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A SELF-ADAPTIVE SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, we propose and analyze a type of subgradient extragradient algorithm for the approximation of a solution of variational inequality problem which is also a common fixed point of an infinite family of relatively nonexpansive mappings in 2-uniformly convex Banach spaces which are uniformly smooth. By using the generalized projection operator, we prove a strong convergence theorem which does not require the prior knowledge of the Lipschitz constant of cost operator. We further applied our result to constrained convex minimization problem, convex feasibility problem and infinite family of equilibrium problems. Our results improve and complement related results in 2-uniformly convex and uniformly smooth Banach spaces and Hilbert spaces.

Key words and phrases: Generalized projection; Subgradient extragradient algorithm, Lyapunov functional, Variational inequality problem; Fixed point problem; Banach spaces.

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1. INTRODUCTION

Let E be a Banach space and let $B_E = \{x \in E : ||x|| = 1\}$, then E is said to be strictly convex if for any $x, y \in B_E$ and $x \neq y$ implies $\frac{||x+y||}{2} < 1$. E is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in B_E$, $||x - y|| \geq \epsilon$ implies $\frac{||x+y||}{2} \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \Big|\Big|\frac{x+y}{2}\Big|\Big| : x, y \in B_E; \epsilon = ||x - y||\Big\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is a $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0, 2]$. Clearly, every p-uniformly convex Banach space is uniformly convex. For example, see [3, 36] for more details.

A Banach space E is said to be smooth if the limit $\lim\limits_{t\to 0}\dfrac{||x + ty|| - ||x||}{t}$ exists for all $x, y \in B_E$ and is said to be uniformly smooth if the limit is attained uniformly for $x, y \in B_E$. It is well known that Hilbert and the Lebesgue $L_p(1 < p \leq 2)$ spaces are 2-uniformly convex and uniformly smooth.

The mapping $J_p(x)$ $(p > 1)$ from E to 2^{E^*} defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||||x^*||, ||x^*|| = ||x||^{p-1}\} \; \forall x \in E,$$

is called the generalised duality mapping. If $p = 2$, then $J_2 = J$ is the normalised duality mapping. If E is smooth, strictly convex and reflexive, then $J^* = J^{-1}$, where $J^* : E^* \to 2^E$ is the the normalized duality mapping on E^*. Also, if E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded subsets of E^*. Let E be a reflexive, strictly convex and smooth Banach space and let J be the duality mapping from E into E^*, then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. Some other properties of the normalised duality mappings includes:
(1) For every $x \in E$, Jx is nonempty closed convex and bounded subset of E^*.
(2) If E is smooth or E^* is strictly convex, then J is single-valued.
(3) If E is strictly convex, then J is one-one.
(4) If E is reflexive, then J is onto.
(5) If E is strictly convex, then J is strictly monotone, that is, $\langle x - y; Jx - Jy\rangle > 0$; for all $x; y \in E$ such that $x \neq y$.
For more properties of the normalized duality mapping J, see for example [1, 35].

Let E be a Banach space and let E^* be the topological dual of E, let the duality pairing between E and E^* be denoted $\langle ., .\rangle$. Let C be a nonempty, closed and convex subset of E. In this paper, we consider the following Variational Inequality Problem (VIP) introduced by Stampacchia [23], which is to find a point $\bar{x} \in C$ such that

$$\langle A(\bar{x}), x - \bar{x}\rangle \geq 0, \; \forall x \in C, \tag{1.1}$$

where $A : E \to E^*$ is a single-valued mapping. The solution set of VIP (1.1) shall be denoted by $VI(C, A)$. The VIP is considered invaluable and have been studied extensively due to its applications to numerous problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research and general equilibrium problems in economics and transportation. In both Hilbert and Banach spaces, variant iterative methods have been utilized to study and approximate solutions of VIP (1.1) when A has some monotonicity and Lipschitz continuity properties, (see, for example, [5, 6, 9, 10, 11, 16, 17, 18, 21, 22, 24, 31, 32, 37] and the reference therein.)

An operator A of C into E^* is said to be
(i) monotone if $\langle x-y, A(x)-A(y)\rangle \geq 0, \forall x, y \in C.$
(ii) α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle x-y, A(x)-A(y)\rangle \geq \alpha||A(x)-A(y)||^2, \forall x, y \in C.$$

(iii) L-Lipschitz continuous if there exists a constant $L > 0$ such that $||A(x)-A(y)|| \leq L||x-y||, \forall x, y \in C.$
Clearly, every α-inverse-strongly-monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. But, the converse is not true.
(iv) β-strongly monotone if there exists a positive real number β such that

$$\langle x-y, A(x)-A(y)\rangle \geq \beta||x-y||^2, \forall x, y \in C.$$

The gradient method in which only one projection onto the feasible set is performed is a simple method for finding the approximate solution of variational inequalities. This process is to start with any $x_0 = x \in C$ and generate iteratively the subsequent term x_{n+1} according to the formula

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \tau_n A(x_n)),\ n \geq 0, \tag{1.2}$$

where Π_C, is the generalised projection mapping from E onto C, J is the normalised duality mapping and τ_n is a sequence of positive numbers. However, the convergence of this method requires a slightly strong assumption that operators are strongly monotone or inverse strongly monotone [13].

