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ANALYSIS OF A DYNAMIC ELASTO-VISCOPLASTIC FRICTIONLESS ANTIPLAN CONTACT PROBLEM WITH NORMAL COMPLIANCE

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ABSTRACT. We consider a mathematical model which describes the dynamic evolution of a thermo elasto viscoplastic contact problem between a body and a rigid foundation. The mechanical and thermal properties of the obstacle coating material near its surface. A variational formulation of this dynamic contact phenomenon is derived in the context of general models of thermo elasto viscoplastic materials. The displacements and temperatures of the bodies in contact are governed by the coupled system consisting of a variational inequality and a parabolic differential equation. The proof is based on a classical existence and uniqueness result on parabolic inequalities,differential equations and fixed point arguments.

Key words and phrases: Nonlinear constitutive law; Normal compliance; Elasto-viscoplastic materials; Variational inequality; Weak solution.

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1. INTRODUCTION

Dynamic or quasistatic contact phenomena for elastic-plastic or viscoplastic materials with heat flow appear in many engineering problems [1, 5, 7] and are intensively studied in literature (see references in [1, 2, 3, 4, 8, 10, 12, 14, 17, 21, 22]). General model of thermo-elastic-viscoplastic material is characterized by a rate–type constitutive equation with internal variables modeling their impact on the behavior of real bodies in contact under plastic deformation. The considered internal state variables include, among others, spatial display of dislocations, the work hardening of materials, the temperature or the damage field [3].

The existence of solutions to these contact problems is studied in monographs [11, 13] and papers [1, 3, 9, 12, 15, 16, 18, 19].

The paper is concerned with the analysis and numerical modeling of the rolling contact between a rigid wheel and an elasto-viscoplastic rail lying on a rigid foundation. The contact phenomenon includes also a heat generation and flow through the contact surface [6, 18]. The obstacle is assumed to be covered with functionally graded coating material which properties depending on the spatial variables according to the power law. In the paper the nonhomogeneous plastically graded model of the coating layer rather than elastic one as in [7, 20] is assumed. The existence of solutions for this hyperbolic, parabolic coupling of the boundary value problems is presented in the context of general models of thermoelastic-viscoplastic materials.

The paper is organized as follows. First in Section 2 we formulate the dynamic frictionless contact problem for a body with a normal compliance. Moreover, we introduce some notations and preliminaries which will be used in the next. In Section 3 and by using the monotonicity arguments and fixed point theorem we establish the existence of the solution of the problem considered.

2. MECHANICAL PROBLEM AND VARIATIONAL FORMULATION

Consider a dynamic frictionless contact problem for a body occupying a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz continuous boundary Γ. This boundary is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. Assume $meas(\Gamma_1) > 0$. A body is assumed clamped along the boundary Γ_1, i.e., the displacement vanishes there. Along the boundary $\Gamma_3 \times (0, T)$ the body is assumed to be in contact with the foundation. The surface traction f_2 acts on the boundary Γ_2 $\times(0, T)$.The body is loaded by a volume force of density f_1 in $\Omega \times (0, T)$. The external heat source q is applied in $\Omega \times (0, T)$. The body is assumed to undergo the coupled thermal as well as elastic-viscoplastic deformation with linear isotropic and kinematic hardening.

Let us denote by $u = (u_1, u_2), u = u(x, t), x \in \Omega, t \in (0, T), T > 0$ is given, and by $\theta = \theta(x, t)$ a displacement field and a temperature field of the body, respectively.

The infinitesimal strain tensor is denoted by $\varepsilon(u) = (\varepsilon_{ij}(u))$ and the stress field by $\sigma = (\sigma_{ij})$, where $\varepsilon(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), i, j = 1, 2$.

The divergence operator div is defined as $div(\sigma) = \{\sigma ij, j\}, i, j = 1, 2$, and

$$\sigma ij, j = \frac{\partial \sigma_{i,j}}{\partial x_j}$$

The symbol denotes the derivative with respect the time variable.

$$\dot{u} = \frac{\partial u}{dt} \text{ and } \ddot{u} = \frac{\partial^2 u}{\partial t^2}$$

We denote υ the unit outward normal vector to the boundary Γ.

The normal and tangential components of the displacement field u are denoted by

$$u_\upsilon = u \cdot \upsilon = u_{i.} \upsilon_i \,,\ i = 1, 2 \,,\ u_\tau = u - u_\nu \upsilon$$

Respectively, similarly normal and tangential components of the stress field σ are denoted by $\sigma\nu = \sigma\upsilon{\cdot}\upsilon$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\upsilon$.

