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ANALYSIS OF A FRICTIONAL CONTACT PROBLEM FOR VISCOELASTIC PIEZOELECTRIC MATERIALS

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ABSTRACT. In this paper, we consider a mathematical model that describes the quasi-static process of contact between two thermo-electro-viscoelastic bodies with damage and adhesion. The damage of the materials caused by elastic deformations. The contact is frictional and modeled with a normal compliance condition involving adhesion effect of contact surfaces. Evolution of the bonding field is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations, and fixed point theorem.

Key words and phrases: Damage; Adhesion; Normal compliance; Frictional contact; Piezoelectric materials.

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1. INTRODUCTION

The piezoelectric phenomenon represents the coupling between the mechanical and electrical behavior of a class of materials, called piezoelectric materials. In simplest terms, when a piezoelectric material is squeezed, an electric charge collects on its surface, conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate and polyvinylidene flouride (a polymer film). Piezoelectric materials are used extensively as switches and actually in many engineering systems in radioelectronics, electroacoustics and measuring equipment.

Different models have been developed to describe the interaction between the electric and mechanical fields (see, e.g.[2, 9, 10, 21, 25]). Therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties.

General models for elastic materials with piezoelectric effects can be found in [7, 9, 15, 16] and more recently in [1], viscoelastic piezoelectric materials in [21, 25] or elasto-viscoplastic piezoelectric materials have been studied in [12]. The coupling between the thermal, electric and mechanical fields in piezo electric materials provides a mechanism for sensing thermo mechanical disturbances from measurements of induced electric potentials, and for altering structural responses via applied electric fields.

One of the applications of the thermo-piezoelectric material is to detect the responses of a structure by measuring the electric charge, sensing or to reduce excessive responses by applying additional electric forces or thermal forces actuating. If sensing and actuating can be integrated smartly, a so-called intelligent structure can be designed. The piezoelectric materials are also often used as resonators whose frequencies need to be precisely controlled. The coupling between the thermo-piezoelectric and pyroelectric effects, it is important to qualify the effect of heat dissipation on the propagation of wave at low and high frequencies.

The thermo- piezoelectric theory was first proposed by Mindlin [14], later he derived the governing equations of a thermo-piezoelectric plate [17]. The physical laws for the thermo-piezoelectric materials have been discussed by [19, 20]. Chandrasekhariah [4, 5] presented the generalized theory of thermo-piezoelectricity by taking into account the finite speed of propagation of thermal disturbance. Yang and Batra [28] studied the effect of heat conduction on shift in the frequencies of a freely vibrating linear thermo-piezoelectric body with the help of perturbation methods. Sharma and Walia [24] presented the propagation of straight and circular crested waves in generalized piezo thermoelastic materials. The normal compliance contact condition was first considered in [21, 25, 26] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [23]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [9, 10, 11, 22] and recently in the monographs [18].

In these papers, the bonding field, denoted by ζ, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [6], the bonding field satisfies the restriction $0 \leq \zeta \leq 1$, when $\zeta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\zeta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \zeta < 1$ the adhesion is partial and only a fraction ζ of the bonds is active. The novelty of this work lies in the analysis of a system that contains strong couplings in the multivalued boundary conditions: both the normal compliance and the friction law depend on the adhesion (see (2.15) and (2.16)), and the adhesion be written by the differential equation of the form

$$\dot{\zeta} = H_{ad}\big(\zeta, \alpha_\zeta, R_\nu(u^1_\nu + u^2_\nu), \boldsymbol{R}_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau)\big).$$

Here, H_{ad} is the *adhesion evolution rate function*. Then, the adhesion rate function was assumed to depend, in addition to ζ, $R_\nu(u^1_\nu + u^2_\nu)$, $\boldsymbol{R}_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau)$ and α_ζ, where, the truncation operators R_ν, $\boldsymbol{R}_\tau$ are defined by (2.23), and $\alpha_\zeta(x,t) = \int_0^t \zeta(x,s)ds$. We use it in H_{ad}, since usually when the glue is stretched beyond the limit L it does not contribute more to the bond strength. An example of such a function, used in [6], the following form of the evolution of the bonding field was employed $\dot{\zeta} = -\zeta_+ \gamma_n R_\nu(u_\nu)^2$, where γ_n is the normal rate coefficient and $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide and $\zeta_+ = \max(0, \zeta)$. We note that in this case, only debonding is allowed. A different rate equation for the the evolution of the bonding field is $\dot{\zeta} = -(\zeta\big(\gamma_n R_\nu(u^1_\nu + u^2_\nu)^2 + \gamma_t |\boldsymbol{R}_\tau(\boldsymbol{u}^1_\tau - \boldsymbol{u}^2_\tau)|^2\big) - \varepsilon_a)_+$, see, e.g., [9, 10]. Here, γ_t is the tangential rate coefficient, which may also be interpreted as the tangential stiffness coefficient of the interface when the adhesion is complete $(\zeta = 1)$. However, the bonding cannot exceed $\zeta = 1$ and, moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1 + \alpha^2_\zeta$ in the denominator. In all these papers the damage of the material is described with a damage function ς^ℓ, restricted to have values between zero and one. When $\varsigma^\ell = 1$, there is no damage in the material, when $\varsigma^\ell = 0$, the material is completely damaged, when $0 < \varsigma^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity.

In this paper, we study the quasi-static frictional contact problem between two viscoelastic piezoelectric bodies with damage, adhesion and normal compliance. In Section 2, we describe the mathematical models for the frictional contact problem between two thermo-electro-viscoelastic bodies with long-term memory and damage. The contact is modelled with normal compliance and adhesion. We introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. We prove in Section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on arguments of evolutionary variational equalities, differential equations and Banach fixed point theorem.

2. PROBLEM STATEMENT AND VARIATIONAL FORMULATION

We describe the model for the process, we present its variational formulation. The physical setting is the following. Let us consider two thermo-electro-viscoelastic bodies with long-term memory, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. We put a superscript ℓ to indicate that the quantity is related to the domain Ω^ℓ, $\ell = 1, 2$. In the following, the superscript ℓ ranges between 1 and 2. For each domain Ω^ℓ, ($\ell = 1, 2$) the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ^ℓ_1, Γ^ℓ_2 and Γ^ℓ_3, on one hand, and on two measurable parts Γ^ℓ_a and Γ^ℓ_b, on the other hand, such that $meas\Gamma^\ell_1 > 0$, $meas\Gamma^\ell_a > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}^\ell_0$ forces and volume electric charges of density q^ℓ_0. The bodies are assumed to be clamped on $\Gamma^\ell_1 \times (0, T)$, so the displacement field vanishes there. The surface tractions $\boldsymbol{f}^\ell_2$ act on $\Gamma^\ell_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma^\ell_a \times (0, T)$ and a surface electric charge of density q^ℓ_2 is prescribed on $\Gamma^\ell_b \times (0, T)$. The two bodies are in contact along the common part $\Gamma^1_3 = \Gamma^2_3$, which will be denoted Γ_3 below. The bodies is in adhesive contact, over the contact surface Γ_3, the contact is frictional and is modeled with the normal compliance condition and a version of Coulomb's law of friction. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasi-static; i.e., the inertial terms in the momentum balance equations are neglected. We denote by $\boldsymbol{u}^\ell$ the displacement field, by $\boldsymbol{\sigma}^\ell$ the stress field and by $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ the linearized strain tensor. We use an thermo-electro-viscoelastic constitutive law with damage given by

(2.1)
$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \tau^\ell, \varsigma^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \tau^\ell(s), \varsigma^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\xi^\ell),$$

where $\mathcal{A}^\ell$ is a given nonlinear operator, $\mathcal{Q}^\ell$ is the relaxation operator, $\mathcal{B}^\ell$ represents the elasticity operator, where τ^ℓ represents the absolute temperature and ς^ℓ is the damage field. $E(\xi^\ell) = -\nabla\xi^\ell$ is the electric field, $\mathcal{E}^\ell$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. We study a quasi-static Coulomb's frictional contact problem between two thermo-electro-viscoelastic bodies with long-term memory and damage. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

In (2.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. It follows from (2.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}^\ell_V(t) + \boldsymbol{\sigma}^\ell_E(t) + \boldsymbol{\sigma}^\ell_R(t)$, where $\boldsymbol{\sigma}^\ell_V(t) = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^\ell_E(t) = (\mathcal{E}^\ell)^*\nabla\xi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}^\ell_R(t)$ satisfies a rate-type elastic relation

$$\boldsymbol{\sigma}^\ell_R(t) = \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \tau^\ell(t), \varsigma^\ell(t)\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \tau^\ell(s), \varsigma^\ell(s)\big)\, ds.$$

Note also that when $\mathcal{Q}^\ell = 0$ the constitutive law (2.1) becomes the Kelvin-Voigt viscoelastic piezoelectric with damage and thermal effects constitutive relation,

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \tau^\ell(t), \varsigma^\ell(t)\big) + (\mathcal{E}^\ell)^*\nabla\xi^\ell(t).$$

Quasistatic evolution of damage in viscoplastic materials has been studied in [13]. According to Batra and Yang [1] the following constitutive law is employed for the electric potential:

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\xi^\ell)\big), \tag{2.2}$$

where $\boldsymbol{D}^\ell$ is the electric displacement field and $\mathcal{G}^\ell$ is the electric permittivity tensor.

