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OBLIQUE PROJECTORS FROM THE SIMPSON DISCRETE FOURIER TRANSFORMATION MATRIX

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ABSTRACT. In this paper we examine the projectors of the Simpson Discrete Fourier Transform matrix of dimension two modulus four and show how they decompose the complex vector space into a direct sum of oblique eigenspaces. These projection operators are used to define a Simpson Discrete Fractional Fourier Transform (SDFRFT)

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1. INTRODUCTION

The Simpson DFT matrix is defined by [4]

$$\mathbf{U} = \frac{2}{3}\begin{bmatrix} \mathbf{A} & 2\mathbf{DA} \\ \mathbf{A} & -2\mathbf{DA} \end{bmatrix}\mathbf{P} \tag{1.1}$$

where the $\left(\frac{N}{2}\times\frac{N}{2}\right)$ matrix $\mathbf{A}$ is defined by

$$\mathbf{A}_{ij} = \omega^{2ij}, \text{ for } i,j = 0,1,\cdots,\tfrac{N}{2}-1 \tag{1.2}$$

the matrix $\mathbf{D} = diag(1,\omega,\cdots,\omega^{\frac{N}{2}-1})$ where $\omega = e^{-\frac{2\pi i}{N}}$, the $(N\times N)$ permutation matrix $\mathbf{P}$ is defined by its components $P_{j,2j-1} = 1$ and $P_{N/2+j,2j} = 1$ for $j = 0,1,\cdots,\frac{N}{2}-1$. The matrix $\mathbf{U}$ is simple or diagonalizable since its minimal polynomial has linear elementary divisors, furthermore it has eight distinct eigenvalues [6]. The Simpson Discrete Fourier Transform of a vector $\mathbf{f} = [f(0), f(1),\cdots,f(N-1)]^T \in \mathbb{C}^N$ maybe written as

$$\mathbf{Uf} = \mathbf{F}_0 + \mathbf{F}_1, \tag{1.3}$$

where

$$F_0(k) = \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k2j}f(2j) \tag{1.4}$$

and

$$F_1(k) = \frac{4}{3}\sum_{j=0}^{\frac{N}{2}-1}\omega^{k(2j+1)}f(2j+1), \tag{1.5}$$

for $k = 0,1,\cdots,N-1$.
From the Spectral theorem for matrices[2] it follows that if $h(\lambda)$ is a function defined on the spectrum $\sigma(\mathbf{U})$ of $\mathbf{U}$ and $p(\lambda)$ is the interpolating polynomial of minimum degree determined by the values of h on $\sigma(\mathbf{U})$ then $p(\mathbf{U}) = h(\mathbf{U})$ where

$$p(\lambda) = \sum_{n=1}^{8} l_n(\lambda)\, h(\lambda_n) \tag{1.6}$$

and

$$l_n(\lambda) = \prod_{\substack{s=1 \\ s\neq n}}^{8}\frac{(\lambda-\lambda_s)}{(\lambda_n-\lambda_s)} \tag{1.7}$$

are the Lagrange basis polynomials. If $h(\lambda) = 1$ we obtain

$$1 = \sum_{n=1}^{8} l_n(\lambda) \tag{1.8}$$

from which the resolution of the identity matrix follows

$$\mathbf{I} = \sum_{n=1}^{8} l_n(\mathbf{U}) \tag{1.9}$$

If $h(\lambda) = \lambda$ we obtain a decomposition of the matrix

Eigenvalue	**Multiplicity**
$\lambda_{10} = r_1$	$m_{10} = \frac{1}{2}\left[m + 1 - \cos\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{12} = -r_1$	$m_{12} = \frac{1}{2}\left[m + 1 + \cos\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{11} = ir_1$	$m_{11} = \frac{1}{2}\left[m - \sin\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{13} = -ir_1$	$m_{13} = \frac{1}{2}\left[m + \sin\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{20} = r_2$	$m_{20} = \frac{1}{2}\left[m + 1 + \cos\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{22} = -r_2$	$m_{22} = \frac{1}{2}\left[m + 1 - \cos\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{21} = ir_2$	$m_{21} = \frac{1}{2}\left[m + \sin\left(\frac{m\pi}{2}\right)\right]$
$\lambda_{23} = -ir_2$	$m_{23} = \frac{1}{2}\left[m - \sin\left(\frac{m\pi}{2}\right)\right]$

Table 2.1: Eigenvalue decomposition

$$\mathbf{U} = \sum_{n=1}^{8} \lambda_n l_n(\mathbf{U}) = \sum_{n=1}^{8} \lambda_n \mathbf{E}_n \tag{1.10}$$

where $\mathbf{E}_n = l_n(\mathbf{U})$ are the projection operators onto the nullspace $\mathcal{N}(\mathbf{U} - \lambda_n \mathbf{I})$. By the term projection we mean a matrix that is idempotent.