Many authors have succeeded to remove the assumption of strongly monotone or inverse strongly monotone in frame works of both Hilbert and Banach spaces by adapting the extrgradient method proposed by Korpelevich [21] for saddle point problems to variational inequality problems. More precisely, the Korpelevich's extragradient method for a monotone and L-Lipschitz continuous operator $A : E \to E^*$ is designed as follows:

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \mu A(x_n)), \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \mu A(y_n)). \end{cases} \tag{1.3}$$

where $\mu \in (0, \frac{1}{L})$. If the solution set $VI(C, A)$ is nonempty then the sequence $\{x_n\}$ generated by process (1.3) converges weakly to an element in $VI(C, A)$ (May we point out here that the original Korpelevich's extragradient method was in the frame work of Hilbert spaces where the generalised projection reduces to the metric projection and the normalised duality mapping J is the identity operator on the Hilbert space). In recent years, the extragradient method has received great attention and many authors have come out with some improved version of it, see, e.g., [6, 7, 8, 9, 10, 12, 14, 15, 17, 20, 26, 27, 30, 33, 34, 40, 41, 46] and the references therein. The extragradient method has its own drawback due to the requirement to calculate two projections onto the feasible set C. Since the projection onto a closed convex set C is related to a minimum distance problem, if C has a complex structure, this might be costly with respect to the amount of computation time.

One of the notable iterative algorithm that have be used to overcome this drawback is the subgradient extragradient method (see, e.g. [8, 9, 10, 25]). In the subgradient extragradient method, the second projection in Korpelevich's extragradient method is replaced by a projection onto a half-space which is computed explicitly. Precisely, the subgradient extragradient method in Banach spaces is given as follows:

$$\left\{\begin{array}{l} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \tau_n A(x_n)), \\ T_n = \{w \in E : \langle Jx_n - \tau_n A(x_n) - Jy_n, w - y_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{T_n} J^{-1}(Jx_n - \tau_n A(y_n)). \end{array}\right. \tag{1.4}$$

Using this type of iterative algorithm (with $\Pi_C = P_C$ and J the indentity operator), Censor et al. [10], obtained a weak convergence result in Hilbert space.

Liu [25] presented a modified subgradient extragradient algorithm in Banach spaces for finding a solution of the variational inequality (1.1) which is also a fixed point of a given relatively nonexpansive mapping. His algorithm is as follows: For mappings $A, S : E \to E$ and a closed and convex subset C of E, define three iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by:

$$\left\{\begin{array}{l} x_0 \in E, \\ y_n = \Pi_C J^{-1}(x_n - \tau_n A(x_n)), \\ T_n = \{w \in E : \langle w - y_n, Jx_n - \tau_n A(x_n) - Jy_n\rangle \leq 0\}, \\ w_n = \Pi_{T_n} J^{-1}(x_n - \tau_n A(y_n)), \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS(z_n)). \end{array}\right. \tag{1.5}$$

Under the condition that A is monotone and Lipschitz, he obtained a strong convergence result in 2-uniformly convex and uniformly smooth Banach spaces, where S is a relatively nonexpansive mapping.

We observe here that the results of Liu [25] requires the prior knowledge of the Lipschitz constant of the cost operator A, which is sometimes very difficult to compute. This raises a very natural and important question of the possibility of an iterative algorithm for approximating a common solution of variational inequality (1.1) and a fixed point problem for a relatively nonexpansive mapping which does not depend on the prior knowledge of the Lipschitz constant of the cost operator A. This question has been answered in the frame work of Hilbert spaces, for example see Thong and Hieu [39].

Motivated by the works of Thong and Hieu [39] and Liu [25], we contribute to the ongoing research by proposing a self adaptive iterative method without linesearch which is independent of the Lipschitz constant of the cost operator for finding a solution of variational inequality (1.1) which is also a common fixed point of an infinite family of relatively nonexpansive mappings in the frame work 2-uniformly convex and uniformly smooth Banach spaces.

2. PRELIMINARIES

Let E be a smooth Banach space, Alber [2], introduced the following Lyapunov functional $\phi : E \times E \to \mathbb{R}$ defined as:

$$\phi(x;y) = ||x||^2 - 2\langle x; Jy\rangle + ||y||^2;$$

for all $x; y \in E$. Observe that, in a Hilbert space H, $\phi(x;y) = ||x-y||^2$ for all $x; y \in H$. It is clear from the definition of ϕ that for all $x; y; z; w \in E$,

$$(||x|| - ||y||)^2 \leq \phi(x;y) \leq (||x|| + ||y||)^2. \tag{2.1}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle. \tag{2.2}$$

$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.3}$$

$$\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq ||x||||Jx - Jy|| + ||y - x||||y||. \tag{2.4}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \text{ if and only if } x = y. \tag{2.5}$$

Lemma 2.1. *(see* [29]*) Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2. *(see* [32]*) Let E be a 2-uniformly convex and smooth Banach space. Then, for every $x, y \in E$, $\phi(x,y) \geq c_1||x-y||^2$, where $c_1 > 0$ is the 2-uniformly convexity constant of E.*

The following mapping $V : E \times E^* \to \mathbb{R}$ was studied in Alber [2]:

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2, \tag{2.6}$$

for all $x \in E$ and $x^* \in E^*$. Clearly, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into $\mathbb{R}$.

Lemma 2.3. *(see* [2]*) Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.6)*. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E, then for each $x \in E$ (see Alber [2]), there exists a unique element $\bar{x} \in C$ such that $\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : E \to C$, defined by $\Pi_C(x) = \bar{x}$, is called the generalized projection mapping from E onto C and $\bar{x}$ is called the generalized projection of x. If E is a Hilbert space, then the generalized projection Π_C coincides with the metric projection P_C.