The differential inclusion used for the evolution of the damage field is

$$\dot{\varsigma}^\ell - \kappa^\ell\Delta\varsigma^\ell + \partial\psi_{K^\ell}(\varsigma^\ell) \ni \Psi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \varsigma^\ell\big) \text{ in } \Omega^\ell\times(0,T), \tag{2.3}$$

where K^ℓ denotes the set of admissible damage functions defined by

$$K^\ell = \{\alpha\in H^1(\Omega^\ell);\ 0\le\alpha\le 1, \text{ a.e. in } \Omega^\ell\}, \tag{2.4}$$

κ^ℓ is a positive coefficient, $\partial\psi_{K^\ell}$ represents the subdifferential of the indicator function of the set K^ℓ and ϕ^ℓ is a given constitutive function which describes the sources of the damage in the system. When $\varsigma^\ell = 1$, there is no damage in the material, when $\varsigma^\ell = 0$, the material is completely damaged, when $0 < \varsigma^\ell < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [9, 25, 26].

The thermo-electro-viscoelastic constitutive law (2.1) includes a temperature effects described by the parabolic aquation given by

$$\dot{\tau}^\ell - \kappa_0^\ell\Delta\tau^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \tau^\ell, \varsigma^\ell\big) + \rho^\ell. \tag{2.5}$$

With these assumptions, the classical formulation of the quasi-static problem for frictional contact problem between two thermo-electro-viscoelastic bodies with damage, normal compliance and adhesion is the following.

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{S}^d$, , a temperature $\tau^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}$, a damage field $\varsigma^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}$, an electric potential field $\xi^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}$, a bonding field $\zeta : \Gamma_3 \times (0,T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \tau^\ell(t), \varsigma^\ell(t)\big) + \\ \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \tau^\ell(s), \varsigma^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\xi^\ell(t)),\end{aligned} \quad \text{in } \Omega^\ell \times (0,T), \tag{2.6}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\xi^\ell)\big), \quad \text{in } \Omega^\ell \times (0,T), \tag{2.7}$$

$$\dot{\tau}^\ell - \kappa_0^\ell \Delta\tau^\ell = \Theta^\ell\big(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \tau^\ell, \varsigma^\ell\big) + \rho^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{2.8}$$

$$\dot{\varsigma}^\ell - \kappa^\ell \Delta\varsigma^\ell + \partial\psi_{K^\ell}(\varsigma^\ell) \ni \Psi^\ell\big(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \varsigma^\ell\big) \text{ in } \Omega^\ell \times (0,T), \tag{2.9}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.10}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.11}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{2.12}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{2.13}$$

$$\dot{\zeta} = H_{ad}\big(\zeta, \alpha_\zeta, R_\nu([u_\nu]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big), \quad \text{on } \Gamma_3 \times (0,T), \tag{2.14}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \text{ where } \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu \zeta^2 R_\nu([u_\nu]) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.15}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \zeta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| \le \mu p_\nu([u_\nu]), \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \zeta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| < \mu p_\nu([u_\nu]) \Rightarrow [\dot{\boldsymbol{u}}_\tau] = 0, \quad \text{on } \Gamma_3 \times (0,T), \\ \big\|\boldsymbol{\sigma}_\tau + \gamma_\tau \zeta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\big\| = \mu p_\nu([u_\nu]) \Rightarrow \exists \lambda \ge 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau + \gamma_\tau \zeta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]) = -\lambda[\dot{\boldsymbol{u}}_\tau] \end{cases} \tag{2.16}$$

$$\xi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.17}$$

$$\boldsymbol{D}^\ell . \boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.18}$$

$$\kappa_0^\ell \frac{\partial^\ell \tau^\ell}{\partial \nu^\ell} + \lambda_0^\ell \tau^\ell = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.19}$$

$$\frac{\partial \varsigma^\ell}{\partial \nu^\ell} = 0 \quad \text{on } \Gamma^\ell \times (0,T), \tag{2.20}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \ \ \tau^\ell(0) = \tau_0^\ell, \ \ \varsigma^\ell(0) = \varsigma_0^\ell \quad \text{in } \Omega^\ell, \tag{2.21}$$

$$\zeta(0) = \zeta_0 \quad \text{on } \Gamma_3. \tag{2.22}$$

Here and below $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$, whereas "." and $\|.\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively; ν^ℓ

is the unit outer normal vector on Γ^ℓ, equations (2.6) and (2.7) represent the thermo-electro-viscoelastic constitutive law with long term-memory and damage. Equation (2.8) represents the energy conservation where Θ^ℓ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ^ℓ is a given volume heat source. Inclusion (2.9) describes the evolution of the damage field. Equations (2.10) and (2.11) are the equilibrium equations for the stress and electric-displacement fields, respectively. Next, the equations (2.12) and (2.13) represent the displacement and traction boundary condition, respectively. Condition (2.15) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient, p_ν is a given positive function which will be described below and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is $[u_\nu]$ can be positive on Γ_3.

$$(2.23) \qquad R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases} \qquad \boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [22]). Condition (2.16) are a non local Coulomb's friction law conditions coupled with adhesive, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. Equation (2.14) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [26, 27]. (2.17) and (2.18) represent the electric boundary conditions. The relation (2.19) represent a Fourier boundary condition for the temperature on Γ^ℓ. The relation (2.20) represents a homogeneous Neumann boundary condition for the damage field on Γ^ℓ. Finally the functions $\boldsymbol{u}_0, \tau_0, \varsigma_0$ and ζ_0 in (2.21)-(2.22) are the initial data.

We now proceed to obtain a variational formulation of Problem P. For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices i and j run between 1 and d, the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let $H^\ell = L^2(\Omega^\ell)^d$, $H_1^\ell = H^1(\Omega^\ell)^d$, $\mathcal{H}^\ell = L^2(\Omega^\ell)_s^{d\times d}$, $\mathcal{H}_1^\ell = \{\boldsymbol{\theta}^\ell = (\theta_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\theta}^\ell \in H^\ell\}$. The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx, \quad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla\boldsymbol{u}^\ell.\nabla\boldsymbol{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\theta}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\theta}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\theta}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\theta}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\,\boldsymbol{\sigma}^\ell.\,\mathrm{Div}\,\boldsymbol{\theta}^\ell dx,$$

and the associated norms $\|.\|_{H^\ell}$, $\|.\|_{H_1^\ell}$, $\|.\|_{\mathcal{H}^\ell}$, and $\|.\|_{\mathcal{H}_1^\ell}$ respectively.
We introduce for the bonding field the set

$$\mathcal{Z} = \left\{\beta \in L^\infty\big(0, T; L^2(\Gamma_3)\big);\ 0 \le \beta(t) \le 1\ \ \forall t \in [0, T],\ \text{a.e. on}\ \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \left\{\boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell\right\}.$$

Since $meas\Gamma_1^\ell > 0$, the following Korn's inequality holds (see [18]) :

$$(2.24) \qquad \|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \ge c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell.$$

Over the space V^ℓ we consider the inner product given by

$$(2.25) \qquad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell,$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (2.24) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|.\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2.25), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{2.26}$$