2. Oblique Projections

An orthogonal projection is Hermitian, hence its nullspace is orthogonal to the range. Oblique projections are those projections that are not orthogonal. Denote the eigenvalues of matrix $\mathbf{U}$ by λ_{pl} and their corresponding multiplicities by m_{pl}, $p = 1, 2 : l = 0, 1, 2, 3$. The eigenvalue decomposition of matrix $\mathbf{U}$ is summarized in table 1[6]. where $r_1 = \frac{\sqrt{N}}{3}\sqrt{9 + \sqrt{17}}$, $r_2 = \frac{\sqrt{N}}{3}\sqrt{9 - \sqrt{17}}$ and $N = 4m + 2$. In accordance with the notation in table 1 we shall label the projection operators as $\mathbf{E}_{pl}$ corresponding to λ_{pl} and denote its range, equivalently the nullspace by $\mathcal{N}_{pl} = \mathcal{N}(\mathbf{U} - \lambda_{pl}\mathbf{I})$, $p = 0, 1$ and $l = 0, 1, 2, 3$. The projectors $\mathbf{E}_{pl}^2 = \mathbf{E}_{pl}$ have eigenvalues 0 and 1 and hence are rank deficient furthermore $\mathbf{E}_{pl}\mathbf{E}_{mn} = \mathbf{0}$ if $p \neq m$ and $l \neq n$. The rank of $\mathbf{E}_{pl}$ is the same as the multiplicity of the eigenvalue λ_{pl}. Since $\mathbf{E}_{pl} \neq \mathbf{E}_{pl}^*$ as a consequence of $\mathbf{U} \neq \mathbf{U}^*$ it follows that these are oblique projection operators[3]. The analysis of these constitutent matrices of $\mathbf{U}$ provides an insight into the decomposition of the space $\mathbb{C}^N$. Define $\mathbb{C}_{\mathtt{E}}$ to be the subspace of $\mathbb{C}^N$ consisting of all even vectors. Let $\mathbf{B}$ be a $N \times (\frac{N}{2} + 1)$ whose columns comprise of the basis vectors of an arbitrary orthogonal basis of $\mathbb{C}_{\mathtt{E}}$. The range $\mathcal{R}(\mathbf{P}_{\mathtt{E}})$ of the orthogonal projection

$$\mathbf{P}_{\mathtt{E}} = \mathbf{B}\left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T \tag{2.1}$$

is $\mathbb{C}_{\mathrm{E}}$. Hence $\mathbb{C}^N = \mathbb{C}_{\mathrm{E}} \oplus \mathbb{C}_0$, where $\mathbb{C}_0$ denotes the nullspace $\mathcal{N}(\mathbf{P}_{\mathrm{E}})$ consisting of all odd vectors.
Consider the projection operator

$$\mathbf{E}_{10} = \frac{(\mathbf{U}+r_1\mathbf{I})\left(\mathbf{U}^2-r_2^2\mathbf{I}\right)\left(\mathbf{U}^2+r_2^2\mathbf{I}\right)\left(\mathbf{U}^2+r_1^2\mathbf{I}\right)}{4r_1^3\left(r_1^4-r_2^4\right)} \tag{2.2}$$

obtained from (1.7) and define the odd annhilator operator $\mathbf{Z}_{\mathrm{E}}$ by

$$\mathbf{Z}_{\mathrm{E}} = \left(\mathbf{U}^2+r_2^2\mathbf{I}\right)\left(\mathbf{U}^2+r_1^2\mathbf{I}\right). \tag{2.3}$$

Theorem 2.1. *For every* $\mathbf{f} \in \mathbb{C}^N$, $\mathbf{Z}_{\mathrm{E}}\mathbf{f}$ *is even.*

Proof. The even indexed components satisfy

$$\mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(2k) = \mathbf{U}^4\mathbf{f}\,(2k) + \left(r_1^2+r_2^2\right)\mathbf{U}^2\mathbf{f}\,(2k) + r_1^2r_2^2 f\,(2k) \tag{2.4}$$

$$= \frac{4}{3}N^2\left[f\,(2k)+f\,(-2k)\right] - \frac{8}{9}N^2\left[f\left(2k+\frac{N}{2}\right)+f\left(-2k+\frac{N}{2}\right)\right] \tag{2.5}$$

where (2.5) is obtained from (2.4) using the duality property[5]. Similarly the odd indexed components satisfy

$$\begin{aligned}&\mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(2k+1)\\&= \frac{8}{3}N^2\left[f\,(2k+1)+f\,(-2k-1)\right] - \frac{4}{9}N^2\left[f\left(2k+1+\frac{N}{2}\right)+f\left(-2k-1+\frac{N}{2}\right)\right].\end{aligned} \tag{2.6}$$