Lemma 2.4. *(see* [16, 29]*) Let C be a nonempty closed and convex subset of a smooth Banach space E and $x \in E$. Then, $\bar{x} = \Pi_C(x)$ if and only if $\langle \bar{x} - y, Jx - J\bar{x}\rangle \geq 0,\ \forall y \in C$.*

Lemma 2.5. *(see* [16, 29]*) Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x), \forall y \in C$.*

Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E and T be a mapping from C into itself. A point $x \in C$ is said to be a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.1. ([29, 32]) A mapping T of C into itself is said to be relatively nonexpansive if the following conditions are satisfied:
(i) $F(T)$ is nonempty;
(ii) $\phi(u, Tx) \leq \phi(u, x), \forall u \in F(T), x \in C$;
(iii) $\hat{F}(T) = F(T)$.

Lemma 2.6. *(see* [29]*) Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0,1]$ into E^* defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*.

Lemma 2.7. *(see* [16]*) Let C be a nonempty, closed and convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Then $VI(C, A) = \{u \in C : \langle v - u, A(v)\rangle \geq 0,\ \forall v \in C\}$.*

It is obvious from Lemma 2.7 that the set $VI(C, A)$ is a closed and convex subset of C.

Lemma 2.8. *(see* [28]*) Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{n_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.9. *(see* [43]*) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\delta_n\}$ is a sequence of real numbers satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\limsup \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *(see* [44]*) Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$||tx + (1-t)y||^2 \leq t||x||^2 + (1-t)||y||^2 - t(1-t)g(||x - y||),$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$, where $B_r(0) = \{z \in E : ||z|| \leq r\}$.

3. MAIN RESULTS

In this section, we will always assume the following conditions.

- A1. E is a 2-uniformly convex and uniformly smooth Banach space with the 2-uniformly convexity constant c_1 and C is a nonempty closed convex subset of E.
- A2. The mapping $A : E \to E^*$ is monotone and Lipschitz continuous on C with Lipschitz constant $L > 0$.
- A3. $T_j : E \to E$ (For each $j \geq 1$) is a relatively nonexpansive mapping.
- A4. $VI(C, A) \cap (\cap_{j=1}^{\infty} F(T_j)) \neq \emptyset$.

We now present a viscosity type subgradient extragradient algorithm for finding a point in the solution set of a variational inequality problem which is also a common fixed point of a infinite family of relatively nonexpansive mappings in 2-uniformly convex Banach spaces which are uniformly smooth. We further state and prove a strong convergence result with the proposed algorithm.

Algorithm 3.1. Initialization: *Given $\tau_0 > 0$, $\mu \in (0, c_1)$, and arbitrary $x_0 \in E$.*
Step 1. *Compute $y_n = \Pi_C J^{-1}(Jx_n - \tau_n A(x_n))$.*
Step 2. *Compute $z_n = \Pi_{T_n} J^{-1}(Jx_n - \tau_n A(y_n))$, where*

$$T_n = \{w \in E : \langle w - y_n, Jx_n - \tau_n A(x_n) - Jy_n\rangle \leq 0\}.$$

Step 3. *Compute* $x_{n+1} = J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n))$
and

$$\tau_{n+1} = \begin{cases} \min\Big\{\frac{\mu||x_n - y_n||}{||A(x_n) - A(y_n)||}, \tau_n\Big\}, & \text{if } A(x_n) \neq A(y_n), \\ \tau_n, & \text{Otherwise.} \end{cases} \tag{3.1}$$

Lemma 3.2. *The sequence* $\{\tau_n\}$ *generated by* (3.1) *is a nonincreasing sequence and*

$$\lim_{n\to\infty} \tau_n = \lambda \geq \min\Big\{\tau_0, \frac{\mu}{L}\Big\}.$$

Proof. See the proof of Lemma 3.1 in [45]. ∎

Lemma 3.3. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences generated in Algorithm 3.1 and* $u \in VI(C, A) \cap (\cap_{j=1}^{\infty} F(T_j))$. *Then,*

$$\phi(u, z_n) \leq \phi(u, x_n) - \Big(1 - \frac{\mu\tau_n}{c_1\tau_{n+1}}\Big)\phi(y_n, x_n) - \Big(1 - \frac{\mu\tau_n}{c_1\tau_{n+1}}\Big)\phi(z_n, y_n). \tag{3.2}$$

Proof.

$$\begin{aligned} \phi(u, z_n) &\leq \phi(u, J^{-1}(Jx_n - \tau_n A(y_n))) - \phi(z_n, J^{-1}(Jx_n - \tau_n A(y_n))) \\ &= \phi(u, x_n) + \phi(x_n, J^{-1}(Jx_n - \tau_n A(y_n))) + 2\langle u - x_n, \tau_n A(y_n)\rangle \\ &\quad -\phi(z_n, x_n) - \phi(x_n, J^{-1}(Jx_n - \tau_n A(y_n))) - 2\langle z_n - x_n, \tau_n A(y_n)\rangle \\ &= \phi(u, x_n) - \phi(z_n, x_n) - 2\tau_n\langle z_n - u, A(y_n)\rangle. \end{aligned} \tag{3.3}$$

