ANALYSIS OF A FRICTIONAL CONTACT PROBLEM FOR VISCOELASTIC PIEZOELECTRIC MATERIALS

MEZIANE SAID AMEUR, TEDJANI HADJ AMMAR AND LAID MAIZA

Received 7 September, 2019; accepted 5 March, 2020; published 31 March, 2020.

DEPARTEMENT OF MATHEMATICS, EL OUED UNIVERSITY, P.O. BOX 789, 39000 EL OUED, ALGERIA.
said-ameur-meziane@univ-eloued.dz

DEPARTEMENT OF MATHEMATICS, EL OUED UNIVERSITY, P.O. BOX 789, 39000 EL OUED, ALGERIA.
hadjammar-tedjani@univ-eloued.dz

DEPARTMENT OF MATHEMATICS, KASDI MERBAH UNIVERSITY, 30000 OUARGLA, ALGERIA.
maiza.laid@univ-ouargla.dz

ABSTRACT. In this paper, we consider a mathematical model that describes the quasi-static process of contact between two thermo-electro-viscoelastic bodies with damage and adhesion. The damage of the materials caused by elastic deformations. The contact is frictional and modeled with a normal compliance condition involving adhesion effect of contact surfaces. Evolution of the bonding field is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations, and fixed point theorem.

Key words and phrases: Damage; Adhesion; Normal compliance; Frictional contact; Piezoelectric materials.

2010 Mathematics Subject Classification. Primary 74M10, 74M15. Secondary 47H10, 49J40.
1. Introduction

The piezoelectric phenomenon represents the coupling between the mechanical and electrical behavior of a class of materials, called piezoelectric materials. In simplest terms, when a piezoelectric material is squeezed, an electric charge collects on its surface, conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate and polyvinylidene fluoride (a polymer film). Piezoelectric materials are used extensively as switches and actually in many engineering systems in radioelectronics, electroacoustics and measuring equipment.

Different models have been developed to describe the interaction between the electric and mechanical fields (see, e.g. [2, 9, 10, 21, 25]). Therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties.

General models for elastic materials with piezoelectric effects can be found in [7, 9, 15, 16] and more recently in [1]. viscoelastic piezoelectric materials in [21, 25] or elasto-viscoplastic piezoelectric materials have been studied in [12]. The coupling between the thermal, electric and mechanical fields in piezo electric materials provides a mechanism for sensing thermo mechanical disturbances from measurements of induced electric potentials, and for altering structural responses via applied electric fields.

One of the applications of the thermo-piezoelectric material is to detect the responses of a structure by measuring the electric charge, sensing or to reduce excessive responses by applying additional electric forces or thermal forces actuating. If sensing and actuating can be integrated smartly, a so-called intelligent structure can be designed. The piezoelectric materials are also often used as resonators whose frequencies need to be precisely controlled. The coupling between the thermo-piezoelectric and pyroelectric effects, it is important to qualify the effect of heat dissipation on the propagation of wave at low and high frequencies.

The thermo-piezoelectric theory was first proposed by Mindlin [14], later he derived the governing equations of a thermo-piezoelectric plate [17]. The physical laws for the thermo-piezoelectric materials have been discussed by [19, 20]. Chandrasekhariah [4, 5] presented the generalized theory of thermo-piezoelectricity by taking into account the finite speed of propagation of thermal disturbance. Yang and Batra [28] studied the effect of heat conduction on shift in the frequencies of a freely vibrating linear thermo-piezoelectric body with the help of perturbation methods. Sharma and Walia [24] presented the propagation of straight and circular crested waves in generalized piezo thermoelastic materials. The normal compliance contact condition was first considered in [21, 25, 26] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [23]. This condition allows the interpenetration of the body’s surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [9, 10, 11, 22] and recently in the monographs [18].

In these papers, the bonding field, denoted by \( \zeta \), it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [6], the bonding field satisfies the restriction \( 0 \leq \zeta \leq 1 \), when \( \zeta = 1 \) at a point of the contact surface, the adhesion is complete and all the bonds are active, when \( \zeta = 0 \) all the bonds are inactive, severed, and there is no adhesion, when \( 0 < \zeta < 1 \) the adhesion is partial and only a fraction \( \zeta \) of the bonds is active. The novelty of this work lies in the analysis
of a system that contains strong couplings in the multivalued boundary conditions: both the normal compliance and the friction law depend on the adhesion (see (2.15) and (2.16)), and the adhesion be written by the differential equation of the form

$$\dot{\varsigma} = H_{ad}(\varsigma, \alpha_\varsigma, R_w(u_w^1 + u_w^2), R_s(u_s^1 - u_s^2)),$$

Here, $H_{ad}$ is the adhesion evolution rate function. Then, the adhesion rate function was assumed to depend, in addition to $\varsigma$, $R_w(u_w^1 + u_w^2)$, $R_s(u_s^1 - u_s^2)$ and $\alpha_\varsigma$, where, the truncation operators $R_w$, $R_s$ are defined by (2.23), and $\alpha_\varsigma(x, t) = \int_0^t \varsigma(x, s)ds$. We use it in $H_{ad}$, since usually when the glue is stretched beyond the limit $L$ it does not contribute more to the bond strength. An example of such a function, used in [6], the following form of the evolution of the bonding field was employed $\dot{\varsigma} = -\varsigma_n^2 \gamma_n R_w(u_w^2)^2$, where $\gamma_n$ is the normal rate coefficient and $\gamma_n L$ is the maximal tensile normal traction that the adhesive can provide and $\varsigma_n = \max(0, \varsigma)$. We note that in this case, only debonding is allowed. A different rate equation for the the evolution of the bonding field is $\dot{\varsigma} = - (\varsigma_n R_w(u_w^1 + u_w^2)^2 + \gamma_t |R_s(u_s^1 - u_s^2)|^2 - \varsigma_n^2) + \epsilon$, see, e.g., [9] [10]. Here, $\gamma_t$ is the tangential rate coefficient, which may also be interpreted as the tangential stiffness coefficient of the interface when the adhesion is complete ($\varsigma = 1$). However, the bonding cannot exceed $\varsigma = 1$ and, moreover, the rebonding becomes weaker as the process goes on, which is represented by the factor $1 + \alpha_\varsigma^2$ in the denominator. In all these papers the damage of the material is described with a damage function $\varsigma$, restricted to have values between zero and one. When $\varsigma = 1$, there is no damage in the material, when $\varsigma = 0$, the material is completely damaged, when $0 < \varsigma < 1$ there is partial damage and the system has a reduced load carrying capacity.