Clearly $\mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(-2k) = \mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(2k)$ and also $\mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(-2k-1) = \mathbf{Z}_{\mathrm{E}}\mathbf{f}\,(2k+1)$ which implies that $\mathbf{Z}_{\mathrm{E}}\mathbf{f}$ is even. Furthermore if $\mathbf{f} = \mathbf{f}_0 \in \mathbb{C}_0$ is odd then $\mathbf{Z}_{\mathrm{E}}\mathbf{f}_0 = \mathbf{0}$ Thus $\mathbf{Z}_{\mathrm{E}}$ annihilates every odd vector. This also applies to every eigenvector corresponding to purely imaginary eigenvalues since the latter is always odd[6]. The effect of $\mathbf{Z}_{\mathrm{E}}$ in (2.2) is to map $\mathbf{f} \in \mathbb{C}^N$ into the even subspace $\mathbb{C}_{\mathrm{E}}$. The effect of $(\mathbf{U}+r_1\mathbf{I})\left(\mathbf{U}^2-r_2^2\mathbf{I}\right)$ is to map $\mathbf{Z}_{\mathrm{E}}\mathbf{f}$ into the even subspace $\mathcal{N}_{10}$ or generate an eigenvector corresponding to r_1. The denominator in (2.2) is a normalizing factor that ensures that $\mathbf{E}_{10}$ is a projection operator. ∎

Theorem 2.2. *If* $\mathbf{f} \in \mathcal{N}\left(\mathbf{Z}_{\mathrm{E}}\right)$ *then* $\mathbf{f}$ *is odd.*

Proof. If $\mathbf{Z}_{\mathrm{E}}\mathbf{f} = \mathbf{0}$ then from (2.5) and (2.7) it follows that

$$f\,(2k)+f\,(-2k) = \frac{2}{3}\left[f\left(2k+\frac{N}{2}\right)+f\left(-2k+\frac{N}{2}\right)\right] \tag{2.7}$$

and

$$f\,(2k+1)+f\,(-2k-1) = \frac{1}{6}\left[f\left(2k+1+\frac{N}{2}\right)+f\left(-2k-1+\frac{N}{2}\right)\right] \tag{2.8}$$

since $2k+\frac{N}{2}$ is odd, replace k by $k+\frac{N-2}{4}$ in (2.8) to obtain

$$f\left(2k+\frac{N}{2}\right)+f\left(-2k+\frac{N}{2}\right) = \frac{1}{6}\left[f\,(2k)+f\,(-2k)\right] \tag{2.9}$$

Equations (2.7) and (2.9) imply that

$$f\,(2k)+f\,(-2k) = 0. \tag{2.10}$$

Since $2k+1+\frac{N}{2}$ is even one may replace k by $k+\frac{N+2}{4}$ in (2.7) yielding

$$f\left(2k+1+\frac{N}{2}\right)+f\left(-2k-1+\frac{N}{2}\right) = \frac{2}{3}\left[f\,(2k+1)+f\,(-2k-1)\right]. \tag{2.11}$$

Equations (2.8) and (2.11) imply that

$$f(2k+1)+f(-2k-1)=0 \tag{2.12}$$

From (2.10) and (2.12) we conclude that

$$f(-k)=-f(k)\,. \tag{2.13}$$

∎

From theorem 1 and theorem 2 we conclude that $\mathcal{N}(\mathbf{Z}_{\mathtt{E}})=\mathbb{C}_0$, which has dimension $\frac{N}{2}-1$. The range $\mathcal{R}(\mathbf{Z}_{\mathtt{E}})=\mathbb{C}_{\mathtt{E}}$ has dimension $\frac{N}{2}+1$, hence rank $(\mathbf{Z}_{\mathtt{E}})$ is $\frac{N}{2}+1$. Furthermore $\mathbf{Z}_{\mathtt{E}}$ has eigenvalues:

i) $2r_1^2(r_1^2+r_2^2)$ of multiplicity $m+1$ which it inherits from the eigenvalues of $\pm r_1$ of $\mathbf{U}$
ii) $2r_2^2(r_1^2+r_2^2)$ of multiplicity $m+1$ which it inherits from the eigenvalues of $\pm r_2$ of $\mathbf{U}$
iii) 0 of multiplicity $2m$ which it inherits from the eigenvalues of $\pm ir_1$ and $\pm ir_2$ of $\mathbf{U}$.