Since A is monotone, we have $2\tau_n\langle A(y_n) - A(u), y_n - u\rangle \geq 0$. Therefore, adding $2\tau_n\langle A(y_n) - A(u), y_n - u\rangle$ to the right hand side of (3.3), we obtain

$$\begin{aligned} \phi(u, z_n) &\leq \phi(u, x_n) - \phi(z_n, x_n) - 2\tau_n\langle z_n - u, A(y_n)\rangle + 2\tau_n\langle A(y_n) - A(u), y_n - u\rangle \\ &= \phi(u, x_n) - \phi(z_n, x_n) + 2\tau_n\langle A(y_n), y_n - u\rangle \\ &\quad -2\tau_n\langle A(u), y_n - u\rangle - 2\tau_n\langle z_n - u, A(y_n)\rangle \\ &= \phi(u, x_n) - \phi(z_n, x_n) + 2\tau_n\langle y_n - z_n, A(y_n)\rangle - 2\tau_n\langle A(u), y_n - u\rangle \\ &= \phi(u, x_n) - \phi(z_n, x_n) + 2\tau_n\langle y_n - z_n, A(y_n) - A(x_n)\rangle \\ &\quad +2\tau_n\langle A(x_n), y_n - z_n\rangle - 2\tau_n\langle A(u), y_n - u\rangle. \end{aligned} \tag{3.4}$$

But by the Cauchy Schwartz inequality and the definition of τ_n, we have

$$\begin{aligned} 2\tau_n\langle y_n - z_n, A(y_n) - A(x_n)\rangle &\leq 2\tau_n||A(y_n) - A(x_n)||||y_n - z_n|| \\ &\leq 2\mu\frac{\tau_n}{\tau_{n+1}}||y_n - x_n||||y_n - z_n|| \\ &\leq \frac{\mu\tau_n}{\tau_{n+1}}||y_n - x_n||^2 + \frac{\mu\tau_n}{\tau_{n+1}}||y_n - z_n||^2 \\ &\leq \frac{\mu\tau_n}{c_1\tau_{n+1}}\phi(y_n, x_n) + \frac{\mu\tau_n}{c_1\tau_{n+1}}\phi(z_n, y_n). \end{aligned} \tag{3.5}$$

Again, by the definition of T_n, we have

$$\langle z_n - y_n, Jx_n - \tau_n A(x_n) - Jy_n\rangle \leq 0,$$

which implies

$$\begin{aligned} 2\tau_n\langle A(x_n), y_n - z_n\rangle &\leq 2\langle Jy_n - Jx_n, z_n - y_n\rangle \\ &= \phi(z_n, x_n) - \phi(z_n, y_n) - \phi(y_n, x_n). \end{aligned} \tag{3.6}$$

It then follows from (3.4), (3.5) and (3.6) that

$$
\begin{aligned}
\phi(u, z_n) \;\leq\;& \phi(u, x_n) - \phi(z_n, x_n) + \frac{\mu\tau_n}{c_1\tau_{n+1}}\phi(y_n, x_n) + \frac{\mu\tau_n}{c_1\tau_{n+1}}\phi(z_n, y_n) + \phi(z_n, x_n) \\
& -\phi(z_n, y_n) - \phi(y_n, x_n) - 2\tau_n\langle A(u), y_n - u\rangle. \qquad (3.7)
\end{aligned}
$$

Obviously, from $u \in VI(C, A)$, we have $\langle A(u), y_n - u\rangle \geq 0$, Thus, we have from (3.7) that

$$
\phi(u, z_n) \leq \phi(u, x_n) - \Big(1 - \frac{\mu\tau_n}{c_1\tau_{n+1}}\Big)\phi(y_n, x_n) - \Big(1 - \frac{\mu\tau_n}{c_1\tau_{n+1}}\Big)\phi(z_n, y_n). \qquad (3.8)
$$

∎

Theorem 3.4. *Let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_{n,j}\}_{j=1}^{\infty}$, *be sequences chosen in* $(0, 1)$ *such that*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,,
(iii) $0 < a \leq \beta_n, \sum_{j=1}^{\infty} \gamma_{n,j} < b < 1$ *and*
(iv) $\alpha_n + \beta_n + \sum_{j=1}^{\infty} \gamma_{n,j} = 1$.

Suppose the conditions A1-A4 hold, then the sequence $\{x_n\}$ *generated by Algorithm* (3.1) *converges strongly to* $p = \Pi_{VI(C,A)\cap(\cap_{j=1}^{\infty} F(T_j))} v$.

Proof. First, we show that $\{x_n\}$ is bounded.
Then From Lemma 3.2, we have

$$
\lim_{n\to\infty}\Big(1 - \frac{\mu\tau_n}{c_1\tau_{n+1}}\Big) = 1 - \frac{\mu}{c_1} > 0.
$$

This implies that there exists $n_0 \in \mathbb{N}$ such that $1 - \frac{\mu\tau_n}{c_1\tau_{n+1}} > 0, \forall n \geq n_0$. Thus from Lemma 3.3, we have $\phi(p, z_n) \leq \phi(p, x_n), \forall n \geq n_0$. Therefore, for all $n \geq n_0$, we have

$$
\begin{aligned}
\phi(p, x_{n+1}) &= \phi\Big(p, J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty}\gamma_{n,j}JT_j(z_n))\Big) \\
&\leq \alpha_n\phi(p, v) + \beta_n\phi(p, z_n) + \sum_{j=1}^{\infty}\gamma_{n,j}\phi(p, T_j(z_n)) \\
&\leq \alpha_n\phi(p, v) + \beta_n\phi(p, z_n) + \sum_{j=1}^{\infty}\gamma_{n,j}\phi(p, z_n) \\
&= \alpha_n\phi(p, v) + (1 - \alpha_n)\phi(p, z_n) \\
&\leq \alpha_n\phi(p, v) + (1 - \alpha_n)\phi(p, x_n) \\
&\leq \max\{\phi(p, v), \phi(p, x_n)\} \\
&\;\;\vdots \\
&\leq \max\{\phi(p, v), \phi(p, x_{n_0})\}. \qquad (3.9)
\end{aligned}
$$

Hence the sequence $\{\phi(p, x_n)\}$ is bounded and consequently, we have that $\{x_n\}$ is bounded.