In this paper, we study the quasi-static frictional contact problem between two viscoelastic piezoelectric bodies with damage, adhesion and normal compliance. In Section 2, we describe the mathematical models for the frictional contact problem between two thermo-electro-viscoelastic bodies with long-term memory and damage. The contact is modelled with normal compliance and adhesion. We introduce some notation, list the assumptions on the problem’s data, and derive the variational formulation of the model. We prove in Section 3 the existence and uniqueness of the solution, where it is carried out in several steps and is based on arguments of evolutionary variational equalities, differential equations and Banach fixed point theorem.

2. PROBLEM STATEMENT AND VARIATIONAL FORMULATION

We describe the model for the process, we present its variational formulation. The physical setting is the following. Let us consider two thermo-electro-viscoelastic bodies with long-term memory, occupying two bounded domains $\Omega_1^\ell$, $\Omega_2^\ell$, of the space $\mathbb{R}^d (d = 2, 3)$. We put a superscript $\ell$ to indicate that the quantity is related to the domain $\Omega_1^\ell$, $\ell = 1, 2$. In the following, the superscript $\ell$ ranges between 1 and 2. For each domain $\Omega_1^\ell$, $\ell = 1, 2$, the boundary $\Gamma_1^\ell$ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts $\Gamma_1^\ell, \Gamma_2^\ell, \Gamma_3^\ell$, on one hand, and on two measurable parts $\Gamma_{a1}^\ell$ and $\Gamma_{a2}^\ell$, on the other hand, such that $\text{meas} \Gamma_{a1}^\ell > 0$, $\text{meas} \Gamma_{a2}^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The $\Omega_1^\ell$ body is submitted to $f_0^1$ forces and volume electric charges of density $q_0^1$. The bodies are assumed to be clamped on $\Gamma_3^\ell \times (0, T)$, so the displacement field vanishes there. The surface tractions $f_2^1$ act on $\Gamma_3^\ell \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_3^\ell \times (0, T)$ and a surface electric charge of density $q_0^3$ is prescribed on $\Gamma_3^\ell \times (0, T)$. The two bodies are in contact along the common part $\Gamma_3^1 = \Gamma_3^2$, which will be denoted $\Gamma_3$ below. The bodies is in adhesive contact, over the contact surface $\Gamma_3$, the contact is frictional and is modeled with the normal compliance condition and a version of Coulomb’s law of friction. The process is assumed to be
isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasi-static; i.e., the inertial terms in the momentum balance equations are neglected. We denote by \( \mathbf{u}^t \) the displacement field, by \( \sigma^t \) the stress field and by \( \varepsilon(\mathbf{u}^t) \) the linearized strain tensor. We use an thermo-electro-viscoelastic constitutive law with damage given by

\[
\sigma^t = A^t \varepsilon(\mathbf{u}^t) + B^t(\varepsilon(\mathbf{u}^t), \tau^t, \varsigma^t) + \int_0^t Q^t \left( t - s, \varepsilon(\mathbf{u}^t(s)), \tau^t(s), \varsigma^t(s) \right) ds - (E^t)^* E^t(\xi^t),
\]

where \( A^t \) is a given nonlinear operator, \( Q^t \) is the relaxation operator, \( B^t \) represents the elasticity operator, where \( \tau^t \) represents the absolute temperature and \( \varsigma^t \) is the damage field. \( E(\xi^t) = -\nabla \xi^t \) is the electric field, \( E^t \) represents the third order piezoelectric tensor, \( (E^t)^* \) is its transposition. We study a quasi-static Coulomb’s frictional contact problem between two thermo-electro-viscoelastic bodies with long-term memory and damage. The contact is modelled with normal compliance where the adhesion of the contact surfaces is taken into account and is modelled with a surface variable, the bonding field. We derive a variational formulation of the problem and prove the existence of a unique weak solution.

In (2.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable \( t \). It follows from (2.1) that at each time moment, the stress tensor \( \sigma^t(t) \) is split into three parts: \( \sigma^t(t) = \sigma^t_v(t) + \sigma^t_e(t) + \sigma^t_d(t) \), where \( \sigma^t_v(t) = A^t \varepsilon(\mathbf{u}^t(t)) \) represents the purely viscous part of the stress, \( \sigma^t_e(t) = (E^t)^* \nabla \xi^t(t) \) represents the electric part of the stress and \( \sigma^t_d(t) \) satisfies a rate-type elastic relation

\[
\sigma^t_d(t) = B^t(\varepsilon(\mathbf{u}^t(t)), \tau^t(t), \varsigma^t(t)) + \int_0^t Q^t \left( t - s, \varepsilon(\mathbf{u}^t(s)), \tau^t(s), \varsigma^t(s) \right) ds.
\]

Note also that when \( Q^t = 0 \) the constitutive law (2.1) becomes the Kelvin-Voigt viscoelastic piezoelectric with damage and thermal effects constitutive relation,

\[
\sigma^t(t) = A^t \varepsilon(\mathbf{u}^t(t)) + B^t(\varepsilon(\mathbf{u}^t(t)), \tau^t(t), \varsigma^t(t)) + (E^t)^* \nabla \xi^t(t).
\]

Quasistatic evolution of damage in viscoplastic materials has been studied in [13]. According to Batra and Yang [11] the following constitutive law is employed for the electric potential:

\[
D^t = E^t \varepsilon(\mathbf{u}^t) + G^t(\varepsilon(\mathbf{u}^t)),
\]

where \( D^t \) is the electric displacement field and \( G^t \) is the electric permittivity tensor.

The differential inclusion used for the evolution of the damage field is

\[
\varsigma^t - \kappa^t \Delta \varsigma^t + \partial \psi_K(\varsigma^t) \ni \Psi(\sigma^t - A^t \varepsilon(\mathbf{u}^t), \varepsilon(\mathbf{u}^t), \varsigma^t) \text{ in } \Omega^t \times (0, T),
\]

where \( K^t \) denotes the set of admissible damage functions defined by

\[
K^t = \{ \alpha \in H^1(\Omega^t); \ 0 \leq \alpha \leq 1, \ \text{a.e. in } \Omega^t \},
\]

\( \kappa^t \) is a positive coefficient, \( \partial \psi_K(\varsigma^t) \) represents the subdifferential of the indicator function of the set \( K^t \) and \( \phi^t \) is a given constitutive function which describes the sources of the damage in the system. When \( \varsigma^t = 1 \), there is no damage in the material, when \( \varsigma^t = 0 \), the material is completely damaged, when \( 0 < \varsigma^t < 1 \) there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [9] [25] [26].