Consider the projector operator

$$\mathbf{E}_{11}=\frac{(\mathbf{U}+ir_1\mathbf{I})(\mathbf{U}^2+r_2^2\mathbf{I})(\mathbf{U}^2-r_1^2\mathbf{I})(\mathbf{U}^2-r_2^2\mathbf{I})}{-4r_1^3i(r_1^4-r_2^4)} \tag{2.14}$$

Define the even annhilator operator

$$\mathbf{Z}_0=\left(\mathbf{U}^2-r_1^2\mathbf{I}\right)\left(\mathbf{U}^2-r_2^2\mathbf{I}\right) \tag{2.15}$$

Theorem 2.3. *For every* $\mathbf{f}\in\mathbb{C}^N$, $\mathbf{Z}_0\mathbf{f}$ *is odd.*

Proof.

$$\begin{aligned}\text{(2.16)}\quad \mathbf{Z}_0\mathbf{f}(2k) &= \mathbf{U}^4\mathbf{f}(2k)-\left(r_1^2+r_2^2\right)\mathbf{U}^2\mathbf{f}(2k)+r_1^2r_2^2f(2k)\\ &= N^2\left[\frac{4}{3}(f(2k)-f(-2k))-\frac{8}{9}\left(-f\left(-2k-\frac{N}{2}\right)+f\left(2k+\frac{N}{2}\right)\right)\right]\end{aligned}$$

$$\begin{aligned}\text{(2.17)}\quad \mathbf{Z}_0\mathbf{f}(2k+1) &= \mathbf{U}^4\mathbf{f}(2k+1)-\left(r_1^2+r_2^2\right)\mathbf{U}^2\mathbf{f}(2k+1)+r_1^2r_2^2f(2k+1)\\ &= N^2\left[\frac{8}{3}(f(2k+1)-f(-2k-1))-\frac{4}{9}\left(-f\left(-2k-1-\frac{N}{2}\right)\right.\right.\\ &+\left.\left. f\left(2k+1+\frac{N}{2}\right)\right)\right]\end{aligned}$$

which follows from the duality property. Clearly $\mathbf{Z}_0\mathbf{f}(-2k)=-\mathbf{Z}_0\mathbf{f}(2k)$ and $\mathbf{Z}_0\mathbf{f}(-2k-1)=-\mathbf{Z}_0\mathbf{f}(2k+1)$ which implies that $\mathbf{Z}_0\mathbf{f}$ is odd. Furthermore if $\mathbf{f}=\mathbf{f}_{\mathtt{E}}\in\mathbb{C}^N$ is even then $\mathbf{Z}_0\mathbf{f}_{\mathtt{E}}=\mathbf{0}$ or

$$\left(\mathbf{U}^2-r_1^2\mathbf{I}\right)\left(\mathbf{U}^2-r_2^2\mathbf{I}\right)\mathbf{f}_{\mathtt{E}}=\mathbf{0}. \tag{2.18}$$

∎

Thus $\mathbf{Z}_0$ annihilates every even vector. Since every eigenvector corresponding to real eigenvalues is real it follows that every eigenvector satisfies (2.18). The effect of $\mathbf{Z}_0$ is to map $\mathbf{f}\in\mathbb{C}^N$ into the odd subspace $\mathbb{C}_0$. The effect of $(\mathbf{U}^2+ir_1\mathbf{I})(\mathbf{U}^2+r_2^2\mathbf{I})$ is to map $\mathbf{Z}_0\mathbf{f}$ into the odd subspace $\mathcal{N}_{11}$ or to generate an eigenvector corresponding to ir_1. The denominator in (2.14) is a normalizing factor that ensures that $\mathbf{E}_{11}$ is a projection operator. An analysis of the remaining projection operators reveals that $\mathbf{Z}_{\mathtt{E}}$ is a factor of $\mathbf{E}_{pl}$, $p=1,2$:$l=0,2$ and $\mathbf{Z}_0$ is a factor of $\mathbf{E}_{pl}$, $p=1,2$: $l=1,3$.

Theorem 2.4. *If* $\mathbf{f} \in \mathcal{N}(\mathbf{Z}_0)$ *then* $\mathbf{f}$ *is even.*

Proof. If $\mathbf{Z}_0\mathbf{f} = \mathbf{0}$ then from (2.17) and (2.18) it follows that

$$f(2k) - f(-2k) = \frac{2}{3}\left[f\left(2k + \frac{N}{2}\right) - f\left(-2k - \frac{N}{2}\right)\right] \tag{2.19}$$

$$f(2k+1) - f(-2k-1) = \frac{1}{6}\left[f\left(2k + 1 + \frac{N}{2}\right) - f\left(-2k - 1 - \frac{N}{2}\right)\right] \tag{2.20}$$

