n)}x_2 = \frac{[x_2, v(t_n)]}{[v(t_n), v(t_n)]}v(t_n) = \frac{t_n + s_0(1-t_n)}{t_n{}^2 + 2s_0t_n(1-t_n)}v(t_n),$$

which implies

$$\begin{aligned}[P_n{}^+x_2, x_2] &= \frac{t_n + s_0(1-t_n)}{t_n{}^2 + 2s_0t_n(1-t_n)}[v(t_n), x_2] \\ &= 1 + \frac{s_0{}^2}{t_n{}^2 + 2s_0t_n(1-t_n)} + \frac{t_n - 2}{t_n{}^2 + 2s_0t_n(1-t_n)}.\end{aligned}$$

And hence $\|x\|_{J_n} = |\sqrt{[x,x]}|\|x_2\|_{J_n} \to \infty$ as $n \to \infty$. Thus $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.

We now discuss the case when $[x,x] < 0$. We take $y_1 = \frac{x}{\sqrt{|[x,x]|}}$ so that $[y_1, y_1] = -1$. Choose x_2 such that $[x_2, x_2] = 1$. Proceeding as above we find $\|y\|_{J_n} = [y,y]^{\frac{1}{2}}$ for all $n > 2$ and $\|y_1\|_{J_n}$ as follows. We have

$$P_n{}^+y_1 = P_{v(t_n)}y_1 = \frac{[y_1, v(t_n)]}{[v(t_n), v(t_n)]}v(t_n) = \frac{t_n + s_0(1-t_n)}{t_n{}^2 + 2s_0t_n(1-t_n)}v(t_n).$$

which implies

$$\begin{aligned}[P_n{}^+y_1, y_1] &= \frac{s_0 - 1}{t_n{}^2 + 2s_0t_n(1-t_n)}[v(t_n), y_1] \\ &= 1 + \frac{s_0 - 1^2}{t_n{}^2 + 2s_0t_n(1-t_n)} + \frac{t_n - 2}{t_n{}^2 + 2s_0t_n(1-t_n)}.\end{aligned}$$

Thus $\|x\|_{J_n} = |\sqrt{|[x,x]|}|\|y_1\|_{J_n} \to \infty$ as $n \to \infty$. And hence $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.
∎

Theorem 3.10. *Let x and y be linearly independent elements of a Krein space $\mathcal{K}$ which are non-orthogonal. If y is neutral, there exists a sequence of fundamental symmetries (J_n) such that $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.*

Proof. Since $[y,y] = 0$, there exists a sequence of fundamental symmetries (J_n) such that $\|y\|_{J_n} \to 0$ as $n \to \infty$. We first discuss the case when x is non-neutral. By Theorem 3.1 we have

$$\left\{\|x\|_J : J \text{ is a fundamental symmetry}\right\} = [|[x,x]|^{\frac{1}{2}}, \infty).$$

So for all n, $\|x\|_{J_n} \geq |[x,x]|^{\frac{1}{2}}$. We get $\left(\frac{1}{\|x\|_{J_n}}\right)$ is bounded and hence we can conclude that $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.

We now discuss the case when x is neutral. Let $[x,y] = k$. Since x and y are non-orthogonal, $k \neq 0$. By replacing y by $\frac{y}{k}$ we get $[x,y] = 1$. Let

$$x_1 = \frac{1}{\sqrt{2}}(x+y), y_1 = \frac{1}{\sqrt{2}}(x-y),$$

then

$$x = \frac{1}{\sqrt{2}}(x_1+y_1), y = \frac{1}{\sqrt{2}}(x_1-y_1), [x_1,x_1] = 1, [y_1,y_1] = -1, [x_1,y_1] = 0.$$