We now continue with the rest of the proof. From Lemma 2.3, we have

$$
\begin{aligned}
\phi(p, x_{n+1}) &= V\Big(p, \alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n)\Big) \\
&\leq V(p, \alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n) - \alpha_n(Jv - Jp)) \\
&\quad -\langle J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n) - p, -\alpha_n(Jv - Jp)\rangle \\
&= V\Big(p, \alpha_n Jp + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n)\Big) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle \\
&= \phi\Big(p, J^{-1}(\alpha_n Jp + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n))\Big) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle \\
&\leq \alpha_n \phi(p, p) + \beta_n(p, z_n) + \sum_{j=1}^{\infty} \gamma_{n,j} \phi(p, T_j(z_n)) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle \\
&\leq \beta_n(p, z_n) + \sum_{j=1}^{\infty} \gamma_{n,j} \phi(p, z_n) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle \\
&= (1 - \alpha_n)\phi(p, z_n) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle \\
&\leq (1 - \alpha_n)\phi(p, x_n) + \alpha_n \langle x_{n+1} - p, Jv - Jp\rangle, \ \forall n \geq n_0. \qquad (3.10)
\end{aligned}
$$

Let us now consider two cases.

Case 1: Assume that there exists $n_1 \in \mathbb{N}$ such that $\phi(p, x_{n+1}) \leq \phi(p, x_n)$ for all $n \geq n_1$. Then $\{\phi(p, x_n)\}$ converges and $\lim_{n\to\infty}(\phi(p, x_{n+1}) - \phi(p, x_n)) = 0$. Set $w_n = J^{-1}\Big(\frac{\beta_n}{1-\alpha_n} Jz_n + \frac{\sum_{j=1}^{\infty} \gamma_{n,j}}{1-\alpha_n} JT_j(z_n)\Big)$. Then

$$
x_{n+1} = J^{-1}(\alpha_n Jv + (1 - \alpha_n)Jw_n).
$$

Now, since $\{x_n\}$ and $\{z_n\}$ are bounded, there exists $r > 0$ such that $\{x_n\}, \{z_n\} \in B_r(0)$. Therefore, by Lemma 2.10, there exists a continuous, strictly increasing and convex function

$g : [0, r] \to \mathbb{R}$ with $g(0) = 0$ such that

$$\begin{aligned}
\phi(p, w_n) &= \phi\Big(p, J^{-1}\Big(\frac{\beta_n}{1-\alpha_n} J z_n + \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} JT_j(z_n)\Big)\Big) \\
&= ||p||^2 + \Big|\Big|\frac{\beta_n}{1-\alpha_n} J z_n + \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} JT_j(z_n)\Big|\Big|^2 \\
&\quad -2\frac{\beta_n}{1-\alpha_n}\Big\langle p, Jz_n\Big\rangle - 2\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big\langle p, JT_j(z_n)\Big\rangle \\
&\leq ||p||^2 + \frac{\beta_n}{1-\alpha_n}\Big|\Big|z_n\Big|\Big|^2 + \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big|\Big|T_j z_n\Big|\Big|^2 \\
&\quad -\frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big) \\
&\quad -2\frac{\beta_n}{1-\alpha_n}\Big\langle p, Jz_n\Big\rangle - 2\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big\langle p, JT_j(z_n)\Big\rangle \\
&= \Big(1 - \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big)\Big|\Big|p\Big|\Big|^2 + \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big|\Big|p\Big|\Big|^2 + \Big(1 - \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big)\Big|\Big|z_n\Big|\Big|^2 \\
&\quad +\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big|\Big|T_j z_n\Big|\Big|^2 \\
&\quad -2\Big(1 - \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big)\Big\langle p, Jz_n\Big\rangle - 2\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big\langle p, JT_j(z_n)\Big\rangle \\
&\quad -\frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big) \\
&= \Big(1 - \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\Big)\phi(p, z_n) + \frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n}\phi(p, T_j(z_n)) \\
&\quad -\frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big) \\
&\leq \phi(p, z_n) - \frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big). \qquad (3.11)
\end{aligned}$$

Therefore, from (3.8) and (3.11), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \alpha_n\phi(p, v) + (1-\alpha_n)\phi(p, w_n) \\
&\leq \alpha_n\phi(p, v) + \phi(p, z_n) - \frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big) \\
&\leq \alpha_n\phi(p, v) + \phi(p, x_n) - \Big(1 - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\phi(y_n, x_n) - \Big(1 - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\phi(z_n, y_n) \\
&\quad -\frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big). \qquad (3.12)
\end{aligned}$$

which implies

$$\begin{aligned}
&\Big(1 - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\phi(y_n, x_n) + \Big(1 - \frac{\mu\tau_n}{\tau_{n+1}}\Big)\phi(z_n, y_n) \\
&+\frac{\beta_n}{1-\alpha_n}\frac{\sum_{j=1}^{\infty}\gamma_{n,j}}{1-\alpha_n} g\Big(\Big|\Big|JT_j(z_n) - Jz_n\Big|\Big|\Big) \leq \alpha_n\phi(p, v) + \phi(p, x_n) - \phi(p, x_{n+1}). \qquad (3.13)
\end{aligned}$$