Replacing k by $k + \frac{N-2}{4}$ in (2.20) and combining the result with (2.19) one can show that

$$f(2k) - f(-2k) = 0 \tag{2.21}$$

Replacing k by $k + \frac{N+2}{4}$ in (2.19) and combining the result with (2.20) one can show that

$$f(2k+1) - f(-2k-1) = 0 \tag{2.22}$$

∎

A similar procedure was used in proving theorem 2. This implies that $\mathbf{f} \in \mathbb{C}_{\mathtt{E}}$. Thus $\mathcal{N}(\mathbf{Z}_0) = \mathbb{C}_{\mathtt{E}}$ of dimension $\frac{N}{2} + 1$ and $\mathcal{R}(\mathbf{Z}_0) = \mathbb{C}_0$ of dimension $\frac{N}{2} - 1$ which imply that rank of $\mathbf{Z}_0$ is $\frac{N}{2} - 1$. Furthermore $\mathbf{Z}_0$ has eigenvalues:

i) $2r_1^2(r_1^2 + r_2^2)$ of multiplicity m which it inherits from the eigenvalues of $\pm ir_1$ of $\mathbf{U}$
ii) $2r_2^2(r_1^2 + r_2^2)$ of multiplicity m which it inherits from the eigenvalues of $\pm ir_2$ of $\mathbf{U}$
iii) 0 of multiplicity $2m+2$ which it inherits from the eigenvalues of $\pm r_1$ and $\pm r_2$ of $\mathbf{U}$

3. STRUCTURAL DECOMPOSITION

The space $\mathbb{C}^N$ is fragmented into four even subspaces $\mathcal{N}_{pl}$,$p = 1, 2$: $l = 0, 2$ using the oblique projection operators $\mathbf{E}_{pl}$,$p = 1, 2$: $l = 0, 2$ and into four odd subspaces $\mathcal{N}_{pl}$,$p = 1, 2$: $l = 1, 3$ using the oblique projection operators $\mathbf{E}_{pl}$ $p = 1, 2$: $l = 1, 3$. This is depicted in the figure 1. Hence $\mathbb{C}_{\mathtt{E}} = \mathcal{N}_{10} \oplus \mathcal{N}_{12} \oplus \mathcal{N}_{20} \oplus \mathcal{N}_{22}$ and $\mathbb{C}_0 = \mathcal{N}_{11} \oplus \mathcal{N}_{13} \oplus \mathcal{N}_{21} \oplus \mathcal{N}_{23}$.

Theorem 3.1. *The projection matrices* $\mathbf{E}_{pl}$, $p = 1, 2$*:* $l = 0, 1, 2, 3$ *are real.*

Proof. We prove the result for $\mathbf{E}_{11}$ defined in (2.14). Let $\mathbf{f} \in \mathbb{C}^N$ be an arbitrary vector then

$$\mathbf{E}_{11}\mathbf{f} = \frac{(\mathbf{U} + ir_1\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{Z}_0\mathbf{f}}{-4r_1^3 i(r_1^4 - r_2^4)} \tag{3.1}$$

Conjugating $\mathbf{E}_{11}$ and operating on $\mathbf{f}$ we obtain

$$\overline{\mathbf{E}}_{11}\mathbf{f} = \frac{\left(\overline{\mathbf{U}} - ir_1\mathbf{I}\right)\left(\overline{\mathbf{U}^2} + r_2^2\mathbf{I}\right)\overline{\mathbf{Z}}_0\mathbf{f}}{4r_1^3 i(r_1^4 - r_2^4)} \tag{3.2}$$

Using the fact that $\mathbf{U}^2$ is real[5] it follows that $\overline{\mathbf{Z}}_0$ is real. From theorem 3 and the fact that the transform of an odd signal is odd[7] we conclude that the vector

$$\hat{\mathbf{f}} = \frac{(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{Z}_0\mathbf{f}}{4r_1^3 i(r_1^4 - r_2^4)} \tag{3.3}$$

is odd. Rewriting (3.1) and (3.2) results in

$$\mathbf{E}_{11}\mathbf{f} = (-\mathbf{U} - ir_1\mathbf{I})\hat{\mathbf{f}} \tag{3.4}$$

$$\overline{\mathbf{E}}_{11}\mathbf{f} = (\overline{\mathbf{U}} - ir_1\mathbf{I})\hat{\mathbf{f}} \tag{3.5}$$