We also introduce the spaces

$$L_0^\ell = L^2(\Omega^\ell), \quad L_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \left\{\psi^\ell \in L_1^\ell;\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell\right\},$$
$$\mathcal{W}^\ell = \left\{\boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell)\right\}.$$

Since $meas \Gamma_a^\ell > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla \psi^\ell\|_{W^\ell} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{2.27}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ. Over the space W^ℓ, we consider the inner product given by

$$(\xi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla \xi^\ell . \nabla \psi^\ell dx \tag{2.28}$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (2.27) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|.\|_{W^\ell})$ is areal Hilbert space. The space $\mathcal{W}^\ell$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{\Phi}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell . \boldsymbol{\Phi}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell . \operatorname{div} \boldsymbol{\Phi}^\ell dx,$$

where $\operatorname{div} \boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|.\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V} = V^1 \times V^2,\ H = H^1 \times H^2,\ H_1 = H_1^1 \times H_1^2,\ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,\ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2,$$
$$L_0 = L_0^1 \times L_0^2,\ L_1 = L_1^1 \times L_1^2,\ W = W^1 \times W^2,\ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, L_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_{L_1}$, $(.,.)_W$ and $(.,.)_{\mathcal{W}}$.

In the study of the Problem **P**, we consider the following assumptions:

The *viscosity function* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} : \forall\, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \\ \quad |\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_2)| \leq L_{\mathcal{A}^\ell} |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|, \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} : \forall\, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \\ \quad (\mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_2)) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geq m_{\mathcal{A}^\ell} |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|^2, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{\omega}) \text{ is measurable on } \Omega^\ell,\ \forall \boldsymbol{\omega} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x}, \boldsymbol{0}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases} \tag{2.29}$$

The *elasticity operator* $\mathcal{B}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{B}^\ell} > 0 \text{ such that} : \forall \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\ \qquad |\mathcal{B}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_1, r_1, d_1) - \mathcal{B}^\ell(\boldsymbol{x}, \boldsymbol{\omega}_2, r_2, d_2)| \leq L_{\mathcal{B}^\ell} \big(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |r_1 - r_2| + \\ \qquad |d_1 - d_2|\big), \text{ a.e.} \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x}, \boldsymbol{\omega}, r, d) \text{ is measurable in } \Omega^\ell, \ \forall \boldsymbol{\omega} \in \mathbb{S}^d, r, d \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x}, \boldsymbol{0}, 0, 0) \text{ belongs to } \mathcal{H}^\ell. \end{cases} \tag{2.30}$$

The *relaxation function* $\mathcal{Q}^\ell : \Omega^\ell \times (0,T) \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$(2.31)\quad \begin{cases} \text{(a) There exists } L_{\mathcal{Q}^\ell} > 0 \text{ such that } : \forall \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\ \qquad |\mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\omega}_1, r_1, d_1) - \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\omega}_2, r_2, d_2)| \le L_{\mathcal{Q}^\ell}\big(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + \\ \qquad |r_1 - r_2| + |d_1 - d_2|\big), \text{ for all } t \in (0,T), \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\omega}, r, d) \text{ is measurable in } \Omega^\ell, \\ \qquad \text{for any } t \in (0,T),\ \boldsymbol{\omega} \in \mathbb{S}^d,\ \ r, d \in \mathbb{R}. \\ \text{(c) The mapping } t \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \boldsymbol{\omega}, r, d) \text{ is continuous in } (0,T), \\ \qquad \text{for any } \boldsymbol{\omega} \in \mathbb{S}^d,\ \ r, d \in \mathbb{R},\ \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{Q}^\ell(\boldsymbol{x}, t, \mathbf{0}, 0, 0) \text{ belongs to } \mathcal{H}^\ell, \forall t \in (0,T). \end{cases}$$

The *energy function* $\Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$(2.32)\quad \begin{cases} \text{(a) There exists } L_{\Theta^\ell} > 0 \text{ such that } : \forall\, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2, d_1, d_2 \in \mathbb{R}, \\ \quad |\Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, \alpha_1, d_1) - \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, \alpha_2, d_2)| \le L_{\Theta^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + \\ \quad |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |\alpha_1 - \alpha_2| + |d_1 - d_2|\big), \ \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha, d) \text{ is measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d \text{ and } \alpha, d \in \mathbb{R}, \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Theta^\ell(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0, 0) \text{ belongs to } L^2(\Omega^\ell), \\ \text{(d) } \Theta^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha, d) \text{ is bounded for all } \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d, \alpha, d \in \mathbb{R} \ \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases}$$

The *adhesion rate function* $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

$$(2.33)\quad \begin{cases} \text{(a) There exists } L_{ad} > 0 \text{ such that } : \forall\, \zeta_1, \zeta_2, \omega_1, \omega_2, r_1, r_2 \in \mathbb{R}, d_1, d_2 \in \mathbb{R}^{d-1}, \\ \quad |H_{ad}(\boldsymbol{x}, \zeta_1, \omega_1, r_1, d_1) - H_{ad}(\boldsymbol{x}, \zeta_2, \omega_2, r_2, d_2)| \le L_\nu|\zeta_1 - \zeta_2| + |\omega_1 - \omega_2| + \\ \quad |r_1 - r_2| + |d_1 - d_2|, \ \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto H_{ad}(\boldsymbol{x}, \zeta, \omega, r, d) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } \zeta, \omega, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1}, \\ \text{(c) The mapping } (\zeta, \omega, r, d) \mapsto H_{ad}(\boldsymbol{x}, \zeta, \omega, r, d) \text{ is continuous on} \\ \quad \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(d) } H_{ad}(\boldsymbol{x}, 0, \omega, r, d) = 0, \forall\, \xi, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(e) } H_{ad}(\boldsymbol{x}, \zeta, \omega, r, d) \ge 0, \quad \forall\, \zeta \le 0, \omega, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \text{ and} \\ \quad H_{ad}(\boldsymbol{x}, \zeta, \omega, r, d) \le 0, \quad \forall\, \zeta \ge 1, \omega, r \in \mathbb{R},\ d \in \mathbb{R}^{d-1}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$(2.34)\quad \begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x}, \tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i, j, k \le d. \end{cases}$$

The *damage source function* $\Psi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$(2.35)\quad \begin{cases} \text{(a) There exists } L_{\Psi^\ell} > 0 \text{ such that } : \forall\, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}, \\ \quad |\Psi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, \alpha_1) - \Psi^\ell(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, \alpha_2)| \le L_{\Psi^\ell}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + \\ \quad |\alpha_1 - \alpha_2|\big), \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \Psi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha) \text{ is measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}, \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Psi^\ell(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0) \text{ belongs to } L^2(\Omega^\ell), \\ \text{(d) } \Psi^\ell(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha) \text{ is bounded}, \forall \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d, \alpha \in \mathbb{R} \ \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases}$$

The *electric permittivity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$, satisfies:

$$(2.36)\quad \begin{cases} \text{(a) } \mathcal{G}^\ell(\boldsymbol{x}, \mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j),\ b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i, j \le d. \\ \text{(b) There exists } m_{\mathcal{G}^\ell} > 0 \text{ such that :} \\ \quad \mathcal{G}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{G}^\ell}|\mathbf{E}|^2,\ \forall \mathbf{E} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{cases}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\text{(2.37)} \qquad \begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that } : \forall\, r_1, r_2 \in \mathbb{R}, \\ \quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu |r_1 - r_2|, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0, \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \ \forall r \in \mathbb{R}. \\ \text{(d) } p_\nu(\boldsymbol{x}, r) = 0, \text{ for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

The adhesion coefficients γ_ν and γ_τ satisfy the conditions

$$\text{(2.38)} \qquad \gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \gamma_\nu, \gamma_\tau \ge 0, \text{ a.e. on } \Gamma_3.$$

The *forces, tractions* have the regularity

$$\text{(2.39)} \qquad \begin{gathered} \mathbf{f}_0^\ell \in C(0,T; L^2(\Omega^\ell)^d), \quad \mathbf{f}_2^\ell \in C(0,T; L^2(\Gamma_2^\ell)^d), \\ q_0^\ell \in C(0,T; L^2(\Omega^\ell)), \quad q_2^\ell \in C(0,T; L^2(\Gamma_b^\ell)), \quad \rho^\ell \in C(0,T; L^2(\Omega^\ell)). \end{gathered}$$