Hence, since $\lim_{n\to\infty}\left(1-\frac{\mu\tau_n}{c_1\tau_{n+1}}\right)=\left(1-\frac{\mu}{c_1}\right)>0$, we have

$$\lim_{n\to\infty}||JT_j(z_n)-Jz_n||=0. \tag{3.14}$$

and

$$\lim_{n\to\infty}\phi(y_n,x_n)=\lim_{n\to\infty}\phi(z_n,y_n)=0. \tag{3.15}$$

From (3.15), we get

$$\lim_{n\to\infty}||y_n-x_n||=0 \tag{3.16}$$

and

$$\lim_{n\to\infty}||y_n-z_n||=0. \tag{3.17}$$

$$||x_n-z_n||\le||x_n-y_n||+||y_n-z_n||\to 0, n\to\infty. \tag{3.18}$$

Furthermore, since J^{-1} is uniformly norm to norm continuous, we have from (3.14) that

$$\lim_{n\to\infty}||T_j(z_n)-z_n||=0. \tag{3.19}$$

Therefore, from (2.4), (3.14) and (3.19), we get

$$\begin{aligned}\phi(z_n,T_j(z_n)) &= \langle z_n, Jz_n-JT_j(z_n)\rangle+\langle T_j(z_n)-z_n, Jz_n\rangle\\ &\le ||z_n||||Jz_n-JT_j(z_n)||+||T_jz_n-z_n||||T_j(z_n)||\to 0, n\to\infty.\end{aligned} \tag{3.20}$$

Hence,

$$\begin{aligned}\phi(z_n,x_{n+1}) &= \phi\Big(z_n, J^{-1}(\alpha_nJv+\beta_nJz_n+\sum_{j=1}^{\infty}\gamma_{n,j}JT_j(z_n)\Big)\\ &\le \alpha_n(z_n,v)+\beta_n\phi(z_n,z_n)+\sum_{j=1}^{\infty}\gamma_{n,j}\phi(z_n,T_j(z_n))\to 0,\to\infty,\end{aligned} \tag{3.21}$$

which implies $\lim_{n\to\infty}||z_n-x_{n+1}||=0$. Thus, from (3.18), we have

$$||x_n-x_{n+1}||\le||x_n-z_n||+||z_n-x_{n+1}||\to 0, n\to\infty. \tag{3.22}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_{n+1}-p, Jv-Jp\rangle=\lim_{k\to\infty}\langle x_{n_k}-p, Jv-Jp\rangle.$$

and $x_{n_k}\rightharpoonup q$ for some $q\in E$.

Next, we show that $q\in VI(C,A)\cap(\cap_{j=1}^{\infty}F(T_j))$. Let $x\in C$. Since $y_n=\Pi_CJ^{-1}(Jx_n-\tau_nA(x_n))$, then by Lemma 2.4, we have

$$\langle y_n-x, Jx_n-\tau_nA(x_n)-Jy_n\rangle\ge 0, \forall n\ge 0.$$

Thus,

$$\begin{aligned}\langle x_n-x,\tau_nA(x_n)\rangle &= \langle x_n-y_n,\tau_nA(x_n)\rangle+\langle y_n-x,\tau_nA(x_n)\rangle\\ &= \langle x_n-y_n,\tau_nA(x_n)\rangle-\langle y_n-x, Jx_n-\tau_nA(x_n)-Jy_n\rangle\\ &\quad +\langle y_n-x, Jx_n-Jy_n)\rangle\\ &\le \langle x_n-y_n,\tau_nA(x_n)\rangle+\langle y_n-x, Jx_n-Jy_n)\rangle\\ &\le \tau_n||A(x_n)||||x_n-y_n||+||Jx_n-Jy_n||||y_n-x||.\end{aligned} \tag{3.23}$$

Since $A(x_n)$ is bounded, $\lim_{n\to\infty}||x_n - y_n|| = 0$ and J is norm to norm uniformly continuous, we have from (3.23) that $\limsup_{n\to\infty}\langle x_n - x, \tau_n A(x_n)\rangle \leq 0$. Thus, from the monotonicity of A, we have that

$$\begin{aligned}\langle q - x, \tau_n A(x)\rangle &= \limsup_{n\to\infty}\langle x_n - x, \tau_n A(x)\rangle \\ &\leq \limsup_{n\to\infty}\langle x_n - x, \tau_n A(x_n)\rangle \leq 0, \forall x \in C. \end{aligned} \tag{3.24}$$

Since $x_n \rightharpoonup q$ and $\lim_{n\to\infty}||x_n - y_n|| = 0$, we have $y_n \rightharpoonup q$. Noting that C is closed and convex and $y_n \in C, \forall n \geq 0$, then from Lemma 2.7 and (3.24), we conclude that $q \in VI(C, A)$. Furthermore, from the definition of relatively nonexpansive mapping, (3.18) and (3.19), we have $q \in F(T_j),\ j = 1, 2, \cdots$. That is $q \in \cap_{j=1}^{\infty}F(T_j)$. Hence $q \in VI(C, A) \cap (\cap_{j=1}^{\infty}F(T_j))$.