Denote by $\mathcal{S}_2$ the space of second-order symmetric tensors on $\mathbb{R}^2$. Moreover

$$Q = \left\{q = (q_{ij})_{2\times 2} : q_{ij} = q_{ji} \quad q_{ij} \in L^2(\Omega)\right\}$$

and

$$Q_0 = \{q \in Q : tr(q) = 0\}$$

is closed subspace of Q. Additive small strain plasticity model is used Additive small strain plasticity model is used [10, 11, 12] where ε^p denotes the plastic part of the strain tensor.

We denote by $(\sigma, \mathcal{X}) \in Q \times [L^2(\Omega)]^2$ and $(\varepsilon^p, \zeta) \in Q_0 \times [L^2(\Omega)]^2$ the generalized stress and strain tensors respectively. For a given yield function ϕ the set of admissible generalized stresses K is defined by

$$K = \{(\sigma, \mathcal{X}) : \phi(\sigma, \mathcal{X}) \leq 0\}$$

Denote by N_K a normal cone to the set K at a point $(\sigma,\ \mathcal{X})$ and by ϕ the support function of the set K called the dissipation function [11] as well as by $Kp = dom\phi.$

Problem 1. ***P**. find the displacement field* $u : \Omega \times\ [0,T] \to \mathbb{R}^2$ *,the stress field* $\sigma : \Omega \times [0,T] \to \mathbb{S}^2$ *, the internal field* $(\varepsilon^p$,$\zeta) : \Omega \times [0,T] \to \mathbb{R}^{2\times 2} \times \mathbb{R}^2$*and the temperature* $\theta : \Omega \times [0,T] \to R$ *satisfying*

$$\rho\ddot{u} = \operatorname{div}\sigma + f_1 \textit{ in } \Omega \times (0,T) \tag{2.1}$$

$$\begin{cases} \sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{E}(\varepsilon(u(t))) + \int\limits_0^1 \mathcal{G}(\sigma(s) - \mathcal{A}(\dot{\varepsilon}(\dot{u}(s)), \varepsilon(u(s)), \theta(s), \zeta(s))ds \\ \qquad \textit{in } \Omega \times (0,T) \end{cases} \tag{2.2}$$

$$\varphi(q,\eta) - \varphi(\dot{\varepsilon}^p, \dot{\zeta}) - \sigma(q - \dot{\varepsilon}^p) - \mathcal{X}(\eta - \dot{\zeta}) \geq 0\ ,\ \forall (q,\eta) \in K_p \textit{ ,in } \Omega \times (0,T) \tag{2.3}$$

$$\rho\dot{\theta} - \theta = \psi(\sigma - \mathcal{A}(\varepsilon(\dot{u}), \varepsilon(u), \theta, \zeta) + g \textit{ in } \Omega \times (0,T) \tag{2.4}$$

$$u = 0 \textit{ on } \Gamma_1 \times (0,T) \tag{2.5}$$

$$\sigma_\upsilon = f_2 \textit{ on } \Gamma_2 \times (0,T) \tag{2.6}$$

$$-\sigma_\upsilon = p_\nu(u_\nu) \textit{ on } \Gamma_3 \times (0,T) \tag{2.7}$$

$$-\sigma_\tau = p_\tau \textit{ on } \Gamma_3 \times (0,T) \tag{2.8}$$

$$\nabla\theta_\nu + k_1\theta = \widetilde{g} \textit{ on } \Gamma_3 \times (0,T) \tag{2.9}$$

$$u(0) = u_0, \varepsilon^p(0) = \varepsilon_0^p, \zeta(0) = \zeta_0, \theta(0) = \theta_0 \tag{2.10}$$

Equation (2.1) represents the motion of the body where ρ denotes the material mass density. The equation (2.2) represent the thermo-elastic-viscoplastic constitutive law with operators $\mathcal{A}$ and $\mathcal{E}$ governing the viscous and the elastic properties of the material as well as with nonlinear constitutive function $\mathcal{G}$ governing viscoplastic properties of the material. The inequality (2.3) describes the plastic flow. Heat flow is governed by the equation (2.4) where ψ is a constitutive function representing the heat generated by the work of internal forces and g is a given volume heat source. Displacement and stress boundary conditions are given by (2.5, 2.6), respectively. Normal compliance condition with a given positive function p_ν is described by (2.7). In (2.8 tangential traction p_τ is a given function. Fourier type boundary condition for temperature is given in (2.9 with a given function $\widetilde{g}$ and constant $k_1 > 0$. Suitable regular initial data functions u_0, u_1, ε_0^p , ζ_0, θ_0 in (2.10 are assumed to be given. Before we formulate initial problem (2.1 - 2.10) in variational form let us introduce the following spaces and subspaces,