The energy coefficient κ_0^ℓ and the microcrack diffusion coefficient κ^ℓ satisfies :

$$\text{(2.40)} \qquad \kappa_0^\ell > 0, \quad \kappa^\ell > 0.$$

Finally, the friction coefficient and the initial data satisfy:

$$\text{(2.41)} \qquad \begin{gathered} \mu \in L^\infty(\Gamma_3), \quad \mu(x) \ge 0 \quad \text{a.e. on } \Gamma_3, \\ \boldsymbol{u}_0^\ell \in \boldsymbol{V}^\ell, \ \varsigma_0^\ell \in K^\ell, \ \tau_0^\ell \in L_1^\ell, \ \zeta_0 \in L^2(\Gamma_3), \ 0 \le \zeta_0 \le 1, \text{ a.e. on } \Gamma_3. \end{gathered}$$

Next, we define the mappings $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to \boldsymbol{V}$, $q = (q^1, q^2) : [0,T] \to W$, $a_0 : L_1 \times L_1 \to \mathbb{R}$, $a : L_1 \times L_1 \to \mathbb{R}$, $j_{ad} : L^2(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{fr} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$\text{(2.42)} \qquad (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}} = \sum_{\ell=1}^2 \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \boldsymbol{v}^\ell \, dx + \sum_{\ell=1}^2 \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \cdot \boldsymbol{v}^\ell \, da \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

$$\text{(2.43)} \qquad (q(t), \phi)_W = \sum_{\ell=1}^2 \int_{\Omega^\ell} q_0^\ell(t) \phi^\ell \, dx - \sum_{\ell=1}^2 \int_{\Gamma_b^\ell} q_2^\ell(t) \phi^\ell \, da \quad \forall \phi \in W,$$

$$\text{(2.44)} \qquad a_0(\tau, \alpha) = \sum_{\ell=1}^2 \kappa_0^\ell \int_{\Omega^\ell} \nabla \tau^\ell . \nabla \alpha^\ell dx + \sum_{\ell=1}^2 \lambda_0^\ell \int_{\Gamma^\ell} \tau^\ell \alpha^\ell da,$$

$$\text{(2.45)} \qquad a(\varsigma, \alpha) = \sum_{\ell=1}^2 \kappa^\ell \int_{\Omega^\ell} \nabla \varsigma^\ell . \nabla \alpha^\ell dx,$$

$$\text{(2.46)} \qquad j_{ad}(\zeta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \zeta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau \zeta^2 \boldsymbol{R}_\tau([\boldsymbol{u}_\tau]).[\boldsymbol{v}_\tau] \Big) \, da,$$

$$\text{(2.47)} \qquad j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da,$$

$$\text{(2.48)} \qquad j_{fr}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p_\nu([u_\nu]) \big\| [v_\tau] \big\| \, da.$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.6)–(2.22).

Problem PV. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, an electric potential field $\xi = (\xi^1, \xi^2) : [0,T] \to W$, a temperature $\tau = (\tau^1, \tau^2) : [0,T] \to L_1$, a damage field $\varsigma = (\varsigma^1, \varsigma^2) : [0,T] \to L_1$, a bonding field

$\zeta : [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0,T] \to \mathcal{W}$ such that, for a.e. $t \in (0,T)$,

(2.49)
$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \tau^\ell, \varsigma^\ell\big) + \int_0^t \mathcal{Q}^\ell\big(t-s, \varepsilon(\boldsymbol{u}^\ell(s)), \tau^\ell(s), \varsigma^\ell(s)\big)\, ds - (\mathcal{E}^\ell)^* E^\ell(\xi^\ell),$$

(2.50)
$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) + \mathcal{G}^\ell\big(E^\ell(\xi^\ell)\big),$$

(2.51)
$$\left.\begin{aligned} &\textstyle\sum_{\ell=1}^2 (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)))_{\mathcal{H}^\ell} + j_{ad}(\zeta(t), \boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j_{fr}(\boldsymbol{u}(t), \boldsymbol{v}) \\ &-j_{fr}(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) + j_{\nu c}(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) \geq (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\boldsymbol{V}},\ \forall \boldsymbol{v} \in \boldsymbol{V}, \end{aligned}\right\}$$

(2.52)
$$\sum_{\ell=1}^2 \Big(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)) + \mathcal{G}^\ell\big(E^\ell(\xi^\ell(t))\big),\ \nabla\phi^\ell\Big)_{H^\ell} = (-q(t), \phi)_W,\ \forall \phi \in W,$$

(2.53)
$$\begin{aligned} &\forall \alpha \in L_1, \quad \sum_{\ell=1}^2 (\dot{\tau}^\ell(t) - \rho^\ell(t), \alpha^\ell)_{L^2(\Omega^\ell)} + a_0(\tau(t), \alpha) = \\ &\sum_{\ell=1}^2 \Big(\Theta^\ell\big(\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \tau^\ell(t), \varsigma^\ell(t)\big), \alpha^\ell\Big)_{L^2(\Omega^\ell)}, \end{aligned}$$

(2.54)
$$\left.\begin{aligned} &\varsigma(t) \in K,\ \forall \alpha \in K,\ \textstyle\sum_{\ell=1}^2 (\dot{\varsigma}^\ell(t), \alpha^\ell - \varsigma^\ell(t))_{L^2(\Omega^\ell)} + a(\varsigma(t), \alpha - \varsigma(t)) \geq \\ &\textstyle\sum_{\ell=1}^2 \Big(\Psi^\ell\big(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \varsigma^\ell(t)\big), \alpha^\ell - \varsigma^\ell(t)\Big)_{L^2(\Omega^\ell)}, \end{aligned}\right\}$$

(2.55)
$$\dot{\zeta}(t) = H_{ad}(\zeta(t), \alpha_\zeta(t), R_\nu([u_\nu(t)]), \boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)]),$$

(2.56)
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \tau(0) = \tau_0, \quad \varsigma(0) = \varsigma_0, \ \ \zeta(0) = \zeta_0.$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution of Problem **PV** is stated and proved in the next section.

Remark 2.1. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0 \leq \zeta \leq 1$. Indeed, equation (2.55) guarantees that $\zeta(x,t) \leq \zeta_0(x)$ and, therefore, assumption (2.41) shows that $\zeta(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\zeta(x,t_0) = 0$ at time t_0, then it follows from (2.55) that $\dot{\zeta}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\zeta(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \zeta(x,t) \leq 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

First, we note that the functional j_{ad} and $j_{\nu c}$ are linear with respect to the last argument and, therefore,

(2.57)
$$\begin{aligned} j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) &= -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}), \\ j_{\nu c}(\boldsymbol{u}, -\boldsymbol{v}) &= -j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned}$$

Next, using (2.47) and (2.37.b) imply

(2.58)
$$j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v}_2) \leq 0,\ \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V},$$

and use (2.48), (2.37)(a), keeping in mind (2.26), we obtain

(2.59)
$$\begin{aligned} &j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_2) - j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_1) + j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ &\leq c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{V}},\ \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}. \end{aligned}$$

3. MAIN RESULTS

The main results are stated by the following theorems.

Theorem 3.1. *Assume that* (2.29)–(2.41) *hold. Then there exists a unique solution of Problem* ***PV****. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0,T;\boldsymbol{V}), \tag{3.1}$$

$$\xi \in C(0,T;W), \tag{3.2}$$

$$\zeta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{3.3}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{3.4}$$

$$\tau \in L^2(0,T;L_1) \cap H^1(0,T;L_0), \tag{3.5}$$

$$\varsigma \in L^2(0,T;L_1) \cap H^1(0,T;L_0), \tag{3.6}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{3.7}$$

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we assume in what follows that (2.29)–(2.41) hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, $\mathcal{A}^\ell$, $\mathcal{B}^\ell$, $\mathcal{G}^\ell$, $\mathcal{Q}^\ell$, $\mathcal{E}^\ell$, H_{ad}, γ_ν, γ_τ, Θ^ℓ, Ψ^ℓ, κ_0^ℓ, κ^ℓ, and T with $\ell = 1, 2$. but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In the first step. Let $(\lambda,\mu) \in C(0,T;L_0 \times L_0)$ and consider the auxiliary problem.