Let $\mathcal{K} = \mathcal{M}^+[\dot{+}]\mathcal{M}^-$ be a fundamental decomposition such that

$$\mathcal{M}^+ = \mathcal{L}^+[\dot{+}]\text{span}\{x_1\}, \mathcal{M}^- = \mathcal{L}^-[\dot{+}]\text{span}\{y_1\}$$

with some subspaces $\mathcal{L}^{\pm}$. Set $v(t_n) = x_1 + t_n y_1$, $t_n \in (-1, 1)$. Then

$$[v(t_n), v(t_n)] = 1 - {t_n}^2.$$

We have $\mathcal{K}^+_{t_n} = \mathcal{L}^+[\dot{+}]\text{span}\{v(t_n)\}$ is a maximal uniformly positive subspace and hence there exists a fundamental decomposition of $\mathcal{K}$ with $\mathcal{K}^+ = \mathcal{K}^+_{t_n}$. Now the projection $P^+_{t_n}$ onto $\mathcal{K}^+_{t_n}$ can be written as

$$P^+_{t_n} = P_{\mathcal{L}_+} + P_{v(t_n)}.$$

Thus

$$[P^+_{t_n} x, x] = \frac{|[x, v(t)]|^2}{[v(t_n), v(t_n)]} = \frac{1 - t_n}{2(1 + t_n)},$$

from which we get

$$\|x\|_{J_n}{}^2 = [J_n x, x] = 2[{P_n}^+ x, x] - [x, x] = \frac{2(1 - t_n)}{2(1 + t_n)} \to \infty$$

if we choose (t_n) such that $t_n \to -1$ as $n \to \infty$. Similarly we get

$$\|y\|_{J_n}{}^2 = \frac{1 + t_n}{1 - t_n} \to 0$$

if we choose (t_n) such that $t_n \to -1$ as $n \to \infty$. Thus we see that $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.
∎

Example 3.2. *Consider the two dimensional real Minkowski space $\mathcal{K} = \mathbb{R}^2$ with the inner product $[x, y] = x_1 y_1 - x_2 y_2$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Consider the fundamental decompositions with $\mathcal{K}^+_n = span\{(\frac{n+1}{n}, \frac{n-1}{n})\}$ and $\mathcal{K}^-_n = span\{(\frac{n-1}{n}, \frac{n+1}{n})\}$ where $n > 1$. Then we get*

$$\|x\|^2_{J_n} = \frac{1}{4}[(2n + 2/n)(x_1^2 + y_1^2) + 4x_1 y_1(1/n - n)].$$

Let $y = (1, 1)$ and $x = (1, 0)$. Then $\|y\|^2_{J_n} = \frac{2}{n}$ and $\|x\|^2_{J_n} = \frac{1}{2}(n + \frac{1}{n})$. Thus $\frac{\|y\|_{J_n}}{\|x\|_{J_n}} \to 0$ as $n \to \infty$.

4. CONCLUSION

Different fundamental decompositions on a Krein space induce different norms and there are fundamental symmetries corresponding to given fundamental decompositions. Hence the norm of a single element actually depends upon the choice of fundamental decomposition. Several estimates of norms of elements in the Krein space have been derived with illustrative examples and a few results of Bognar are also generalized in the paper.

REFERENCES

[1] MATTIAS LANGER and ANNEMARIE LUGER, On norms in indefinite inner product spaces, *Operator Theory: Advances and Applications*, **198**(2009), pp. 259-264.

[2] J. BOGNAR, Various norms on indefinite inner product spaces, *Periodica Mathematlca Hungarica*, **6** (1975), pp. 309-321.

[3] J. BOGNAR, *Indefinite inner product spaces*, Springer-Verlag, 1974.

[4] T. Ya. AZIZOV and I. S. IOKHVIDOV, *Linear operators in spaces with an indefinite metric*, A Wiley-Interscience Publication, 1989.

[5] MICHEAL KALTENBACK and HARALD WORACEK, Theory of Pontryagin spaces: Geometry and operators.

[6] DIRAC P. A. M., The physical interpretation of quantum mechanics, *Proc. Roy. Soc. London Ser. A.*, **180**(1942), pp. 1-40.

[7] FRANK HANSEN, Selfpolar norms on an indefinite inner product space, *Publ. RIMS, Kyoto Univ*, **16**(1980), pp. 889-913.

[8] MICHAEL A. DRITSCHEL and JAMES ROVNYAK, *Operators on indefinite inner product spaces*, 1991.