$$H = \left\{ \{u_i\}_i^2 , i = 1, 2 : u_i \in L^2(\Omega) \right\} = \left[L^2(\Omega) \right]^2 \tag{2.11}$$

$$\mathcal{H} = \left\{ \sigma = \{\sigma_i, j\}_{i,j=1}^2 : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\} \tag{2.12}$$

$$H_1 = \{u \in H : \varepsilon(u) \in \mathcal{H}\} \tag{2.13}$$

$$\mathcal{H}_1 = \{\sigma = \mathcal{H} : \operatorname{div}(\sigma) \in H\} \tag{2.14}$$

$$V = H^1(\Omega) \, , \mathcal{V} = \left\{ v \in H^1(\Omega) : v = 0 \text{ , on } \Gamma_1 \right\} \tag{2.15}$$

The spaces H, $\mathcal{H}$, H_1, $\mathcal{H}_1$, V are endowed with the canonical inner products

$$(u \, , v)_H = \int_\Omega u_i v_i dx, i = 1, 2$$

The inner product on the space $\mathcal{V}$ is equal to

$$(u, v)_{\mathcal{V}} = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$$

and let $\|.\|_{\mathcal{V}}$ be the associated norm, defined by

$$\|v\|_{\mathcal{V}} = \|\varepsilon(v)\|_{\mathcal{H}}$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_i \tau_i dx \, , i = 1, 2$$

$$(u, v)_{H_1} = (u \, , v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$$

$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}_1} + (Div(\sigma), Div(\tau))_H$$

It follows from Korn's inequality that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on $\mathcal{V}$. Therefore $(V; \|.\|_V)$is a real Hilbert space [12, 13].

Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^n} = C_0 \|v\|_{\mathcal{V}} \quad \forall v \in \mathcal{V}$$

Furthermore, if $\sigma \in \mathcal{H}_1$ there exists an element $\sigma\nu \in H'_\Gamma$ such that the following Green formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div(\sigma), v)_H = \int_\Gamma \sigma\nu.\gamma v \quad \forall v \in H_1$$

$$(f, g)_V = (f, g)_{L^2(\Omega)} + (f_{x_i}, g_{x_i})_{L^2(\Omega)}$$

Let $\mathcal{V}'$ and V' denote dual spaces to the spaces $\mathcal{V}$ and V, respectively. We have the inclusions

$$\mathcal{V} \subset H \subset \mathcal{V}' \; ; \; V \subset L^2(\Omega) \subset V'$$

Let us introduce the following assumptions.

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^2 \to \mathbb{S}^2$ satisfies

(2.16)
$$\begin{cases} (a) \text{ There exixts a constant } L_1 > 0 \text{ such that} \\ \mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2) \leq L_1 |\varepsilon_1 - \varepsilon_2|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^2, \text{a.e.} x \in \Omega \\ (b) \text{ There exists a constant } m_1 \text{ such that} \\ \mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2) \geq m_1 |\varepsilon_1 - \varepsilon_2|^2, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^2, \text{a.e.} x \in \Omega \\ (c) \text{ The mapping } x \to \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \forall \varepsilon \in \mathbb{S}^2 \\ (d) \text{ The mapping } x \to \mathcal{A}(x, 0) \in \mathcal{H} \end{cases}$$

The elasticity operator $\mathcal{E} : \Omega \times \mathbb{S}^2 \to \mathbb{S}^2$ satisfies:

(2.17)
$$\begin{cases} (a) \text{ There exists a constant } L_2 > 0 \text{ such that:} \\ \mathcal{E}(x, \varepsilon_1) - \mathcal{E}(x, \varepsilon_2) \leq L_2 |\varepsilon_1 - \varepsilon_2|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^2, \text{a.e.} x \in \Omega \\ (b) \text{The mapping } x \to \mathcal{E}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \forall \varepsilon \in \mathbb{S}^2 \\ (c) \text{ The mapping } x \to \mathcal{E}(x, 0) \in \mathcal{H} \end{cases}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$ is assumed to satisfy:

(2.18)
$$\begin{cases} (a) \text{ There exists a constant } L_3 > 0 \text{ such that:} \\ \mathcal{G}(x, \sigma_1, \varepsilon_1, \theta_1, \zeta_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2, \theta_2, \zeta_2) \leq L_3(|\sigma_1 - \sigma_2| + \\ |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\zeta_1 - \zeta_2|), \forall \sigma_1, \sigma_2 \in \mathcal{S}^2, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^2 \\ \forall \theta_1, \theta_2 \in \mathbb{R}, \forall \zeta_1, \zeta_2 \in \mathbb{R}.a.e\ .x \in \Omega \\ (b)\text{The mapping } x \to \mathcal{G}(x, \sigma, \varepsilon, \theta, \zeta) \text{ is Lebesgue measurable on } \Omega\ , \\ \forall \sigma, \varepsilon \in \mathbb{S}^2, \theta, \zeta \in \mathbb{R}. \\ (c)\text{The mapping } x \to \mathcal{G}(x, 0, 0, 0, 0) \in \mathcal{H}. \end{cases}$$

The dissipation function $\varphi : \Omega \times \mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as well as the set K_p of the admissible states and the hardening modulus H satisfy

(2.19)
$$\begin{cases} (a)\varphi \text{ is a proper,convex and lower semi-continuous function,} \\ (b)K_p \text{ is nonempty,closed and convex set in } L^2(\Omega; \mathbb{R}^{2\times 2} \times \mathbb{R}^2), \\ (c)\text{the hardening modulus } H \text{ is symmetric,positive definite} \\ \text{and linear operator from } \mathbb{R}^2 \text{ into } \mathbb{R}^2. \end{cases}$$

The function $\psi : \Omega \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$(2.20)\quad \begin{cases} (a) \text{ There exists a constant } L_4 > 0 \text{ such that:} \\ \psi(x,\sigma_1,\varepsilon_1,\theta_1,\zeta_1) - \psi(x,\sigma_2,\varepsilon_2,\theta_2,\zeta_2) \le L_4(|\sigma_1-\sigma_2| + \\ |\varepsilon_1-\varepsilon_2| + |\theta_1-\theta_2| + |\zeta_1-\zeta_2|), \forall\sigma_1,\sigma_2 \in \mathcal{S}^2, \forall\varepsilon_1,\varepsilon_2 \in \mathbb{S}^2 \\ \forall\theta_1,\theta_2 \in \mathbb{R}, \forall\zeta_1,\zeta_2 \in \mathbb{R}.a.e\ .x \in \Omega \\ (b)\text{The mapping } x \to \psi(x,\sigma,\varepsilon,\theta,\zeta) \text{ is Lebesgue measurable on } \Omega\ , \\ \forall\sigma,\varepsilon \in \mathbb{S}^2, \theta,\zeta \in \mathbb{R} \\ \text{The mapping } x \to \psi(x,0,0,0,0) \in \mathcal{H} \end{cases}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}+$ is assumed to satisfy:

$$(2.21)\quad \begin{cases} (a) \text{ There exists a constant } L_5 > 0 \text{ such that:} \\ p_\nu(x,z_1) - p_\nu(x,z_2) \le L_5 |z_1 - z_2|\,, \forall z_1,z_2 \in \mathbb{R}, .a.e.x \in \Gamma_3 \\ (b)\text{The mapping } x \to p_\nu(x,z) \text{ is Lebesgue measurable on } \Gamma_3, \forall z \in \mathbb{R} \\ (c)\text{The mapping } x \to p_\nu(x,z) = 0, \forall z \le 0, a.e.x \in \Gamma_3. \end{cases}$$

We shall also assume:

$$(2.22)\quad f_1 \in L(0,T,H),\ \ f_2 \in L^2(0,T,\left[L^2(\Gamma_2)\right]^2)$$

$$(2.23)\quad g \in L^2(0,T,L^2(\Omega))\ ,\ \widetilde{g} \in L^2(\Gamma_3)\ ,\ k_1 > 0\ ,\ p_\nu \in L^\infty(\Gamma_3)$$

$$(2.24)\quad \rho \in L^\infty(\Omega)\ ;\ (\varepsilon_0^p,\zeta_0) \in K_p, u_0 \in \mathcal{V}\ ,\ u_1 \in H\ ,\ \theta_0 \in V$$

Let us define the following bilinear and linear forms:

$$\begin{cases} a_\theta : V \times V \to \mathbb{R} \\ a_\theta(\zeta,\xi) = \int\limits_\Omega \rho\nabla\zeta\nabla\xi dx + k_1 \int\limits_{\Gamma_3} \zeta\xi ds \end{cases}$$

$$\langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \int\limits_\Omega f_1(t)v dx + \int\limits_{\Gamma_2} f_2 v dx\ \ ,\ f(t) \in L^2(0,T,\mathcal{V}')$$

$$\begin{cases} j_c : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \\ j_c(u,v) = \int\limits_{\Gamma_3} p_\nu(u_\nu)v_\nu + p_\tau v_\tau) ds \end{cases}$$

$$\begin{cases} j_p : Q_0 \times H \to \mathbb{R} \\ j_p(q,\zeta) = \int \phi(q,\zeta) dx \end{cases}$$

3. AN ABSTRACT EXISTENCE AND UNIQUENESS RESULT

Using Green's formula it is straightforward to derive the following variational formulation of Problem P.