Problem PV$_{(\lambda,\mu)}$. Find $\tau_\lambda : [0,T] \to L_0$, and $\varsigma_\mu : [0,T] \to L_0$, such that $\varsigma_\mu(t) \in K$

$$\sum_{\ell=1}^2 (\dot{\tau}_\lambda^\ell(t) - \lambda^\ell(t) - \rho^\ell(t), \alpha^\ell)_{L^2(\Omega^\ell)} + a_0(\tau_\lambda^\ell(t), \alpha) = 0, \ \forall \alpha \in L_0, \tag{3.8}$$

$$\sum_{\ell=1}^2 (\dot{\varsigma}_\mu^\ell(t) - \mu^\ell(t), \alpha^\ell - \varsigma_\mu^\ell(t))_{L^2(\Omega^\ell)} + a(\varsigma_\mu(t), \alpha - \varsigma_\mu(t)) \geq 0, \ \forall \alpha \in K, \tag{3.9}$$

$$\tau_\lambda(0) = \tau_0, \quad \varsigma_\mu(0) = \varsigma_0, \tag{3.10}$$

where $K = K^1 \times K^2$.

Lemma 3.2. *There exists a unique solution* $\{\tau_\lambda, \varsigma_\mu\}$ *to the auxiliary problem* ***PV***$_{(\lambda,\mu)}$ *satisfying* (3.5)–(3.6).

Proof. Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant $c_0 > 0$ such that

$$\int_{\Omega^\ell} |\nabla\alpha|^2 dx + \frac{\lambda_0^\ell}{\kappa_0^\ell} \int_{\Gamma^\ell} |\alpha|^2 da \geq c_0 \int_{\Omega^\ell} |\alpha|^2 dx, \quad \forall \alpha \in L_1^\ell, \ \ell = 1, 2.$$

Thus, we obtain

$$a_0(\alpha,\alpha) \geq c_1 \|\alpha\|_{L_1}^2, \quad \forall \alpha \in L_1,$$

where $c_1 = \kappa_0 \min(1, c_0)/2$, which implies that a_0 is L_1-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.9) has a unique solution τ_λ satisfying $\tau_\lambda(0) = \tau_0$ and the regularity (3.5).

On the other hand, we know that the form a is not L_1-elliptic. To solve this problem we introduce the functions

$$\tilde{\varsigma}_\mu^\ell(t) = e^{-\kappa^\ell t} \varsigma_\mu^\ell(t), \quad \tilde{\alpha}^\ell(t) = e^{-\kappa^\ell t} \alpha^\ell(t), \quad \ell = 1, 2.$$

We remark that if ς_μ^ℓ, $\alpha^\ell \in K^\ell$ then $\tilde{\varsigma}_\mu^\ell$, $\tilde{\alpha}^\ell \in K^\ell$. Consequently, (3.9) is equivalent to the inequality

$$\begin{aligned}\tilde{\varsigma}_\mu \in K, \quad & \sum_{\ell=1}^{2} (\dot{\tilde{\varsigma}}_\mu^\ell(t) - e^{-\kappa^\ell t}\mu^\ell(t), \tilde{\alpha}^\ell - \tilde{\varsigma}_\mu^\ell(t))_{L^2(\Omega^\ell)} + a(\tilde{\varsigma}_\mu(t), \tilde{\alpha} - \tilde{\varsigma}_\mu(t)) + \\ & \sum_{\ell=1}^{2} \kappa^\ell (\tilde{\varsigma}_\mu^\ell, \tilde{\alpha}^\ell - \tilde{\varsigma}_\mu^\ell(t))_{L^2(\Omega^\ell)} \geq 0, \ \forall \tilde{\alpha} \in K, \ \text{a.e. } t \in (0,T).\end{aligned} \tag{3.11}$$

The fact that

$$a(\tilde{\alpha}, \tilde{\alpha}) + \sum_{\ell=1}^{2} \kappa^\ell (\tilde{\alpha}^\ell, \tilde{\alpha}^\ell)_{L^2(\Omega^\ell)} \geq \sum_{\ell=1}^{2} \kappa^\ell \|\tilde{\alpha}^\ell\|_{L_1^\ell}^2 \quad \forall \tilde{\alpha} \in L_1, \tag{3.12}$$

and using classical arguments of functional analysis concerning parabolic inequalities [3, 8], implies that (3.11) has a unique solution $\tilde{\varsigma}_\mu$ having the regularity (3.6). ∎

In the second step. Let $(\lambda, \mu, \eta) \in C(0,T; L_0 \times L_0 \times \boldsymbol{V})$, we use the $\{\tau_\lambda, \varsigma_\mu\}$ obtained in Lemma 3.2 and consider the auxiliary problem.

Problem PV$_{(\lambda,\mu,\eta)}$. Find $\boldsymbol{u}_{\lambda\mu\eta} : [0,T] \to \boldsymbol{V}$, $\xi_{\lambda\mu\eta} : [0,T] \to W$, and $\zeta_{\lambda\mu\eta} : [0,T] \to L^2(\Gamma_3)$ such that

$$\left.\begin{aligned} & \textstyle\sum_{\ell=1}^{2} \Big(\mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\lambda\mu\eta}^\ell) + \mathcal{B}^\ell(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\lambda\mu\eta}^\ell), \tau_\lambda^\ell, \varsigma_\mu^\ell), \ \varepsilon(\boldsymbol{v}^\ell) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\lambda\mu\eta}^\ell(t))\Big)_{\mathcal{H}^\ell} \\ & + j_{\nu c}(\boldsymbol{u}_{\lambda\mu\eta}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\mu\eta}(t)) + j_{fr}(\boldsymbol{u}_{\lambda\mu\eta}(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_{\lambda\mu\eta}(t), \dot{\boldsymbol{u}}_{\lambda\mu\eta}(t)) \\ & + (\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\lambda\mu\eta}(t))_{\boldsymbol{V}} \geq (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\mu\eta}(t))_{\boldsymbol{V}}, \ \ \forall \boldsymbol{v} \in \boldsymbol{V}, \end{aligned}\right\} \tag{3.13}$$

$$\textstyle\sum_{\ell=1}^{2} \left(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_{\lambda\mu\eta}^\ell(t)) + \mathcal{G}^\ell E^\ell(\xi_{\lambda\mu\eta}^\ell(t)), \nabla\phi^\ell\right)_{H^\ell} = (-q(t), \phi)_W, \quad \forall \phi \in W, \tag{3.14}$$

$$\dot{\zeta}_{\lambda\mu\eta}(t) = H_{ad}\big(\zeta_{\lambda\mu\eta}(t), \alpha_{\zeta_{\lambda\mu\eta}}, R_\nu([u_{\lambda\mu\eta\nu}(t)]), \boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\mu\eta\boldsymbol{\tau}}(t)])\big), \tag{3.15}$$

$$\boldsymbol{u}_{\lambda\mu\eta}(0) = \boldsymbol{u}_0, \ \ \zeta_{\lambda\mu\eta}(0) = \zeta_0. \tag{3.16}$$

We have the following result

Lemma 3.3. (1) *Problem* ***PV***$_{(\lambda,\mu,\eta)}$ *has a unique solution* $\{\boldsymbol{u}_{\lambda\mu\eta}, \xi_{\lambda\mu\eta}, \zeta_{\lambda\mu\eta}\}$ *which satisfies the regularity* (3.1)–(3.3)*.*

(2) *If* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *are two solutions of* (3.13) *and* (3.16) *corresponding to the data* $(\lambda_1, \mu_1, \eta_1)$, $(\lambda_2, \mu_2, \eta_2) \in C(0,T; L_0 \times \boldsymbol{V})$*, then there exists* $c > 0$ *such that, for* $t \in [0,T]$,

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_{\boldsymbol{V}} \leq c\left(\|\eta_1(t) - \eta_2(t)\|_{\boldsymbol{V}} + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}}\right). \tag{3.17}$$