The thermo-electro-viscoelastic constitutive law (2.1) includes a temperature effects described by the parabolic aquation given by

\[
\dot{\varsigma}^t - \kappa^t_0 \Delta \varsigma^t = \Theta^t(\sigma^t, \varepsilon(\mathbf{u}^t), \tau^t, \varsigma^t) + \rho^t.
\]
With these assumptions, the classical formulation of the quasi-static problem for frictional contact problem between two thermo-electro-viscoelastic bodies with damage, normal compliance and adhesion is the following.

**Problem P.** For \( \ell = 1, 2 \), find a displacement field \( u^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d \), a stress field \( \sigma^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{S}^d \), a temperature \( \tau^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R} \), a damage field \( \zeta^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R} \), an electric potential field \( \xi^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R} \), a bonding field \( \zeta : \Gamma_3 \times (0, T) \longrightarrow \mathbb{R} \) and an electric displacement field \( D^\ell : \Omega^\ell \times (0, T) \longrightarrow \mathbb{R}^d \) such that

\[
\begin{align*}
\sigma^\ell(t) &= \mathcal{A}^\ell \varepsilon(u^\ell(t)) + \mathcal{B}^\ell (\varepsilon(u^\ell(t)), \tau^\ell(t), \zeta^\ell(t)) + \\
\int_0^t \mathcal{Q}^\ell(t-s, \varepsilon(u^\ell(s)), \tau^\ell(s), \zeta^\ell(s)) \, ds - (\mathcal{E}^\ell)^*E^\ell(\xi^\ell(t)), \quad \text{in } \Omega^\ell \times (0, T),
\end{align*}
\]

\[\mathcal{D}^\ell = \mathcal{E}^\ell \varepsilon(u^\ell) + \mathcal{G}^\ell (E^\ell(\xi^\ell)), \quad \text{in } \Omega^\ell \times (0, T),\]

\[\tau^\ell - \kappa_0 \Delta \tau^\ell = \Theta^\ell (\sigma^\ell, \varepsilon(u^\ell), \tau^\ell, \zeta^\ell) + \rho^\ell \quad \text{in } \Omega^\ell \times (0, T),\]

\[\zeta^\ell - \kappa^\ell \Delta \zeta^\ell + \partial \psi_K(\nu^\ell) \equiv \Psi^\ell (\sigma^\ell - \mathcal{A}^\ell \varepsilon(u^\ell), \varepsilon(u^\ell), \zeta^\ell) \quad \text{in } \Omega^\ell \times (0, T),\]

\[\text{Div } \sigma^\ell + f^\ell_0 = 0 \quad \text{in } \Omega^\ell \times (0, T),\]

\[\text{Div } D^\ell - q^\ell_0 = 0 \quad \text{in } \Omega^\ell \times (0, T),\]

\[u^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0, T),\]

\[\sigma^\ell \nu^\ell = f^\ell_2 \quad \text{on } \Gamma_2^\ell \times (0, T),\]

\[\zeta = H_{\alpha d}(\zeta, \alpha, R_{\nu}([u^\nu]), R_{\tau}([u^\tau])), \quad \text{on } \Gamma_3 \times (0, T),\]

\[\sigma^\nu = \sigma^\nu_1 \equiv \sigma_{\nu}, \quad \text{where } \sigma_{\nu} = -p_{\nu}([u^\nu]) + \gamma_{\nu} \zeta^2 R_{\nu}([u^\nu]) \quad \text{on } \Gamma_3 \times (0, T),\]

\[\begin{cases}
\sigma^\nu_1 = -\sigma^\nu_2 \equiv \sigma_{\tau}, \\
\|\sigma^\tau + \gamma_{\tau} \zeta^2 R_{\tau}([u^\tau])\| \leq \mu p_{\nu}([u^\nu]), \\
\|\sigma^\tau + \gamma_{\tau} \zeta^2 R_{\tau}([u^\tau])\| < \mu p_{\nu}([u^\nu]) \Rightarrow [\hat{u}^\tau] = 0, \quad \text{on } \Gamma_3 \times (0, T), \\
\|\sigma^\tau + \gamma_{\tau} \zeta^2 R_{\tau}([u^\tau])\| = \mu p_{\nu}([u^\nu]) \Rightarrow \exists \lambda \geq 0 \\
\text{such that } \sigma^\tau + \gamma_{\tau} \zeta^2 R_{\tau}([u^\tau]) = -\lambda [\hat{u}^\tau],
\end{cases}\]

\[\begin{align*}
\xi^\ell &= 0 \quad \text{on } \Gamma_{\alpha}^\ell \times (0, T), \\
\mathcal{D}^\ell \cdot \nu^\ell &= q^\ell_2 \quad \text{on } \Gamma_{\beta}^\ell \times (0, T), \\
\kappa_0 \frac{\partial \tau^\ell}{\partial t^\ell} + \lambda_0 \tau^\ell &= 0 \quad \text{on } \Gamma^\ell \times (0, T), \\
\frac{\partial \xi^\ell}{\partial t^\ell} &= 0 \quad \text{on } \Gamma^\ell \times (0, T), \\
\tau^\ell(0) &= \tau^\ell_0, \quad \zeta^\ell(0) = \zeta^\ell_0 \quad \text{in } \Omega^\ell, \\
\zeta(0) &= \zeta_0 \quad \text{on } \Gamma_3.
\end{align*}\]
is the unit outer normal vector on \( \Gamma^\ell \), equations (2.6) and (2.7) represent the thermo-electro-viscoelastic constitutive law with long term-memory and damage. Equation (2.8) represents the energy conservation where \( \Theta^\ell \) is a nonlinear constitutive function which represents the heat generated by the work of internal forces and \( \rho^\ell \) is a given volume heat source. Inclusion (2.9) describes the evolution of the damage field. Equations (2.10) and (2.11) are the equilibrium equations for the stress and electric-displacement fields, respectively. Next, the equations (2.12) and (2.13) represent the displacement and traction boundary condition, respectively. Condition (2.15) represents the normal compliance conditions with adhesion where \( \gamma^\ell \) is a given positive function which will be described below and \( [u^\ell] = u^\ell_1 - u^\ell_2 \) stands for the displacements in normal direction, in this condition the interpenetrability between two bodies, that is \([u^\ell]\) can be positive on \( \Gamma_3 \).