Now, from Lemma 2.4, we have

$$\limsup_{n\to\infty}\langle x_{n+1} - p, Jv - Jp\rangle = \lim_{k\to\infty}\langle x_{n_k} - p, Jv - Jp\rangle = \langle q - p, Jv - Jp\rangle \leq 0.$$

Therefore, applying Lemma 2.9 to (3.10), we obtain that $\lim_{n\to\infty}\phi(p, x_n) = 0$, which implies $||x_n - p|| \to 0, n \to \infty$. That is $x_n \to p = \Pi_{VI(C,A)\cap(\cap_{j=1}^{\infty}F(T_j))}v$.

Case 2. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\phi(p, x_{n_i}) \leq \phi(p, x_{n_i+1})\ \forall i \in \mathbb{N}.$$

From Lemma 2.8, there exists a nondecreasing sequence $\{n_l\}$ of $\mathbb{N}$ such that $\lim_{l\to\infty} n_l = \infty$ and the following inequalities hold for all $l \in \mathbb{N}$:

$$\phi(p, x_{n_l}) \leq \phi(p, x_{n_l+1}) \tag{3.25}$$

and

$$\phi(p, x_l) \leq \phi(p, x_{n_l+1}). \tag{3.26}$$

Thus from (3.13), we have

$$\begin{aligned}&\Big(1 - \frac{\mu\tau_{n_l}}{c_1\tau_{n_l+1}}\Big)\phi(y_{n_l}, x_{n_l}) + \Big(1 - \frac{\mu\tau_{n_l}}{c_1\tau_{n_l+1}}\Big)\phi(z_{n_l}, y_n) \\ &+\frac{\beta_{n_l}}{1-\alpha_{n_l}}\frac{\sum_{j=1}^{\infty}\gamma_{n_l,j}}{1-\alpha_{n_l}}g\Big(\Big|\Big|JT_j(z_{n_l}) - Jz_{n_l}\Big|\Big|\Big) \leq \alpha_{n_l}\phi(p, v) + \phi(p, x_{n_l}) - \phi(p, x_{n_l+1}).\end{aligned} \tag{3.27}$$

Hence, since $\lim_{n\to\infty}\Big(1 - \frac{\mu\tau_{n_k}}{c_1\tau_{n_k+1}}\Big) = \Big(1 - \frac{\mu}{c_1}\Big) > 0$, we have

$$\lim_{l\to\infty}||JT_j(z_{n_l}) - Jz_{n_l}|| = 0. \tag{3.28}$$

and

$$\lim_{l\to\infty}\phi(y_{n_l}, x_{n_l}) = \lim_{l\to\infty}\phi(z_{n_l}, y_{n_l}) = 0. \tag{3.29}$$

Using similar argument as in case 1, we obtain

$$\limsup_{l\to\infty}\langle x_{n_l+1} - p, Jv - Jp\rangle \leq 0.$$

Furthermore, from (3.10), we have

$$\phi(p, x_{n_l+1}) \leq (1 - \alpha_{n_l})\phi(p, x_{n_l}) + \alpha_{n_l}\langle x_{n_l+1} - p, Jv - Jp\rangle,\ \forall l \geq n_0.$$

It therefore follows from (3.25) that

$$\phi(p, x_{n_l+1}) \leq (1 - \alpha_{n_l})\phi(p, x_{n_l+1}) + \alpha_{n_l}\langle x_{n_l+1} - p, Jv - Jp\rangle. \tag{3.30}$$

Combining (3.26) and (3.30), we obtain

$$\phi(p, x_l) \leq \langle x_{n_l+1} - p, Jv - Jp \rangle,$$

which gives $\limsup_{l\to\infty} \phi(p, x_l) = 0$ and thus $x_l \to p$. ∎

4. APPLICATIONS

4.1. **Constrained Minimization Problem.** In this subsection, we give an application of our result to constrained minimization problem.

Consider the constrained convex minimization problem:

$$\min\{f(x) : x \in C\}, \tag{4.1}$$

where C is a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E and $f : E \to \mathbb{R}$ is a real valued convex function. Assume that f is continuously Fréchet differentiable with Lipschitz continuous gradient:

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \tag{4.2}$$

for all $x, y \in E$, where L is a positive constant. It is well known that the minimization problem (4.1) is equivalent to the following variational inequality problem:

$$x \in C,\ \langle \nabla f(x), x - x^* \rangle \geq 0,\ \forall x \in C. \tag{4.3}$$

Moreover, the gradient of a convex and continuously Fréchet differentiable function is monotone. Letting $A = \nabla f$, we obtain from Algorithm 3.1, the following algorithm for finding a solution of (4.1) which is also a common fixed point of an infinite family of relatively nonexpansive mappings.