Proof. To prove (3.13) and (3.17), we use an abstract existence and unique result which may be found in [21, Lemma 4.2.]. Next, we consider the form $G : W \times W \to \mathbb{R}$,

$$G(\xi, \phi) = \sum_{\ell=1}^{2} (\mathcal{G}^\ell \nabla\xi^\ell, \nabla\phi^\ell)_{H^\ell} \quad \forall \xi, \phi \in W. \tag{3.18}$$

We use (2.27), (2.28), (2.36) and (3.18) to show that the form G is bilinear continuous, symmetric and coercive on W, moreover using (2.43) and the Riesz representation theorem we may define an element $w_{\lambda\mu\eta} : [0,T] \to W$ such that

$$(w_{\lambda\mu\eta}(t), \phi)_W = (q(t), \phi)_W + \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}_{\lambda\mu\eta}^\ell(t)), \nabla\phi^\ell)_{H^\ell} \quad \forall \phi \in W, t \in (0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\xi_{\mu\eta}(t) \in W$ such that

$$G(\xi_{\lambda\mu\eta}(t), \phi) = (w_{\lambda\mu\eta}(t), \phi)_W \quad \forall \phi \in W. \tag{3.19}$$

It follows from (3.19) that $\xi_{\lambda\mu\eta}$ is a solution of the equation (3.14). Let $t_1, t_2 \in [0, T]$, it follows from (3.14) that

$$\|\xi_{\lambda\mu\eta}(t_1) - \xi_{\lambda\mu\eta}(t_2)\|_W \leq C\big(\|\boldsymbol{u}_{\lambda\mu\eta}(t_1) - \boldsymbol{u}_{\lambda\mu\eta}(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big). \tag{3.20}$$

Now, from (2.39), (3.20) and $\boldsymbol{u}_{\lambda\mu\eta} \in C^1(0, T; \boldsymbol{V})$, we obtain that $\xi_{\lambda\mu\eta} \in C(0, T; W)$.

On the other hand, we consider the mapping $H_{\lambda\mu\eta} : [0, T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_{\lambda\mu\eta}(t, \zeta) = H_{ad}\big(\zeta(t), \alpha_\zeta, R_\nu([u_{\lambda\mu\eta\boldsymbol{\nu}}(t)]), \boldsymbol{R}_\tau([\boldsymbol{u}_{\lambda\mu\eta\boldsymbol{\tau}}(t)])\big),$$

for all $t \in [0, T]$ and $\zeta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that $H_{\lambda\mu\eta}$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\zeta \in L^2(\Gamma_3)$, the mapping $t \to H_{\lambda\mu\eta}(t, \zeta)$ belongs to $L^\infty(0, T; L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz Theorem (see [26, p.48], we deduce that there exists a unique function $\zeta_{\lambda\mu\eta} \in W^{1,\infty}(0, T; L^2(\Gamma_3))$ solution of the equation (3.15). Also, the arguments used in Remark 2.1 show that $0 \leq \zeta_{\lambda\mu\eta}(t) \leq 1$ for all $t \in [0, T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\zeta_{\lambda\mu\eta} \in \mathcal{Z}$. This completes the proof. ∎

In the third step, let us consider the element

$$\Pi(\eta, \lambda, \mu)(t) = \big(\Pi^1(\eta, \lambda, \mu)(t),\ \Pi^2(\eta, \lambda, \mu)(t),\ \Pi^3(\eta, \lambda, \mu)(t)\big) \in \boldsymbol{V} \times L_0 \times L_0, \tag{3.21}$$

defined by the equations

$$\begin{aligned}(\Pi^1(\eta, \lambda, \mu)(t), \boldsymbol{v})_{\boldsymbol{V}} &= -\sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^* E^\ell(\xi^\ell_{\lambda\mu\eta}),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + j_{ad}(\zeta_{\lambda\mu\eta}(t), \boldsymbol{u}_{\lambda\mu\eta}(t), \boldsymbol{v}) \\ &+ \sum_{\ell=1}^{2} \left(\int_0^t \mathcal{Q}^\ell\big(t - s, \varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(s)), \tau^\ell_\lambda(s), \varsigma^\ell_\mu(s)\big)\, ds,\ \varepsilon(\boldsymbol{v}^\ell) \right)_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{3.22}$$

$$\Pi^2(\eta, \lambda, \mu) = \Big(\Theta^1\big(\boldsymbol{\sigma}^1_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^1_{\lambda\mu\eta}), \tau^1_\lambda, \varsigma^1_\mu\big),\ \Theta^2\big(\boldsymbol{\sigma}^2_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^2_{\lambda\mu\eta}), \tau^2_\lambda, \varsigma^2_\mu\big)\Big), \tag{3.23}$$

$$\Pi^3(\eta, \lambda, \mu) = \Big(\Psi^1\big(\boldsymbol{\sigma}^1_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^1_{\lambda\mu\eta}), \varsigma^1_\mu\big),\ \Psi^2\big(\boldsymbol{\sigma}^2_{\lambda\mu\eta}, \boldsymbol{\varepsilon}(\boldsymbol{u}^2_{\lambda\mu\eta}), \varsigma^2_\mu\big)\Big), \tag{3.24}$$

where the mapping $\boldsymbol{\sigma}^\ell_{\lambda\mu\eta}$ is given by

$$\boldsymbol{\sigma}^\ell_{\lambda\mu\eta} = \mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_{\lambda\mu\eta}), \tau^\ell_\lambda, \varsigma^\ell_\mu\big) + \int_0^t \mathcal{Q}^\ell\big(t - s, \varepsilon(\boldsymbol{u}^\ell_{\lambda\mu\eta}(s)), \tau^\ell_\lambda(s), \varsigma^\ell_\mu(s)\big) ds - (\mathcal{E}^\ell)^* E^\ell(\xi^\ell_{\lambda\mu\eta}). \tag{3.25}$$

Lemma 3.4. *The mapping* Λ *has a fixed point* $(\eta^*, \lambda^*, \mu^*) \in C(0, T; \boldsymbol{V} \times L_0 \times L_0)$.

Proof. Let $(\eta_1, \lambda_1, \mu_1), (\eta_2, \lambda_2, \mu_2) \in C(0, T; \boldsymbol{V} \times L_0 \times L_0)$ and denote by τ_i, ς_i, $\boldsymbol{u}_i$, ξ_i, ζ_i and $\boldsymbol{\sigma}_i$, the functions obtained in Lemmas 3.2, 3.3 and the relation (3.25), for $(\eta, \lambda, \mu) = (\eta_i, \lambda_i, \mu_i)$,

$i = 1, 2$. Let $t \in [0, T]$. We use (2.35), (2.34), (2.46) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$\|\Pi^1(\eta_1,\lambda_1,\mu_1)(t) - \Pi^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}} \le \sum_{\ell=1}^{2} \|(\mathcal{E}^\ell)^*\nabla\xi_1^\ell(t) - (\mathcal{E}^\ell)^*\nabla\xi_2^\ell(t)\|^2_{\mathcal{H}^\ell} +$$

$$\sum_{\ell=1}^{2}\int_0^t \big\|\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_1^\ell(s)),\tau_1^\ell(s),\varsigma_1^\ell(s)\big) - \mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_2^\ell(s)),\tau_2^\ell(s),\varsigma_2^\ell(s)\big)\big\|^2_{\mathcal{H}^\ell}\,ds$$

$$+C\|\zeta_1^2(t)R_\nu([u_{1\nu}(t)]) - \zeta_2^2(t)R_\nu([u_{2\nu}(t)])\|^2_{L^2(\Gamma_3)}$$

$$+C\|\zeta_1^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)]) - \zeta_2^2(t)\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|^2_{L^2(\Gamma_3)}.$$

Therefore,

$$\|\Pi^1(\eta_1,\lambda_1,\mu_1)(t) - \Pi^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}} \le C\bigg(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds +$$

$$\int_0^t \|\tau_1(s) - \tau_2(s))\|^2_{L_0}\,ds \;+ \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0}\,ds +$$

$$\|\xi_1(t) - \xi_2(t)\|^2_W + \|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Gamma_3)}\bigg). \tag{3.26}$$