\begin{align}
R^\ell_v(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0. \end{cases} \\
R^\ell_t(v) = \begin{cases} v & \text{if } |v| \leq L, \\ L \frac{v}{|v|} & \text{if } |v| > L. \end{cases}
\end{align}

Here \( L > 0 \) is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [22]). Condition (2.16) are a non local Coulomb’s friction law conditions coupled with adhesive, where \([u^\ell] = u^\ell_1 - u^\ell_2 \) stands for the jump of the displacements in tangential direction. Equation (2.14) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [26,27]. (2.17) and (2.18) represent the electric boundary conditions. The relation (2.19) represent a Fourier boundary condition for the temperature on \( \Gamma^\ell \). The relation (2.20) represents a homogeneous Neumann boundary condition for the damage field on \( \Gamma^\ell \). Finally the functions \( u^0, \tau^0, s^0 \) and \( \zeta^0 \) in (2.21)-(2.22) are the initial data.

We now proceed to obtain a variational formulation of Problem \( P \). For this purpose, we introduce additional notation and assumptions on the problem data. Here and in what follows the indices \( i \) and \( j \) run between 1 and \( d \), the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. Let \( H^\ell = L^2(\Omega^\ell)^d \), \( H^\ell_1 = H^1(\Omega^\ell)^d \), \( H^\ell = L^2(\Omega^\ell)^d \times d \), \( H^\ell_{1,j} = \{ \theta^\ell \in (\theta^\ell_{1,j}) \in H^\ell; \text{div}\theta^\ell \in H^\ell \}. \) The spaces \( H^\ell, H^\ell_1, H^\ell_1 \) and \( H^\ell_1 \) are real Hilbert spaces endowed with the canonical inner products given by

\begin{align}
(u^\ell, v^\ell)_{H^\ell} &= \int_{\Omega^\ell} u^\ell \cdot v^\ell \, dx, \\
(u^\ell, v^\ell)_{H^\ell_1} &= \int_{\Omega^\ell} u^\ell : v^\ell \, dx + \int_{\Omega^\ell} \nabla u^\ell \cdot \nabla v^\ell \, dx, \\
(\sigma^\ell, \theta^\ell)_{H^\ell_1} &= \int_{\Omega^\ell} \sigma^\ell : \theta^\ell \, dx + \int_{\Omega^\ell} \text{div} \sigma^\ell \, dx, \\
(\sigma^\ell, \theta^\ell)_{H^\ell} &= \int_{\Omega^\ell} \sigma^\ell : \theta^\ell \, dx + \int_{\Omega^\ell} \text{div} \sigma^\ell \, dx, \\
(\sigma^\ell, \theta^\ell)_{H^\ell_1} &= \int_{\Omega^\ell} \sigma^\ell : \theta^\ell \, dx + \int_{\Omega^\ell} \text{div} \sigma^\ell \, dx,
\end{align}

and the associated norms \( \| \cdot \|_{H^\ell}, \| \cdot \|_{H^\ell_1}, \| \cdot \|_{H^\ell_1}, \) and \( \| \cdot \|_{H^\ell_1} \) respectively. We introduce for the bonding field the set

\begin{align}
Z = \{ \beta \in L^\infty(0,T; L^2(\Gamma_3)); \ 0 \leq \beta(t) \leq 1 \ \forall t \in [0,T], \ \text{a.e. on } \Gamma_3 \},
\end{align}

and for the displacement field we need the closed subspace of \( H^\ell_1 \) defined by

\begin{align}
V^\ell = \{ v^\ell \in H^\ell_1; \ v^\ell = 0 \ \text{on } \Gamma_3 \}.
\end{align}

Since \( \text{meas}\Gamma_3^\ell > 0 \), the following Korn’s inequality holds (see [18]):

\begin{align}
\| \varepsilon(v^\ell) \|_{H^\ell} \geq c_K \| v^\ell \|_{H^\ell_1}, \ \forall v^\ell \in V^\ell.
\end{align}

Over the space \( V^\ell \) we consider the inner product given by

\begin{align}
(u^\ell, v^\ell)_{V^\ell} = (\varepsilon(u^\ell), \varepsilon(v^\ell))_{H^\ell_1}, \ \forall u^\ell, v^\ell \in V^\ell.
\end{align}
and let \(\|\cdot\|_{V^\ell}\) be the associated norm. It follows from Korn’s inequality (2.24) that the norms \(\|\cdot\|_{H^1_{\ell}}\) and \(\|\cdot\|_{V^\ell}\) are equivalent on \(V^\ell\). Then \((V^\ell, \|\cdot\|_{V^\ell})\) is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2.25), there exists a constant \(c_0 > 0\), depending only on \(\Omega^\ell\), \(\Gamma^\ell_1\) and \(\Gamma^\ell_3\) such that

\[
\|v^\ell\|_{L^2(\Gamma^3, \S)} \leq c_0 \|v^\ell\|_{V^\ell} \quad \forall v^\ell \in V^\ell.
\]

We also introduce the spaces

\[
L_0^\ell = L^2(\Omega^\ell), \quad L_1^\ell = H^1(\Omega^\ell), \quad W^\ell = \{\psi^\ell \in L_1^\ell; \psi^\ell = 0 \text{ on } \Gamma^\ell_0\},
\]

\[
W^\ell = \{D^\ell = (D_i^\ell); \ D_i^\ell \in L^2(\Omega^\ell), \ \text{div } D^\ell \in L^2(\Omega^\ell)\}.
\]

Since \(\text{meas}\ \Gamma^\ell_0 > 0\), the following Friedrichs-Poincaré inequality holds:

\[
\|\nabla \psi^\ell\|_{W^\ell} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell,
\]

where \(c_F > 0\) is a constant which depends only on \(\Omega^\ell, \Gamma^\ell_0\). Over the space \(W^\ell\), we consider the inner product given by

\[
(L^\ell, \Phi^\ell, W^\ell) = \frac{1}{\Omega^\ell} \int_{\Omega^\ell} D^\ell : \Phi^\ell \, dx + \int_{\Omega^\ell} \text{div } D^\ell, \text{div } \Phi^\ell \, dx,
\]

where \(\text{div } D^\ell = (\text{div } D_i^\ell)\), and the associated norm \(\|\cdot\|_{W^\ell}\).