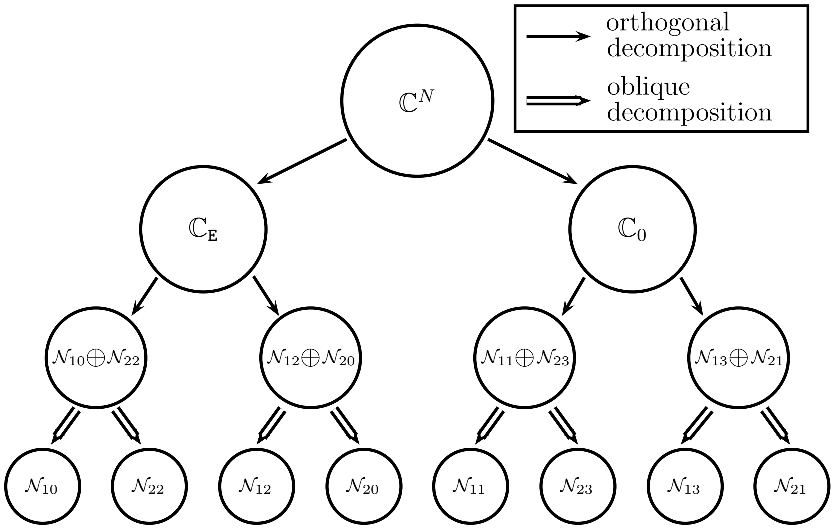

Figure 1: Decomposition of $\mathbb{C}^N$

In order to show that $\mathbf{E}_{11}\mathbf{f} = \overline{\mathbf{E}}_{11}\mathbf{f}$, it remains to be shown that $\overline{\mathbf{U}}\hat{\mathbf{f}} = -\mathbf{U}\hat{\mathbf{f}}$. Transforming the conjugate vector $\overline{\hat{\mathbf{f}}}$ we obtain

(3.6) $$\mathbf{U}\overline{\hat{\mathbf{f}}}(k) = \overline{\mathbf{U}\hat{\mathbf{f}}(-k)}$$

using the conjugation property[7]. Further conjugating (3.6) results in

(3.7) $$\overline{\mathbf{U}}\hat{\mathbf{f}}(k) = \mathbf{U}\hat{\mathbf{f}}(-k) = -\mathbf{U}\hat{\mathbf{f}}(k)$$

where we have used the fact that $\hat{\mathbf{f}}$ is an odd vector. ∎

Theorem 3.2. *The subspace* $\mathcal{N}_{10}$ *is perpendicular to the subspace* $\mathcal{N}_{12}$

Proof. Let $\mathbf{a} = (\mathbf{U} - r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\,\mathbf{f}$ for some $\mathbf{f} \in \mathbb{C}_{\mathtt{E}}$ and $\mathbf{b} = (\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\,\mathbf{g}$ for some $\mathbf{g} \in \mathbb{C}_{\mathtt{E}}$ then taking the inner product we get

(3.8) $$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{\frac{N}{2}-1} a\,(2k)\,\overline{b\,(2k)} + \sum_{k=0}^{\frac{N}{2}-1} a\,(2k+1)\,\overline{b\,(2k+1)},$$

where $a\,(2k) = (\mathbf{U}^3 - r_1\mathbf{U}^2 - r_2^2\mathbf{U} + r_1r_2^2\mathbf{I})\,\mathbf{f}\,(2k)$. Now

(3.9) $$\mathbf{U}\mathbf{f}\,(2k) = F_0\,(2k) + F_1\,(2k)$$

(3.10) $$\mathbf{U}^2\mathbf{f}\,(2k) = \frac{2}{3}Nf\,(2k) - \frac{4}{9}Nf\left(2k + \frac{N}{2}\right)$$

(3.11) $$\mathbf{U}^3\mathbf{f}\,(2k) = \frac{2}{9}NF_0\,(2k) + \frac{10}{9}NF_1\,(2k)$$

where we have made use of the duality property and the fact that $\mathbf{f}$ is even which also implies that $\mathbf{F}_0$ and $\mathbf{F}_1$ defined in (1.4) and (1.5) are even. Using (3.9),(3.10) and (3.11) the components

$a\,(2k)$ simplifies to

$$a\,(2k) = N\left[\alpha_1 F_0\,(2k) + \alpha_2 F_1\,(2k) + \alpha_3 r_1 f\,(2k) + \alpha_4 r_1 f\left(2k + \frac{N}{2}\right)\right]. \tag{3.12}$$

Similarly it can be shown that

$$\overline{b\,(2k)} = N\left[\alpha_1 \overline{G_0\,(2k)} + \alpha_2 \overline{G_1\,(2k)} - \alpha_3 r_1 \overline{g\,(2k)} - \alpha_4 r_1 \overline{g\left(2k + \frac{N}{2}\right)}\right], \tag{3.13}$$

where $\mathbf{G}_0 + \mathbf{G}_1$ denotes the transform of **g**, $\alpha_1 = \frac{-7+\sqrt{17}}{9}$, $\alpha_2 = \frac{1+\sqrt{17}}{9}$, $\alpha_3 = \frac{3-\sqrt{17}}{9}$ and $\alpha_4 = \frac{4}{9}$. Hence the first summation in (3.8) becomes