By similar arguments, from (3.23), (3.25) and (2.32) it follows that

$$\|\Pi^2(\eta_1,\lambda_1,\mu_1)(t) - \Pi^2(\eta_2,\lambda_2,\mu_2)(t)\|^2_{L_0} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\varsigma_1(t) - \varsigma_2(t))\|^2_{L_0} + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0}\,ds$$

$$+\|\tau_1(t) - \tau_2(t))\|^2_{L_0} + \int_0^t \|\tau_1(s) - \tau_2(s))\|^2_{L0}\,ds + \|\xi_1(t) - \xi_2(t)\|^2_W\bigg). \tag{3.27}$$

Similarly, using (2.35) implies

$$\|\Pi^3(\eta_1,\lambda_1,\mu_1)(t) - \Pi^3(\eta_2,\lambda_2,\mu_2)(t)\|^2_{L_0} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\varsigma_1(t) - \varsigma_2(t))\|^2_{L_0} + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0}\,ds$$

$$\|\xi_1(t) - \xi_2(t)\|^2_W\bigg). \tag{3.28}$$

It follows now from (3.26), (3.27) and (3.28) that

$$\|\Pi(\eta_1,\lambda_1,\mu_1)(t) - \Pi(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}\times L_0\times L_0} \le C\bigg(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\varsigma_1(t) - \varsigma_2(t))\|^2_{L_0} + \int_0^t \|\varsigma_1(s) - \varsigma_2(s))\|^2_{L_0}\,ds$$

$$+\|\tau_1(t) - \tau_2(t))\|^2_{L_0} + \int_0^t \|\tau_1(s) - \tau_2(s))\|^2_{L_0}\,ds$$

$$+\|\xi_1(t) - \xi_2(t)\|^2_W + \|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Gamma_3)}\bigg). \tag{3.29}$$

Also, since

$$\boldsymbol{u}_i^\ell(t) = \int_0^t \dot{\boldsymbol{u}}_i^\ell(s)ds + \boldsymbol{u}_0^\ell(t), \quad t \in [0,T], \;\; \ell = 1,2,$$

we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s))\|_{\boldsymbol{V}}\, ds$$

and using this inequality in (3.17) yields

(3.30) $$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \le C\Big(\int_0^t \|\eta_1(s) - \eta_2(s))\|_{\boldsymbol{V}}\, ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_{\boldsymbol{V}}\, ds\Big).$$

Next, we apply Gronwall's inequality to deduce

(3.31) $$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t))\|_{\boldsymbol{V}} \le C\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\boldsymbol{V}}\, ds \quad \forall t \in [0,T].$$

On the other hand, from the Cauchy problem (3.15) we can write

$$\zeta_i(t) = \zeta_0 - \int_0^t H_{ad}\big(\zeta_i(s), \alpha_{\zeta_i}(s), R_\nu([u_{i\nu}(s)]), \boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])\big)ds$$

and, employing (2.23) and (2.33) we obtain that

$$\begin{aligned}\big\|\zeta_1(t) - \zeta_2(t)\big\|_{L^2(\Gamma_3)} &\le C\int_0^t \big\|\zeta_1(s) - \zeta_2(s)\big\|_{L^2(\Gamma_3)} ds \\ &+ C\int_0^t \big\|R_\nu([u_{1\nu}(s)]) - R_\nu([u_{2\nu}(s)])\big\|_{L^2(\Gamma_3)} ds \\ &+ C\int_0^t \big\|\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)]) - \boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])\big\|_{L^2(\Gamma_3)} ds.\end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\zeta_1 = \zeta_1 - \zeta_2 + \zeta_2$, we get

(3.32)

$$\big\|\zeta_1(t) - \zeta_2(t)\big\|_{L^2(\Gamma_3)} \le C\Big(\int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d} ds\Big).$$

Next, we apply Gronwall's inequality and from the Sobolev trace theorem we obtain

(3.33) $$\|\zeta_1(t) - \zeta_2(t)\|^2_{L^2(\Gamma_3)} \le C\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V ds.$$

We use now (3.14), (2.27), (2.34) and (2.36) to find

(3.34) $$\|\xi_1(t) - \xi_2(t)\|^2_W \le C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V.$$

From (3.8) we deduce that

$$(\dot\tau_1 - \dot\tau_2, \tau_1 - \tau_2)_{L_0} + a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) + \big(\lambda_1 - \lambda_2, \theta_1 - \theta_2\big)_{L_0} = 0.$$

We integrate this equality with respect to time, using the initial conditions $\tau_1(0) = \tau_2(0) = \tau_0$ and inequality $a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) \ge 0$, to find

$$\frac{1}{2}\|\tau_1(t) - \tau_2(t)\|^2_{L_0} \le \int_0^t \big(\lambda_1(s) - \lambda_2(s), \tau_1(s) - \tau_2(s)\big)_{L_0} ds,$$

which implies that

$$\|\tau_1(t) - \tau_2(t)\|^2_{L_0} \le \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{L_0} ds + \int_0^t \|\tau_1(s) - \tau_2(s)\|^2_{L_0} ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\tau_1(t)-\tau_2(t)\|^2_{L_0} \le C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{E_0}\,ds \quad \forall t\in[0,T]. \tag{3.35}$$

Moreover, from (3.9) we deduce that $\text{a.e. } t \in (0,T)$,

$$(\dot\varsigma_1-\dot\varsigma_2,\varsigma_1-\varsigma_2)_{L_0}+a(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\le\big(\mu_1-\mu_2,\varsigma_1-\varsigma_2\big)_{L_0},$$

Integrating the previous inequality with respect to time, using the initial conditions $\varsigma_1(0)=\varsigma_2(0)=\varsigma_0$ and inequality $a(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\ge 0$, to find

$$\frac{1}{2}\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le\int_0^t\big(\mu_1(s)-\mu_2(s),\varsigma_1(s)-\varsigma_2(s)\big)_{L_0}\,ds,$$

which implies that

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{L_0}\,ds+\int_0^t\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L_0}ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L_0}\le C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_{L_0}\,ds. \tag{3.36}$$

We substitute (3.17), (3.33)-(3.36) in (3.29) to obtain

$$\|\Pi(\eta_1,\lambda_1,\mu_1)(t)-\Pi(\eta_2,\lambda_2,\mu_2)(t)\|^2_{\boldsymbol{V}\times L_0\times L_0}\le$$
$$C\int_0^t\|(\eta_1,\lambda_1,\mu_1)(s)-(\eta_2,\lambda_2,\mu_2)(s)\|^2_{\boldsymbol{V}\times L_0\times L_0}\,ds.$$

Reiterating this inequality m times we obtain

$$\|\Pi^m(\eta_1,\lambda_1,\mu_1)-\Pi^m(\eta_2,\lambda_2,\mu_2)\|^2_{C(0,T;\boldsymbol{V}\times L_0\times L_0)}\le$$
$$\frac{C^mT^m}{m!}\|(\eta_1,\lambda_1,\mu_1)-(\eta_2,\lambda_2,\mu_2)\|^2_{C(0,T;\boldsymbol{V}\times L_0\times L_0)}.$$

Thus, for m sufficiently large, Π^m is a contraction on the Banach space $C(0,T;\boldsymbol{V}\times L_0\times L_0)$, and so Π has a unique fixed point. ∎

Let $(\eta^*,\lambda^*,\mu^*)\in C(0,T;\boldsymbol{V}\times L_0\times L_0)$, be the fixed point of Λ, and

$$\boldsymbol{u}_*=\boldsymbol{u}_{\lambda^*\mu^*\eta^*},\quad \xi_*=\xi_{\lambda^*\mu^*\eta^*},\quad \zeta_*=\zeta_{\lambda^*\mu^*\eta^*},\quad \tau_*=\tau_{\lambda^*},\quad \varsigma_*=\varsigma_{\mu^*}, \tag{3.37}$$

(3.38)

$$\boldsymbol{\sigma}^\ell_*=\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*)+\mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*),\tau^\ell_*,\varsigma^\ell_*\big)+\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\tau^\ell_*,\varsigma^\ell_*(s)\big)\,ds-(\mathcal{E}^\ell)^*E^\ell(\xi^\ell_*),$$

$$\boldsymbol{D}^\ell_*=\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_*)+\mathcal{G}^\ell\big(E^\ell(\xi^\ell_*)\big). \tag{3.39}$$