In order to simplify the notations, we define the product spaces

\[
V = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2,
\]

\[
L_0 = L_0^1 \times L_0^2, \quad L_1 = L_1^1 \times L_1^2, \quad W = W^1 \times W^2, \quad W = W^1 \times W^2.
\]

The spaces \(V, L_1, W\) and \(W\) are real Hilbert spaces endowed with the canonical inner products denoted by \((\cdot, \cdot)_V, (\cdot, \cdot)_{L_1}, (\cdot, \cdot)_{W}\) and \((\cdot, \cdot)_W\).

In the study of the Problem \(P\), we consider the following assumptions:

The \textit{viscosity function} \(\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d\) satisfies:

\[
\begin{align*}
(a) \text{ There exists } & L_{A^\ell} > 0 \text{ such that: } \forall \omega_1, \omega_2 \in \mathbb{S}^d, \\
& |\mathcal{A}^\ell(x, \omega_1) - \mathcal{A}^\ell(x, \omega_2)| \leq L_{A^\ell} |\omega_1 - \omega_2|, \quad \text{a.e. } x \in \Omega^\ell.
\end{align*}
\]

(b) \text{ There exists } m_{A^\ell} > 0 \text{ such that: } \forall \omega_1, \omega_2 \in \mathbb{S}^d,

\[
|\mathcal{A}^\ell(x, \omega_1) - \mathcal{A}^\ell(x, \omega_2)| (|\omega_1 - \omega_2|) \geq m_{A^\ell} |\omega_1 - \omega_2|^2, \quad \text{a.e. } x \in \Omega^\ell.
\]

(c) \text{ The mapping } x \mapsto \mathcal{A}^\ell(x, \omega) \text{ is measurable on } \Omega^\ell, \forall \omega \in \mathbb{S}^d.

(d) \text{ The mapping } x \mapsto \mathcal{A}^\ell(x, 0) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } x \in \Omega^\ell.

The \textit{elasticity operator} \(\mathcal{B}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d\) satisfies:

\[
\begin{align*}
(a) \text{ There exists } & L_{B^\ell} > 0 \text{ such that: } \forall \omega_1, \omega_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\
& |\mathcal{B}^\ell(x, \omega_1, r_1, d_1) - \mathcal{B}^\ell(x, \omega_2, r_2, d_2)| \leq L_{B^\ell} (|\omega_1 - \omega_2| + |r_1 - r_2| + |d_1 - d_2|), \quad \text{a.e. } x \in \Omega^\ell.
\end{align*}
\]

(b) \text{ The mapping } x \mapsto \mathcal{B}^\ell(x, \omega, r, d) \text{ is measurable in } \Omega^\ell, \forall \omega \in \mathbb{S}^d, r, d \in \mathbb{R}.

(c) \text{ The mapping } x \mapsto \mathcal{B}^\ell(x, 0, 0, 0) \text{ belongs to } \mathcal{H}^\ell.
The relaxation function $\mathcal{Q}^\ell : \Omega^\ell \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

\[
\begin{align*}
(a) \text{ There exists } L_{\mathcal{Q}^\ell} > 0 \text{ such that } &:\forall \omega_1, \omega_2 \in \mathbb{S}^d, r_1, r_2, d_1, d_2 \in \mathbb{R}, \\
&|Q^\ell(x, t, \omega_1, r_1, d_1) - Q^\ell(x, t, \omega_2, r_2, d_2)| \leq L_{\mathcal{Q}^\ell}(|\omega_1 - \omega_2| + |r_1 - r_2| + |d_1 - d_2|), \text{ for all } t \in (0, T), \text{ a.e. } x \in \Omega^\ell.
\end{align*}
\]

\[
(b) \text{ The mapping } x \mapsto Q^\ell(x, t, \omega, r, d) \text{ is measurable in } \Omega^\ell, \\
\text{ for any } t \in (0, T), \omega \in \mathbb{S}^d, r, d \in \mathbb{R}.
\]

\[
(c) \text{ The mapping } t \mapsto Q^\ell(x, t, \omega, r, d) \text{ is continuous in } (0, T), \\
\text{ for any } \omega \in \mathbb{S}^d, r, d \in \mathbb{R}, \text{ a.e. } x \in \Omega^\ell.
\]

\[
(d) \text{ The mapping } x \mapsto Q^\ell(x, t, 0, 0, 0) \text{ belongs to } \mathcal{H}^\ell, \forall t \in (0, T).
\]

The energy function $\Theta^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

\[
\begin{align*}
(a) \text{ There exists } L_{\Theta^\ell} > 0 \text{ such that } &:\forall \eta_1, \eta_2, \omega_1, \omega_2 \in \mathbb{S}^d, \alpha_1, \alpha_2, d_1, d_2 \in \mathbb{R}, \\
&|\Theta^\ell(x, \eta_1, \omega_1, \alpha_1, d_1) - \Theta^\ell(x, \eta_2, \omega_2, \alpha_2, d_2)| \leq L_{\Theta^\ell}(|\eta_1 - \eta_2| + |\omega_1 - \omega_2| + |\alpha_1 - \alpha_2| + |d_1 - d_2|), \text{ a.e. } x \in \Omega^\ell.
\end{align*}
\]

\[
(b) \text{ The mapping } x \mapsto \Theta^\ell(x, \eta, \omega, \alpha, d) \text{ is measurable on } \Omega^\ell, \\
\text{ for any } \eta, \omega \in \mathbb{S}^d \text{ and } \alpha, d \in \mathbb{R}.
\]

\[
(c) \text{ The mapping } x \mapsto \Theta^\ell(x, 0, 0, 0, 0) \text{ belongs to } L^2(\Omega^\ell),
\]

\[
(d) \Theta^\ell(x, \eta, \omega, \alpha, d) \text{ is bounded for all } \eta, \omega \in \mathbb{S}^d, \alpha, d \in \mathbb{R} \text{ a.e. } x \in \Omega^\ell.
\]