(3.14)

$$\sum_{k=0}^{\frac{N}{2}-1} a\,(2k)\,\overline{b\,(2k)} = N^2\left[\alpha_1^2 \sum_{k=0}^{\frac{N}{2}-1} F_0\,(2k)\,\overline{G_0\,(2k)} - \alpha_3^2 r_1^2 \sum_{k=0}^{\frac{N}{2}-1} f\,(2k)\,\overline{g\,(2k)}\right] +$$

$$N^2\left[\alpha_2^2 \sum_{k=0}^{\frac{N}{2}-1} F_1\,(2k)\,\overline{G_1\,(2k)} - \alpha_4^2 r_1^2 \sum_{k=0}^{\frac{N}{2}-1} f\left(2k + \frac{N}{2}\right)\overline{g\left(2k + \frac{N}{2}\right)}\right] + \tag{3.15}$$

$$N^2\left[\alpha_1\alpha_2 \sum_{k=0}^{\frac{N}{2}-1} F_0\,(2k)\,\overline{G_1\,(2k)} - \alpha_3\alpha_4 r_1^2 \sum_{k=0}^{\frac{N}{2}-1} f\,(2k)\,\overline{g\left(2k + \frac{N}{2}\right)}\right] + \tag{3.16}$$

$$N^2\left[\alpha_1\alpha_2 \sum_{k=0}^{\frac{N}{2}-1} F_1\,(2k)\,\overline{G_0\,(2k)} - \alpha_3\alpha_4 r_1^2 \sum_{k=0}^{\frac{N}{2}-1} f\left(2k + \frac{N}{2}\right)\overline{g\,(2k)}\right] + \tag{3.17}$$

$$\alpha_1\alpha_3 r_1 N^2\left[\sum_{k=0}^{\frac{N}{2}-1} f\,(2k)\,\overline{G_0\,(2k)} - \sum_{k=0}^{\frac{N}{2}-1} F_0\,(2k)\,\overline{g\,(2k)}\right] + \tag{3.18}$$

$$r_1 N^2\left[\alpha_1\alpha_4 \sum_{k=0}^{\frac{N}{2}-1} f\left(2k + \frac{N}{2}\right)\overline{G_0\,(2k)} - \alpha_2\alpha_3 \sum_{k=0}^{\frac{N}{2}-1} F_1\,(2k)\,\overline{g\,(2k)}\right] + \tag{3.19}$$

$$r_1 N^2\left[\alpha_2\alpha_3 \sum_{k=0}^{\frac{N}{2}-1} f\,(2k)\,\overline{G_1\,(2k)} - \alpha_1\alpha_4 \sum_{k=0}^{\frac{N}{2}-1} F_0\,(2k)\,\overline{g\left(2k + \frac{N}{2}\right)}\right] + \tag{3.20}$$

$$\alpha_2\alpha_4 r_1 N^2\left[\sum_{k=0}^{\frac{N}{2}-1} f\left(2k + \frac{N}{2}\right)\overline{G_1\,(2k)} - \sum_{k=0}^{\frac{N}{2}-1} F_1\,(2k)\,\overline{g\left(2k + \frac{N}{2}\right)}\right]. \tag{3.21}$$

The first summation in (3.15) can be simplified in the following manner

$$\sum_{k=0}^{\frac{N}{2}-1} F_1\,(2k)\,\overline{G_1\,(2k)} = \frac{16}{9} \sum_{j=0}^{\frac{N}{2}-1} \sum_{p=0}^{\frac{N}{2}-1} \sum_{k=0}^{\frac{N}{2}-1} \omega^{2k(2j-2p)} f\,(2j+1)\,\overline{g\,(2p+1)}$$

$$= \frac{16}{9}\left(\frac{N}{2}\right) \sum_{j=0}^{\frac{N}{2}-1} f\,(2j+1)\,\overline{g\,(2j+1)}, \tag{3.22}$$

since the inner summation is zero unless $p = j$. Changing to the dummy variable k in (3.22) we further obtain

$$\sum_{k=0}^{\frac{N}{2}-1} F_1(2k)\overline{G_1(2k)} = \frac{8}{9}N\sum_{k=0}^{\frac{N}{2}-1} f(2k+1)\overline{g(2k+1)} \tag{3.23}$$

$$= \frac{8}{9}N\sum_{k=\frac{N}{4}-\frac{1}{2}}^{\frac{3N}{4}-\frac{3}{2}} f(2k+1)\overline{g(2k+1)} \tag{3.24}$$

$$= \frac{8}{9}N\sum_{k=0}^{\frac{N}{2}-1} f\left(2k+\frac{N}{2}\right)\overline{g\left(2k+\frac{N}{2}\right)}, \tag{3.25}$$

where (3.24) is obtained from (3.23) by using the periodicity $\frac{N}{2}$ of the argument of the summation [1]. Finally (3.15) reduces to

$$N^2\left[\frac{8}{9}N\alpha_2^2 - \alpha_4^2 r_1^2\right]\sum_{k=0}^{\frac{N}{2}-1} f\left(2k+\frac{N}{2}\right)\overline{g\left(2k+\frac{N}{2}\right)} = 0, \tag{3.26}$$

since the coefficient is zero. Similarly it can be shown that (3.14), (3.16) and (3.17)are zero. Consider the first summation in (3.18), it may be simplified as follows

$$\sum_{k=0}^{\frac{N}{2}-1} f(2k)\overline{G_0(2k)} = \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1} \omega^{-2k(2j)} f(2k)\overline{g(2j)}$$

$$= \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=-\frac{N}{2}+1}^{0} \omega^{-2k(2j)} f(2k)\overline{g(2j)}$$

$$= \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1} \omega^{2k(2j)} f(2k)\overline{g(-2j)}. \tag{3.27}$$