We use : $\Pi^1(\eta^*,\lambda^*,\mu^*)=\eta^*$, $\Pi^2(\eta^*,\lambda^*,\mu^*)=\lambda^*$, and $\Pi^3(\eta^*,\lambda^*,\mu^*)=\mu^*$, it follows:

$$(\eta^*(t),\boldsymbol{v})_{\boldsymbol{V}}=-\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*E^\ell(\xi^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}+j_{ad}(\zeta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})$$

$$+\sum_{\ell=1}^{2}\left(\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}^\ell_*(s)),\tau^\ell_*(s),\varsigma^\ell_*(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell},\quad\forall\boldsymbol{v}\in\boldsymbol{V}, \tag{3.40}$$

$$\lambda^\ell_*(t)=\Theta^\ell\big(\boldsymbol{\sigma}^\ell_*(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)),\tau^\ell_*(t),\varsigma^\ell_*(t)\big),\quad\ell=1,2. \tag{3.41}$$

$$\mu^\ell_*(t)=\Psi^\ell\big(\boldsymbol{\sigma}^\ell_*(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_*(t)),\varsigma^\ell_*(t)\big),\ \ell=1,2. \tag{3.42}$$

Existence. We prove $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \xi_*, \tau_*, \varsigma_*, \zeta_*, \boldsymbol{D}_*\}$ satisfies (2.49)–(2.56) and the regularites (3.1)–(3.7). Indeed, we write (3.13) for $(\eta, \lambda, \mu) = (\eta^*, \lambda^*, \mu^*)$ and use (3.37) to find

$$\begin{aligned}(3.43)\quad &\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}(\mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell),\tau_*^\ell,\varsigma_*^\ell\big),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}\\ &+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})-j_{fr}(\boldsymbol{u}_*(t),\dot{\boldsymbol{u}}_*(t))\\ &+(\eta^*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t)))_{\boldsymbol{V}}\geq(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))_{\boldsymbol{V}},\ \forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned}$$

Substitute (3.40) in (3.43) to obtain

$$\begin{aligned}&\sum_{\ell=1}^{2}(\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell)(t),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}+\sum_{\ell=1}^{2}(\mathcal{B}^\ell\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell),\tau_*^\ell,\varsigma_*^\ell\big),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)))_{\mathcal{H}^\ell}\\ &+\sum_{\ell=1}^{2}\left(\int_0^t\mathcal{Q}^\ell\big(t-s,\varepsilon(\boldsymbol{u}_*^\ell(s)),\tau_*^\ell,\varsigma_*^\ell(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))\right)_{\mathcal{H}^\ell}\\ &+j_{ad}(\zeta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))+j_{fr}(\boldsymbol{u}_*(t),\boldsymbol{v})\\ &-j_{fr}(\boldsymbol{u}_*(t),\dot{\boldsymbol{u}}_*(t))-\sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*E^\ell(\xi_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t))\big)_{\mathcal{H}^\ell}\\ (3.44)\quad &\geq(\mathbf{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_*(t))_{\boldsymbol{V}}\ \ \forall\boldsymbol{v}\in\boldsymbol{V}\quad a.e.\ t\in[0,T],\end{aligned}$$

and we substitute (3.41) in (3.8) to have

$$(3.45)\qquad \sum_{\ell=1}^{2}(\dot{\tau}_*^\ell(t),\alpha^\ell)_{L^2(\Omega^\ell)}+a_0(\tau_*^\ell(t),\alpha)=\sum_{\ell=1}^{2}\big(\lambda_*^\ell(t)+\rho^\ell(t),\alpha^\ell\big)_{L^2(\Omega^\ell)},$$

for all $\alpha\in L_0,$ a.e. $t\in(0,T)$.

Next, substitute (3.42) in (2.30) to obtain $\varsigma_*(t)\in K$, and

$$\begin{aligned}(3.46)\quad &\sum_{\ell=1}^{2}(\dot{\varsigma}_*^\ell(t),\alpha^\ell-\varsigma_*^\ell(t))_{L^2(\Omega^\ell)}+a(\varsigma_*(t),\alpha-\varsigma_*(t))\geq\\ &\sum_{\ell=1}^{2}\Big(\Psi^\ell\big(\boldsymbol{\sigma}_*^\ell(t)-\mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)),\varsigma_*^\ell(t)\big),\alpha^\ell-\varsigma_*^\ell(t)\Big)_{L^2(\Omega^\ell)},\end{aligned}$$

for all $\alpha\in K,$ a.e. $t\in(0,T)$. We write now (3.15) for $(\eta,\lambda,\mu)=(\eta^*,\lambda^*,\mu^*)$ and use (3.37) to see that

$$(3.47)\qquad \sum_{\ell=1}^{2}(\mathcal{G}^\ell E^\ell(\xi_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}+\sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\nabla\phi^\ell)_{H^\ell}=-(q(t),\phi)_W,$$

for all $\phi\in W,$ a.e. $t\in(0,T)$. Additionally, we use $\boldsymbol{u}_{\lambda^*\mu^*\eta^*}$ in (3.15) and (3.37) to find

$$(3.48)\qquad \dot{\zeta}_*(t)=H_{ad}\big(\zeta_*(t),\alpha_{\zeta_*}(t),R_\nu([u_{*\nu}(t)]),\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])\big)$$

a.e. $t\in[0,T]$. The relations (3.43)–(3.48), allow us to conclude now that $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\xi_*,\varsigma_*,\zeta_*,\boldsymbol{D}_*\}$ satisfies (2.49)–(2.55). Next, (2.56) the regularity (3.1)–(3.3) and (3.6) follow from Lemmas 3.2 and 3.3. Since $\boldsymbol{u}_*$, ξ_* and ς_* satisfies (3.1), (3.2) and (3.6), respectively, It follows from (3.38) that

$$(3.49)\qquad \boldsymbol{\sigma}_*\in C(0,T;\mathcal{H}).$$

For $\ell = 1, 2$, we choose $\boldsymbol{v} = \boldsymbol{u} \pm \phi$ in (3.44), with $\phi = (\phi^1, \phi^2)$, $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$, to obtain

$$\text{Div}\, \boldsymbol{\sigma}_*^\ell(t) = -\boldsymbol{f}_0^\ell(t) \quad \forall t \in [0, T], \;\; \ell = 1, 2, \tag{3.50}$$

where $D(\Omega^\ell)$ is the space of infinitely differentiable real functions with a compact support in Ω^ℓ. The regularity (3.4) follows from (2.39), (3.49) and (3.50). Let now $t_1, t_2 \in [0, T]$, from (2.27), (2.34), (2.36) and (3.39), we conclude that there exists a positive constant $C > 0$ verifying

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \leq C\left(\|\xi_*(t_1) - \xi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and ξ_* given by (3.1) and (3.2) implies

$$\boldsymbol{D}_* \in C(0, T; H). \tag{3.51}$$

For $\ell = 1, 2$, we choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (3.47) and using (2.43) we find

$$\text{div}\, \boldsymbol{D}_*^\ell(t) = q_0^\ell(t) \quad \forall t \in [0, T], \;\; \ell = 1, 2. \tag{3.52}$$

Property (3.7) follows from (2.39), (3.51) and (3.52).

Finally we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \xi_*, \tau_*, \varsigma_*, \zeta_*, \boldsymbol{D}_*\}$ of the problem **PV** has the regularity (3.1)–(3.7), which concludes the existence part of Theorem 3.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Pi(., ., .)$ defined by (3.22)-(3.23) and the unique solvability of the Problems $\text{PV}_{(\lambda,\mu)}$, and $\text{PV}_{(\lambda,\mu,\eta)}$. ∎

4. CONCLUSION

We presented a model for the quasi-static process of frictional contact between two thermo-electro-viscoelastic bodies with damage. The contact was modeled with the normal compliance condition and the associated Coulomb's law of dry friction. The new feature in the model was the normal compliance and the friction law depend on the adhesion as presented in the differential equation (2.14). The difficulty of solving this type of problem lies not only in the coupling of viscoelastic, electrical and thermal aspects, but also in the nonlinearity of the boundary conditions modeling this type of physical phenomena (contact and friction conditions), which gives us a quasi-variational inequalities and type of nonlinear, parabolic variational equalities. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities, parabolic inequalities and fixed point theorem.

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