Algorithm 4.1. Initialization: *Given* $\tau_0 > 0$, $\mu \in (0, c_1)$, *and arbitrary* $x_0 \in E$.
Step 1. *Compute* $y_n = \Pi_C J^{-1}(Jx_n - \tau_n \nabla f(x_n))$.
Step 2. *Compute* $z_n = \Pi_{T_n} J^{-1}(Jx_n - \tau_n \nabla f(y_n))$, *where*

$$T_n = \{w \in E : \langle w - y_n, Jx_n - \tau_n \nabla f(x_n) - Jy_n \rangle \leq 0\}.$$

Step 3. *Compute* $x_{n+1} = J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_j(z_n))$
and

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||x_n - y_n||}{||\nabla f(x_n) - \nabla f(y_n)||}, \tau_n\right\}, & \text{if } \nabla f(x_n) \neq \nabla f(y_n), \\ \tau_n, & \text{Otherwise.} \end{cases} \tag{4.4}$$

4.2. **Convex Feasibility Problem.** Let $\{C_j\}_{j=1}^{\infty}$, be nonempty closed and convex subsets of E such that $\cap_{j=1}^{\infty} C_j \neq \emptyset$. The convex feasibility problem (CFP) is to find $x \in \cap_{j=1}^{\infty} C_j$. Obviously $F(\Pi_{C_j}) = C_j$ for all $j \geq 1$. Thus, if set $T_j = \Pi_{C_j}$ in Algorithm 3.1, we obtain the following Algorithm:

Algorithm 4.2. Initialization: *Given* $\tau_0 > 0$, $\mu \in (0, c_1)$, *and arbitrary* $x_0 \in E$.
Step 1. *Compute* $y_n = \Pi_C J^{-1}(Jx_n - \tau_n A(x_n))$.
Step 2. *Compute* $z_n = \Pi_{T_n} J^{-1}(Jx_n - \tau_n A(y_n))$, *where*

$$T_n = \{w \in E : \langle w - y_n, Jx_n - \tau_n A(x_n) - Jy_n \rangle \leq 0\}.$$

Step 3. *Compute* $x_{n+1} = J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} J\Pi_{C_j}(z_n))$
and

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||x_n - y_n||}{||A(x_n) - A(y_n)||}, \tau_n\right\}, & \text{if } A(x_n) \neq A(y_n), \\ \tau_n, & \text{Otherwise.} \end{cases} \tag{4.5}$$

Therefore from Theorem 3.4, we obtain a strong convergence result for approximating a common solution of a variational inequality problem and a convex feasibility problem.

4.3. **Equilibrium Problem.** Let C be a closed and convex subset of a Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for a bifunction f is to find

$$x \in C \text{such that } f(x,y) \geq 0 \text{for all } y \in C. \tag{4.6}$$

The set of solutions above is denoted by $EP(f, C)$, that is

$$x \in EP(f,C) \text{ iff } f(x,y) \geq 0 \forall y \in C. \tag{4.7}$$

To solve the equilibrium problem (4.6), the bifunction f is usually assumed to satisfy the following conditions:
(B1) $f(x, x) = 0$, for all $x \in C$;
(B2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;
(B3) for all $x, y, z \in C, \limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;
(B4) for all $x \in C$, $f(x, .)$ is convex and lower semicontinuous.

Lemma 4.3. *(*[38], *Lemma 2.8) Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r^f : E \to C$ as follows:*

$$T_r^f = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \forall y \in C\} \tag{4.8}$$

for all $x \in E$. Then, the following hold:

(1) *T_r^f is single-valued;*
(2) *T_r^f is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$*

$$\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y\rangle \leq \langle T_r^f x - T_r^f y, Jx - Jy\rangle; \tag{4.9}$$

(3) *$F(T_r^f) = Ep(f, C)$,*
(4) *$EP(f, C)$ is closed and convex and T_r^f is a relatively nonexpansive mapping.*

Letting $T_r^{f_j} = T_j$ in Algorithm 3.1, we obtain the following Algorithm:

Algorithm 4.4. Initialization: *Given $\tau_0 > 0$, $\mu \in (0, c_1)$, and arbitrary $x_0 \in E$.*
Step 1. *Compute $y_n = \Pi_C J^{-1}(Jx_n - \tau_n A(x_n))$.*
Step 2. *Compute $z_n = \Pi_{T_n} J^{-1}(Jx_n - \tau_n A(y_n))$, where*

$$T_n = \{w \in E : \langle w - y_n, Jx_n - \tau_n A(x_n) - Jy_n\rangle \leq 0\}.$$

Step 3. *Compute $x_{n+1} = J^{-1}(\alpha_n Jv + \beta_n Jz_n + \sum_{j=1}^{\infty} \gamma_{n,j} JT_r^{f_j}(z_n))$ and*

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu||x_n - y_n||}{||A(x_n) - A(y_n)||}, \tau_n\right\}, & \text{if } A(x_n) \neq A(y_n), \\ \tau_n, & \text{Otherwise.} \end{cases} \tag{4.10}$$

Thus from Theorem 3.4, we obtain a strong convergence result for approximating a common solution of an infinite family of equilibrium problems which also solves a variational inequality problem.

5. CONCLUSION

We introduce a subgradient extragradient algorithm with self adaptive variable step sizes without line search which does not require a prior knowledge of the Lipschitz constant for the approximation of a solution of variational inequality problem which is also a common fixed point of an infinite family of relatively nonexpansive mappings in 2-uniformly convex Banach spaces which are uniformly smooth. Using the proposed algorithm, we stated and proved a strong convergence result and give some applications in 2-uniformly convex Banach spaces which are uniformly smooth. The result of this paper extends the work of Thong and Hieu [39] from Hilbert spaces to 2-uniformly convex Banach spaces which are uniformly smooth. In our future project, we hope to introduce a new inertial accelerated version of Algorithm 3.1 for finding a solution of variational inequality problem which is also a common fixed point of a family of relatively nonexpansive mappings in 2-uniformly convex Banach spaces which are uniformly smooth.

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