Problem 2. P_V. *Find the stress field* $\sigma : [0,T] \to \mathbb{S}^2$*, the displacement field* $u : [0,T] \to \mathbb{R}$*, the internal variable* $(\varepsilon_p,\zeta) : [0,T] \to \mathbb{R}^{2\times 2} \times \mathbb{R}^2$*, the temperature field* $\theta : [0,T] \to \mathbb{R}$*, such that*

$$(3.1)\quad \begin{cases} \sigma(t) = \mathcal{A}(\varepsilon(\dot u(t))) + \mathcal{E}(\varepsilon(u(t))) + \int\limits_0^t \mathcal{G}(\sigma(s) - \mathcal{A}(\dot\varepsilon(\dot u(s)), \varepsilon(u(s)), \theta(s), \zeta(s)) ds \\ \textit{in } \Omega,\ a.e.t \in (0,T) \end{cases}$$

(3.2) $$\begin{cases} \left\langle \rho\dot{\ddot{u}}, v \right\rangle_{\mathcal{V}'\times\mathcal{V}} + \int\limits_{\Omega} \sigma(t)\varepsilon(v)dx + j_c(u,v) + j_p(q,\zeta) - j_p(\dot{\varepsilon}^p, \dot{\zeta}) \geq \langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \\ \forall(v,q,\eta) \in \mathcal{V}\times K_p \ \ \textit{a.e.}t \in (0,T) \end{cases}$$

(3.3) $$\begin{cases} \left\langle \rho\dot{\theta}, v \right\rangle_{V'\times V} + a_\theta(\theta, v) = \langle \psi(\sigma(t) - \mathcal{A}(\dot{\varepsilon}(\dot{u}(t)), \varepsilon(u(t)), \theta(t), \zeta(t), v\rangle_{V'\times V} + \\ \int\limits_{\Omega} g(t)v dx + \int\limits_{\Gamma_3} \widetilde{g}v ds \ \forall v \in V, a.e.t \in (0,T) \end{cases}$$

(3.4) $$u(0) = u_0, \dot{u}(0) = u_1, \theta(0) = \theta_0, \varepsilon^p(0) = \varepsilon^p_0 \text{ , } \zeta(0) = \zeta_0$$

The existence of a unique solution to contact problem (3.1 - 3.4) is shown in next theorem

Theorem 3.1. *Assume conditions (2.1 - 2.10) and (2.16 - 2.24) hold. There exists a unique solution* $(\sigma, u, \varepsilon_p, \zeta, \theta)$ *to the problem (3.1 - 3.4). Moreover*

(3.5) $$u \in C^0(0,T,\mathcal{V}) \cap C^1(0,T,H) \ \ , \ \dot{u} \in L^2(0,T,\mathcal{V}) \ , \ddot{u} \in L^2(0,T,\mathcal{V}')$$

(3.6) $$\varepsilon^p \in L^2(0,T,V) \cap C^0(0,T,L^2(\Omega)) \text{ ; } \dot{\varepsilon}^p \in L^2(\dot{0},T,V')$$

(3.7) $$\zeta \in L^2(0,T,V) \cap C^0(0,T,L^2(\Omega)), \dot{\zeta} \in L^2(0,T,V')$$

(3.8) $$\sigma \in L^2(0,T,\mathcal{H})$$

(3.9) $$\theta \in L^2(0,T,V) \cap C^0(0,T,L^2(\Omega)) \ \text{ ; } \dot{\theta} \in L^2(\dot{0},T,V')$$

In order to prove Theorem 3.1 we need the following auxiliary problem $\mathbf{P}_\gamma$

Problem 3. ***P***$_\gamma$*.For a given* $\gamma \in L^2(0,T;V)$*,find the displacement field*

$$u_\gamma : [0,T] \times \Omega \to \mathbb{R}^2$$

And the internal variable

$$(\varepsilon^p_\gamma, \zeta_\gamma) : [0,T] \times \Omega \to \mathbb{R}^{2\times 2} \times \mathbb{R}^2$$