The adhesion rate function $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ satisfies:

\[
\begin{align*}
(a) \text{ There exists } L_{H_{ad}} > 0 \text{ such that } &:\forall \zeta_1, \zeta_2, \omega_1, \omega_2, r_1, r_2 \in \mathbb{R}, d_1, d_2 \in \mathbb{R}^{d-1}, \\
&|H_{ad}(x, \zeta_1, \omega_1, r_1, d_1) - H_{ad}(x, \zeta_2, \omega_2, r_2, d_2)| \leq L_{H_{ad}}(|\zeta_1 - \zeta_2| + |\omega_1 - \omega_2| + |r_1 - r_2| + |d_1 - d_2|), \text{ a.e. } x \in \Gamma_3.
\end{align*}
\]

\[
(b) \text{ The mapping } x \mapsto H_{ad}(x, \zeta, \omega, r, d) \text{ is measurable on } \Gamma_3, \\
\text{ for any } \zeta, \omega, r \in \mathbb{R}, d \in \mathbb{R}^{d-1},
\]

\[
(c) \text{ The mapping } (\zeta, \omega, r, d) \mapsto H_{ad}(x, \zeta, \omega, r, d) \text{ is continuous on } \\
\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}, \text{ a.e. } x \in \Gamma_3,
\]

\[
(d) H_{ad}(x, 0, \omega, r, d) = 0, \forall \xi, r, d \in \mathbb{R}^{d-1}, \text{ a.e. } x \in \Gamma_3.
\]

\[
(e) H_{ad}(x, \zeta, \omega, r, d) \geq 0, \forall \zeta \leq 0, \omega, r \in \mathbb{R}, d \in \mathbb{R}^{d-1}, \text{ a.e. } x \in \Gamma_3, \text{ and}
\]

\[
H_{ad}(x, \zeta, \omega, r, d) \leq 0, \forall \zeta \geq 1, \omega, r \in \mathbb{R}, d \in \mathbb{R}^{d-1}, \text{ a.e. } x \in \Gamma_3.
\]

The piezoelectric tensor $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

\[
\begin{align*}
(a) \text{ There exists } L_{\mathcal{E}^\ell} > 0 \text{ such that } &:\forall \ell_{ij}, (\ell_{ij}) \in \mathbb{S}^d \text{ a.e. } x \in \Omega^\ell.
\end{align*}
\]

\[
(b) \ell_{ij} = \epsilon_{ijk} \in L^\infty(\Omega^\ell), 1 \leq i, j, k \leq d.
\]

The damage source function $\Psi^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies:

\[
\begin{align*}
(a) \text{ There exists } L_{\Psi^\ell} > 0 \text{ such that } &:\forall \eta_1, \eta_2, \omega_1, \omega_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}, \\
&|\Psi^\ell(x, \eta_1, \omega_1, \alpha_1) - \Psi^\ell(x, \eta_2, \omega_2, \alpha_2)| \leq L_{\Psi^\ell}(|\eta_1 - \eta_2| + |\omega_1 - \omega_2| + |\alpha_1 - \alpha_2|), \text{ a.e. } x \in \Omega^\ell.
\end{align*}
\]

\[
(b) \text{ The mapping } x \mapsto \Psi^\ell(x, \eta, \omega, \alpha) \text{ is measurable on } \Omega^\ell, \\
\text{ for any } \eta, \omega \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}.
\]

\[
(c) \text{ The mapping } x \mapsto \Psi^\ell(x, 0, 0, 0) \text{ belongs to } L^2(\Omega^\ell),
\]

\[
(d) \Psi^\ell(x, \eta, \omega, \alpha) \text{ is bounded, } \forall \eta, \omega \in \mathbb{S}^d, \alpha \in \mathbb{R} \text{ a.e. } x \in \Omega^\ell.
\]

The electric permittivity operator $G^\ell : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$, satisfies:

\[
\begin{align*}
(a) \text{ There exists } m_{G^\ell} > 0 \text{ such that :} \\
&G^\ell(x, E) \geq m_{G^\ell}|E|^2, \forall E \in \mathbb{R}^d, \text{ a.e. } x \in \Omega^\ell.
\end{align*}
\]
The **normal compliance function** \( p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \) satisfies:

\[
\begin{align*}
\text{(a)} & \quad \text{There exists } L_\nu > 0 \text{ such that } \forall r_1, r_2 \in \mathbb{R}, \\
& \quad |p_\nu(x, r_1) - p_\nu(x, r_2)| \leq L_\nu |r_1 - r_2|, \text{ a.e. } x \in \Gamma_3. \\
\text{(b)} & \quad (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \geq 0, \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\
\text{(c)} & \quad \text{The mapping } x \mapsto p_\nu(x, r) \text{ is measurable on } \Gamma_3, \forall r \in \mathbb{R}. \\
\text{(d)} & \quad p_\nu(x, r) = 0, \text{ for all } r \leq 0, \text{ a.e. } x \in \Gamma_3.
\end{align*}
\]

(2.37)

The adhesion coefficients \( \gamma_\nu \) and \( \gamma_\tau \) satisfy the conditions

\[
\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \gamma_\nu, \gamma_\tau \geq 0, \text{ a.e. on } \Gamma_3.
\]

(2.38)

The forces, tractions have the regularity

\[
\begin{align*}
\textbf{f}_0^\ell & \in C(0, T; L^2(\Omega^\ell)^d), \quad \textbf{f}_2^\ell \in C(0, T; L^2(\Gamma_0^\ell)^d), \\
q_0^\ell & \in C(0, T; L^2(\Omega^\ell)), \quad q_2^\ell \in C(0, T; L^2(\Gamma_0^\ell)), \quad \rho^\ell \in C(0, T; L^2(\Omega^\ell)).
\end{align*}
\]

(2.39)

The energy coefficient \( \kappa_0^\ell \) and the microcrack diffusion coefficient \( \kappa^\ell \) satisfies:

\[
\kappa_0^\ell > 0, \quad \kappa^\ell > 0.
\]

(2.40)

Finally, the friction coefficient and the initial data satisfy:

\[
\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3,
\]

(2.41)

\[
\begin{align*}
\text{Next, we define the mappings } & f = (f^1, f^2) : [0, T] \to \mathbb{V}, \quad q = (q^1, q^2) : [0, T] \to W, \quad a_0 : \\
& L_1 \times L_1 \to \mathbb{R}, \quad a : L_1 \times L_1 \to \mathbb{R}, \quad j_{\text{ad}} : L^2(\Gamma_3) \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \quad j_{\text{bc}} : \mathbb{V} \times \mathbb{V} \to \mathbb{R} \text{ and } \\
& j_{\text{fr}} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \text{ respectively, by }
\end{align*}
\]