Using the fact that g is even (3.27) simplifies to

$$\sum_{k=0}^{\frac{N}{2}-1} f(2k)\overline{G_0(2k)} = \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1} \omega^{2k(2j)} f(2k)\overline{g(2j)}$$

$$= \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1} \omega^{2k(2j)} f(2k)\overline{g(2k)}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} F_0(2k)\overline{g(2k)}. \tag{3.28}$$

Hence (3.18) is identically zero. Similarly it may be shown that (3.21) is zero. In (3.19) the first sum becomes

$$\sum_{k=0}^{\frac{N}{2}-1} f\left(2k+\frac{N}{2}\right)\overline{G_0\left(2k\right)} = \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-2k(2j)} f\left(2k+\frac{N}{2}\right)\overline{g\left(2j\right)}$$
$$= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\sum_{k=\frac{N}{4}+\frac{1}{2}}^{\frac{3N}{4}-\frac{1}{2}}\omega^{-2k2j} f\left(2k+\frac{N}{2}\right)\overline{g\left(2j\right)}$$
$$= \frac{2}{3}\sum_{j=0}^{\frac{N}{2}-1}\sum_{k=0}^{\frac{N}{2}-1}\omega^{-2j(2k+1)} f\left(2k+\frac{N}{2}\right)\overline{g\left(2j\right)}$$
$$= \frac{2}{3}\sum_{k=0}^{\frac{N}{2}-1}\sum_{j=0}^{\frac{N}{2}-1}\omega^{-2k(2j+1)} f\left(2j+1\right)\overline{g\left(2k\right)}$$
$$= \frac{1}{2}\sum_{k=0}^{\frac{N}{2}-1} F_1\left(2k\right)\overline{g\left(2k\right)} \tag{3.29}$$

Finally (3.19) becomes

$$r_1 N^2\left[\frac{\alpha_1\alpha_4}{2}-\alpha_2\alpha_3\right]\sum_{k=0}^{\frac{N}{2}-1} F_1\left(2k\right)\overline{g\left(2k\right)} = 0,$$

since the coefficient is zero. Similarly (3.20) may be shown to be zero. Likewise, it can be shown that $\sum_{k=0}^{\frac{N}{2}-1} a\left(2k+1\right)\overline{b\left(2k+1\right)} = 0$ which implies that $\mathcal{N}_{10}\perp\mathcal{N}_{12}$. ∎

It may be shown that not all subspaces are orthogonal. The relationship between the subspaces $\mathcal{N}_{pl}$,$p=1,2$:$l=0,1,2,3$ is illustrated in figure 1. In fact $\mathbf{P}_{\mathtt{E}}$ from (2.1) is given by

$$\mathbf{P}_{\mathtt{E}} = \mathbf{E}_{10}+\mathbf{E}_{12}+\mathbf{E}_{20}+\mathbf{E}_{22} \tag{3.30}$$

and

$$\mathbf{I}-\mathbf{P}_{\mathtt{E}} = \mathbf{E}_{11}+\mathbf{E}_{13}+\mathbf{E}_{21}+\mathbf{E}_{23} \tag{3.31}$$

4. SIMPSON DISCRETE FRACTIONAL FOURIER TRANSFORM

Using the projection operators of $\mathbf{U}$ a fractional transform is defined as follows:

$$\mathbf{U}^{\alpha} = \sum_{p=1}^{2}\sum_{l=0}^{3} r_p^{\alpha} e^{i\frac{\pi}{2}l\alpha}\mathbf{E}_{pl}, \tag{4.1}$$

where $\alpha\in\mathbb{R}$. With the definition (4.1) the following properties are satisfied:
(a) The fractional transform of an even signal is even.
(b) The fractional transform of an odd signal is odd.
(c) Time reversal:$\mathbf{U}^{\alpha}\mathbf{f}(-k)$ is the fractional transform of the reversed signal $f(-j), j=0,1,\cdots,N-1$.
(d) Additive:$\mathbf{U}^{\alpha}\mathbf{U}^{\beta} = \mathbf{U}^{\alpha+\beta}$. From this we obtain the inverse fractional transform $\mathbf{U}^{-\alpha}$ by setting $\beta=-\alpha$ in conjunction with (4.1).

5. CONCLUSION

We have investigated the constituent matrices of $\mathbf{U}$, shown that they are real and determined the spatial orientation of their respective ranges. These projectors have been used to define a Simpson Discrete Fractional Fourier Transform.

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