Satisfying

(3.10) $$\begin{cases} \langle \rho\ddot{u}_\gamma, v\rangle_{\mathcal{V}'\times\mathcal{V}} + \int\limits_{\Omega} \mathcal{A}(\varepsilon(\dot{u}(t))\varepsilon(v)dx + j_p(q,\eta) - j_p(\dot{\varepsilon}^p_\gamma, \dot{\zeta}_\gamma) + \\ \langle \gamma(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \geq \langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \text{ , } \forall(v,q,\eta) \in \mathcal{V}\times K_p, \textit{a.e.}t \in (0,T) \end{cases}$$

(3.11) $$u_\gamma(0) = u_0 \text{ , } \dot{u}_\gamma(0) = u_1 \text{ , } \varepsilon^p_\gamma(0) = \varepsilon^p_0 \text{ , } \zeta_\gamma(0) = \zeta_0$$

Lemma 3.2. *For all $\gamma \in L^2(0,T;V)$ there exists a unique solution*

$$u_\gamma : [0,T] \times \Omega \to \mathbb{R}^2$$

and

$$(\varepsilon_\gamma^p, \zeta_\gamma) : [0,T] \times \Omega \to \mathbb{R}^{2\times 2} \times \mathbb{R}$$

to the problem $P\gamma$ satisfying(3.5 - 3.9).

Proof. From the assumption (2.16) it follows that operator $\mathcal{A}$ is bounded, semi continuous and coercive on V. Since $\gamma \in L^2(0,T;V)$ and (2.24) holds by standard arguments concerning the parabolic inequalities it results the existence of (u_γ, ρ_γ, ζ_γ) satisfying (3.10, 3.11). For details see [2, 3, 19]. ∎

Let $\alpha \in L^2(0,T;V)$ be given. Define the auxiliary problem P_α.

Problem 4. P_α*: find the temperature $\theta_\alpha : [0,T] \times \Omega \to \mathbb{R}$ satisfying*

$$\left\{ \begin{array}{c} \left\langle \rho\dot{\theta}_\alpha, \upsilon \right\rangle_{V'\times V} + a_\theta(\theta_\alpha, \upsilon) = \langle \alpha, \upsilon\rangle_{V'\times V} + \int\limits_\Omega g\upsilon dx + \int\limits_{\Gamma_3} \tilde{g}\upsilon ds \text{ , } \forall \upsilon \in V \\ \theta_\alpha(0) = \theta_0 \end{array} \right. \tag{3.12}$$

Lemma 3.3. *For all $\alpha \in L^2(0,T;V)$ there exists a unique solution*

$$\theta_\alpha : [0,T] \times \Omega \to \mathbb{R}$$

to the auxiliary problem P_α satisfying (3.6).

Proof. From Poincaré-Friedrich's inequality it follows that the bilinear form a_θ is V-elliptic. Hence by standard arguments the parabolic boundary value problem (3.12) possesses a unique solution $\theta_\alpha : [0,T] \times \Omega \to \mathbb{R}$ satisfying (3.6). For details [3, 19]. ∎

Let us consider the following auxiliary problem $P\gamma$,α

Problem 5. $P\gamma$,α*: find the stress field $\sigma_{\gamma,\alpha}$:$[0,T] \times \Omega \to \mathbb{S}^2$ solving the equation:*

$$\sigma_{\gamma,\alpha}(t) = \mathcal{E}(\varepsilon(u_\gamma(t))) + \int_0^t \mathcal{G}(\sigma_{\gamma,\alpha}(s), \varepsilon(u_\gamma(s)), \theta_\alpha(s), \zeta_\gamma(s))ds \text{ , } \forall t \in (0,T) \tag{3.13}$$

Lemma 3.4. *There exists a unique solution $\sigma_{\gamma,\alpha}$:$[0,T]\times\Omega \to \mathbb{S}^2$ to the problem $P_{\gamma,\alpha}$ satisfying (3.8, 3.9).*

Let for $i = 1, 2$, u_{γ_i}, θ_{α_i} , ζ_{γ_i} and *$\sigma_{\gamma_i,\alpha_i}$ denote the solutions to problems P_{γ_i}, $P\alpha_i$* and *P_{γ_i,α_i}, respectively*.