(2.42)

\[
(f(t), \nu)(\mathbf{v}) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} f_0^\ell(t) \cdot \nu^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_0^\ell} f_2^\ell(t) \cdot \nu^\ell \, da \quad \forall \nu \in \mathbb{V},
\]

(2.43)

\[
(q(t), \phi)(\mathbf{w}) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \phi^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_0^\ell} q_2^\ell(t) \phi^\ell \, da \quad \forall \phi \in W,
\]

(2.44)

\[
a_{0}(\tau, \alpha) = \sum_{\ell=1}^{2} \kappa_0^\ell \int_{\Omega^\ell} \nabla \tau^\ell \cdot \nabla \alpha^\ell \, dx + \sum_{\ell=1}^{2} \lambda_0^\ell \int_{\Gamma^\ell} \tau^\ell \alpha^\ell \, da,
\]

(2.45)

\[
a(\zeta, \alpha) = \sum_{\ell=1}^{2} \kappa^\ell \int_{\Omega^\ell} \nabla \zeta^\ell \cdot \nabla \alpha^\ell \, dx,
\]

(2.46)

\[
\begin{align*}
\int_{\Gamma_3} \left( -\gamma_\nu \zeta^2 R_\nu([u_\nu])[v_\nu] + \gamma_\tau \zeta^2 R_\tau([u_\nu])[v_\tau] \right) \, da,
\end{align*}
\]

(2.47)

\[
\int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da,
\]

(2.48)

\[
\int_{\Gamma_3} \mu p_\nu([u_\nu])[v_\nu] \, da.
\]

By a standard procedure based on Green’s formula we can derive the following variational formulation of the contact problem (2.6)–(2.22).

**Problem PV.** Find a displacement field \( \mathbf{u} = (u^1, u^2) : [0, T] \to \mathbb{V} \), a stress field \( \mathbf{\sigma} = (\mathbf{\sigma}^1, \mathbf{\sigma}^2) : [0, T] \to \mathcal{H} \), an electric potential field \( \mathbf{\xi} = (\mathbf{\xi}^1, \mathbf{\xi}^2) : [0, T] \to W \), a temperature \( \tau = (\tau^1, \tau^2) : [0, T] \to \mathbb{R} \), a damage field \( \zeta = (\zeta^1, \zeta^2) : [0, T] \to \mathbb{R}_+ \), a bonding field.
\[ \zeta : [0, T] \to L^\infty(\Gamma_3) \] and a electric displacement field \[ D = (D^1, D^2) : [0, T] \to \mathcal{W} \] such that, for a.e. \( t \in (0, T) \),

\[
\sigma^t = \mathcal{A}^t \varepsilon(\dot{u}^t) + \mathcal{B}^t(\varepsilon(u^t), \tau^t, \zeta^t) + \int_0^t \mathcal{Q}^t(t - s, \varepsilon(u^t(s)), \tau^t(s), \zeta^t(s)) \, ds - (\mathcal{E}^t)^r(\xi^t),
\]

\[
D^t = \mathcal{E}^t \varepsilon(u^t) + \mathcal{G}^t(\xi^t),
\]

\[
\sum_{t=1}^2 (\sigma^t - \varepsilon(\dot{u}^t))_{st} + j_{ad}(\zeta(t), u(t), v - \dot{u}(t)) + j_{fr}(u(t), v) \geq (f(t), v - \dot{u}(t))_{st}, \quad \forall v \in V,
\]

\[
\sum_{t=1}^2 \left( \mathcal{E}^t \varepsilon(u^t(t)) + \mathcal{G}^t(\xi^t(t)) \right) \bigg|_{st} = (-q(t), \phi)_{st}, \quad \forall \phi \in W.
\]

We notice that the variational Problem PV is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution of Problem PV is stated and proved in the next section.

**Remark 2.1.** We note that, in Problem P and in Problem PV, we do not need to impose explicitly the restriction \( 0 \leq \zeta \leq 1 \). Indeed, equation (2.55) guarantees that \( \zeta(x, t) \leq \zeta_0(x) \) and, therefore, assumption (2.41) shows that \( \zeta(x, t) \leq 1 \) for \( t \geq 0 \), a.e. \( x \in \Gamma_3 \). On the other hand, if \( \zeta(x, t_0) = 0 \) at time \( t_0 \), then it follows from (2.55) that \( \zeta(x, t) = 0 \) for all \( t \geq t_0 \) and therefore, \( \zeta(x, t) = 0 \) for all \( t \geq t_0 \), a.e. \( x \in \Gamma_3 \). We conclude that \( 0 \leq \zeta(x, t) \leq 1 \) for all \( t \in [0, T] \), a.e. \( x \in \Gamma_3 \).

First, we note that the functional \( j_{ad} \) and \( j_{vc} \) are linear with respect to the last argument and, therefore,

\[
j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v),
\]

\[
j_{vc}(u, -v) = -j_{vc}(u, v).
\]

Next, using (2.47) and (2.37) imply

\[
j_{vc}(u_1, v_2) - j_{vc}(u_1, v_1) + j_{vc}(u_2, v_1) - j_{vc}(u_2, v_2) \leq 0, \quad \forall u_1, u_2, v_1, v_2 \in V,
\]

and use (2.48), (2.37)(a), keeping in mind (2.26), we obtain

\[
j_{fr}(u_1, v_2) - j_{fr}(u_1, v_1) + j_{fr}(u_2, v_1) - j_{fr}(u_2, v_2) \leq \varepsilon_0^2 L_v \|\mu\|_{L^\infty(\Gamma_3)} \|u_1 - u_2\|_V \|v_1 - v_2\|_V, \quad \forall u_1, u_2, v_1, v_2 \in V.
\]
3. Main Results

The main results are stated by the following theorems.