Proof. Then there exists constant $C > 0$ such that:

$$\left\{ \begin{array}{c} \left\|\sigma_{\gamma_1,\alpha_1}(t) - \sigma_{\gamma_2,\alpha_2}(t)\right\|_{\mathcal{H}}^2 \leq C(\left\|u_{\gamma_1}(t) - u_{\gamma_1}(t)\right\|_{\mathcal{V}}^2 + \\ \int\limits_0^t \left(\left\|u_{\gamma_1}(s) - u_{\gamma_1}(s)\right\|_{\mathcal{V}}^2 + \left\|\theta_{\alpha_1}(s) - \theta_{\alpha_2}(s)\right\|_V^2 + \left\|\zeta\gamma_1(s) - \zeta\gamma_1(s)\right\|_V^2\right) ds \end{array} \right. \tag{3.14}$$

We denote the mapping

$$\left\{ \begin{array}{c} \Pi_{\gamma,\alpha} : L^2(0,T,\mathcal{H}) \to L^2(0,T,\ \mathcal{H}) \\ \Pi_{\gamma,\alpha}\sigma(t) = \mathcal{E}(\varepsilon(u_\gamma(t))) + \int\limits_0^t \mathcal{G}(\sigma_{\gamma,\alpha}(s), \varepsilon(u_\gamma(s)), \theta_\alpha(s), \zeta_\gamma(s))ds \end{array} \right. \tag{3.15}$$

Assume $\sigma_i \in L^2(0,T;H), i=1,2$, and $t \in (0,T)$. From the assumption (2.18) and Hölder's inequality we obtain

(3.16) $$\|\Pi_{\gamma,\alpha}\sigma_1(t^*) - \Pi_{\gamma,\alpha}\sigma_2(t^*)\|^2_{\mathcal{H}} \le L_3^2 T \int\limits_0^t \|\sigma_1(s) - \sigma_2(s)\|^2_{\mathcal{H}}\, ds$$

Repeating this evaluation k times and integrating on the time interval $(0,T)$ we obtain

(3.17) $$\|\Pi_{\gamma,\alpha}\sigma_1(t^*) - \Pi_{\gamma,\alpha}\sigma_2(t^*)\|^2_{\mathcal{H}} \le \frac{L_3^{2k}T^k}{k!}\|\sigma_1(s) - \sigma_2(s)\|^2_{\mathcal{H}}$$

Hence for k large enough operator $\Pi_{\gamma,\alpha}$ is a contraction on the space$L^2(0,T;H)$. By Banach fixed point theorem there exists a unique solution

$$\sigma_{\gamma,\alpha} \in L^2(0,T;H)$$

to the equation

(3.18) $$\Pi_{\gamma,\alpha}(\sigma_{\gamma,\alpha}) = \sigma_{\gamma,\alpha}$$

which is also a unique solution to problem $P\gamma$,α. Since for $i=1,2$ then $u\gamma_i$, $\theta\alpha_i$, $\zeta\gamma_i$ are solutions to problems (3.10, 3.11,3.12), respectively, applying Young's inequality and (2.18 - 2.20) we obtain (3.14). ∎

Lemma 3.5. *We denote now the mapping* $\Lambda : L^2(0,T;V\times V) \to L^2(0,T;V\times V)$ *defined as follows:* $\Lambda(\gamma(t),\alpha(t)) = (\Lambda_0(\gamma(t),\alpha(t)), \Lambda_1(\gamma(t),\alpha(t)))$*, where*

$$\begin{cases} \Lambda_0(\gamma(t),\alpha(t),\upsilon) = \mathcal{E}(u_\gamma(t);\varepsilon(\upsilon))_{\mathcal{H}} + j_c(u_\gamma(t),\upsilon) + \\ \left(\int\limits_0^t \mathcal{G}(\sigma_{\gamma,\alpha}(s),\varepsilon(u_\gamma(s)),\theta_\alpha(s),\zeta_\gamma(s))ds, \varepsilon(\upsilon)\right)_{\mathcal{H}} ds, \quad \forall \upsilon \in \mathcal{V} \end{cases}$$

And

(3.19) $$\Lambda_1(\gamma(t),\alpha(t)) = \Psi(\sigma_{\gamma,\alpha}(t),\varepsilon(u_\gamma(t)),\theta_\alpha(t),\zeta_\gamma(t))$$

The mapping Λ *has a fixed point* $(\gamma^*,\alpha^*) \in L^2(0,T;\mathcal{V}'\times V')$.

Proof. Using assumptions (2.15 - 2.21) as well as Hölder's and Young's inequalities we show that

(3.20)
$$\|\Lambda(\gamma_1(t),\alpha_1(t)) - \Lambda(\gamma_2(t),\alpha_2(t))\|^2_{\mathcal{V}'\times V'} \le C\left(\|\gamma_1(t)-\gamma_2(t)\|^2_{\mathcal{V}'} + \|\alpha_1(t)-\alpha_2(t)\|^2_{V'}\right)$$

Reiterating this inequality k times results :

(3.21)
$$\left\|\Lambda^k(\gamma_1(t),\alpha_1(t)) - \Lambda^k(\gamma_2(t),\alpha_2(t\right\|^2_{\mathcal{V}'\times V'} \le \frac{C^kT^k}{k!}\left(\|\gamma_1(t)-\gamma_2(t)\|^2_{\mathcal{V}'} + \|\alpha_1(t)-\alpha_2(t)\|^2_{V'}\right)$$

For k large enough operator Λ^k is a contraction on the space $L^2(0,T;V\times V)$. By Banach fixed point theorem it follows that Λ possesses a unique fixed point $(\gamma^*,\alpha^*) \in L^2(0,T;V\times V)$. Using Lemmas (3.2 - 3.5), we prove Theorem 3.1. ∎

Proof. (of Theorem 3.1) Denote by $((\gamma^*,\alpha^*) \in L^2(0,T;V\times V)$ the fixed point of the operator Λ defined by (3.17 - 3.19).

Let:

(3.22) $$u = u_{\gamma^*}\ , \theta = \theta_{\alpha^*}, \varepsilon^p = \varepsilon^p_{\gamma^*}, \zeta = \zeta_{\gamma^*}, \sigma = \mathcal{A}(\varepsilon(\dot{u}) + \sigma_{\gamma^*,\alpha^*}.$$

Setting in (3.13) $\gamma = \gamma^*$, $\alpha = \alpha^*$ and using (3.13) it results that (3.10) holds. From (3.2) with $\gamma = \gamma^*$ and (3.22) we obtain:

$$(3.23)\quad \begin{cases} \langle \rho \ddot{u}_\gamma, v\rangle_{\mathcal{V}'\times\mathcal{V}} + \int_\Omega \mathcal{A}(\varepsilon(\dot{u}(t))\varepsilon(v)dx + \langle \gamma^*(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} + j_p(q,\eta) - j_p(\dot{\varepsilon}^p_\gamma, \dot{\zeta_\gamma}) \\ \qquad \geq \langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \;, \forall(v,q,\eta) \in \; \mathcal{V}\times K_p, \text{a.e.} t \in (0,T) \end{cases}$$

From (3.17 - 3.19), (3.22) as well as:

$$\Lambda_0(\gamma^*, \alpha^*) = \gamma^* \,;\, \Lambda_1(\gamma^*, \alpha^*) = \alpha^*$$

We obtain:

$$(3.24)\quad \begin{cases} \langle \gamma^*, v\rangle_{\mathcal{V}'\times\mathcal{V}} = \int_\Omega \mathcal{E}(\varepsilon(u_\gamma(t)))\varepsilon(v)dx + j_c(u(t), v) + j_p(q,\eta) - \\ j_p(\dot{\varepsilon}^p, \dot{\zeta}) + \int_\Omega \left(\int_0^t \mathcal{G}(\sigma_{\gamma,\alpha}(s), \varepsilon(u_\gamma(s)), \theta_\alpha(s), \zeta_\gamma(s))ds, \varepsilon(v) \right), \forall(v,q,\eta) \in \mathcal{V} \times K_p \end{cases}$$

$$(3.25)\quad \alpha^*(t) = \psi(\sigma(t) - \mathcal{A}(\varepsilon(\dot{u}(t))), \varepsilon(u(t)), \theta(t), \zeta(t)).$$

Inserting (2.24) into (2.23), using (3.1) we obtain that (3.2) is satisfied. Setting $\alpha = \alpha^*$ in (3.12) and using (3.22) as well as (3.25) we conclude that (3.24) is satisfied. From Lemmas (3.2 - 3.5) it results that (3.5 - 3.9) hold. From the uniqueness of solutions to problems (3.10, 3.11), (3.12, 3.13) as well as from the uniqueness of the fixed point of the operator (3.17, 3.18) follows the uniqueness of the solution to the problem (3.1, 3.4). ∎

4. CONCLUSION

This paper deals with a mathematical model which describes the dynamic evolution of a thermo-elasto-viscoplastic contact problem between a body and a rigid foundation. A variational formulation of this dynamic contact phenomenon is derived in the context of general models of thermo elasto viscoplastic materials. The proof is established on several steps based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

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