**Theorem 3.1.** Assume that (2.29)–(2.41) hold. Then there exists a unique solution of Problem PV. Moreover, the solution satisfies

\begin{align}
\mathbf{u} & \in C^1(0, T; \mathbf{V}), \\
\xi & \in C(0, T; W), \\
\zeta & \in W^{1, \infty}(0, T; L^2(\Gamma_3)) \cap Z, \\
\sigma & \in C(0, T; \mathcal{H}_4), \\
\tau & \in L^2(0, T; L_1) \cap H^1(0, T; L_0), \\
\varsigma & \in L^2(0, T; L_1) \cap H^1(0, T; L_0), \\
D & \in C(0, T; \mathcal{W}).
\end{align}

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we assume in what follows that (2.29)–(2.41) hold, and we consider that \( C \) is a generic positive constant which depends on \( \Omega^\ell, \Gamma^\ell_1, \Gamma_3, p^\ell, p_r, \mathcal{A}^\ell, \mathcal{B}^\ell, \mathcal{G}^\ell, \mathcal{Q}^\ell, \mathcal{E}^\ell, H^d, \gamma^\ell, \gamma^\ell_r, \Theta^\ell, \Psi^\ell, \kappa^\ell, \kappa^\ell_r \), and \( T \) with \( \ell = 1, 2 \). but does not depend on \( t \) nor of the rest of input data, and whose value may change from place to place.

In the first step. Let \((\lambda, \mu) \in C(0, T; L_0 \times L_0)\) and consider the auxiliary problem.

**Problem PV\(_{(\lambda, \mu)}\).** Find \( \tau_\lambda : [0, T] \to L_0 \), and \( \varsigma_\mu : [0, T] \to L_0 \), such that \( \varsigma_\mu(t) \in K \)

\begin{align}
&\sum_{\ell=1}^2 (\tau^\ell_\lambda(t) - \lambda^\ell(t) - \rho^\ell(t), \alpha^\ell)_{L^2(\Omega^\ell)} + a_0(\tau^\ell_\lambda(t), \alpha^\ell) = 0, \forall \alpha \in L_0, \\
&\sum_{\ell=1}^2 (\varsigma^\ell_\mu(t) - \mu^\ell(t), \alpha^\ell - \varsigma^\ell_\mu(t))_{L^2(\Omega^\ell)} + a(\varsigma^\ell_\mu(t), \alpha - \varsigma^\ell_\mu(t)) \geq 0, \forall \alpha \in K, \\
&\tau_\lambda(0) = \tau_0, \quad \varsigma_\mu(0) = 0,
\end{align}

where \( K = K^1 \times K^2 \).

**Lemma 3.2.** There exists a unique solution \( \{\tau_\lambda, \varsigma_\mu\} \) to the auxiliary problem PV\(_{(\lambda, \mu)}\) satisfying (3.5)–(3.6).

**Proof.** Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant \( c_0 > 0 \) such that

\[ \int_{\Omega^\ell} |\nabla \alpha|^2 \, dx + \frac{\lambda^\ell_0}{\kappa^\ell_0} \int_{\Gamma^\ell} |\alpha|^2 \, da \geq c_0 \int_{\Omega^\ell} |\alpha|^2 \, dx, \quad \forall \alpha \in L_1^\ell, \quad \ell = 1, 2. \]

Thus, we obtain

\[ a_0(\alpha, \alpha) \geq c_1 \|
\alpha\|^2_{L_1^\ell}, \quad \forall \alpha \in L_1, \]

where \( c_1 = \kappa_0 \min(1, c_0)/2 \), which implies that \( a_0 \) is \( L_1 \)-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.9) has a unique solution \( \tau_\lambda \) satisfying \( \tau_\lambda(0) = \tau_0 \) and the regularity (3.5).

On the other hand, we know that the form \( a \) is not \( L_1 \)-elliptic. To solve this problem we introduce the functions

\[ \tilde{\varsigma}^\ell_\mu(t) = e^{-\kappa^\ell t} \varsigma^\ell_\mu(t), \quad \tilde{\alpha}^\ell(t) = e^{-\kappa^\ell t} \alpha^\ell(t), \quad \ell = 1, 2. \]
We remark that if $\zeta_\mu^\ell, \alpha^\ell \in K^\ell$ then $\tilde{\zeta}_\mu^\ell, \tilde{\alpha}^\ell \in K^\ell$. Consequently, (3.9) is equivalent to the inequality

$$
\sum_{\ell=1}^{2} \left( \tilde{\zeta}_\mu^\ell(t) - e^{-\kappa t} \mu^\ell(t), \tilde{\alpha}^\ell - \zeta_\mu^\ell(t) \right)_{L^2(\Omega^\ell)} + a(\tilde{\alpha}(t), \tilde{\alpha} - \zeta_\mu(t)) + 
$$

$$
\sum_{\ell=1}^{2} \kappa(\tilde{\zeta}_\mu^\ell, \tilde{\alpha}^\ell - \zeta_\mu^\ell(t))_{L^2(\Omega^\ell)} \geq 0, \; \forall \tilde{\alpha}, \tilde{\alpha} \in K, \; \text{a.e.} \; t \in (0, T).
$$

(3.11)

The fact that

$$
a(\tilde{\alpha}, \tilde{\alpha}) + \sum_{\ell=1}^{2} \kappa(\tilde{\alpha}^\ell, \tilde{\alpha}^\ell)_{L^2(\Omega^\ell)} \geq \sum_{\ell=1}^{2} \kappa(\tilde{\alpha}^\ell)_{L^2(\Omega^\ell)} \; \forall \tilde{\alpha} \in L_1,
$$

(3.12)

implies that (3.11) has a unique solution $\hat{\zeta}_\mu$, having the regularity (3.6).

In the second step. Let $(\lambda, \mu, \eta) \in C(0, T; L_0 \times L_0 \times V)$, we use the $\{\tau_\lambda, \varsigma_\mu\}$ obtained in Lemma (3.2) and consider the auxiliary problem.

**Problem PV(\lambda, \mu, \eta).** Find $\mathbf{u}_{\lambda\mu} : [0, T] \to V$, $\xi_{\lambda\mu} : [0, T] \to W$, and $\zeta_{\lambda\mu} : [0, T] \to L^2(\Gamma_3)$ such that

$$
\sum_{\ell=1}^{2} \left( A^\ell \varepsilon(\mathbf{u}_{\lambda\mu}^\ell), \varepsilon(\mathbf{u}_{\lambda\mu}^\ell), \zeta_{\lambda\mu}^\ell \right) + 
$$

$$
\sum_{\ell=1}^{2} \left( \tilde{A}^\ell \varepsilon(\tilde{\mathbf{u}}_{\lambda\mu}^\ell), \varepsilon(\tilde{\mathbf{u}}_{\lambda\mu}^\ell), \tilde{\zeta}_{\lambda\mu}^\ell \right) \geq \sum_{\ell=1}^{2} \kappa(\tilde{\alpha}^\ell)_{L^2(\Omega^\ell)} \; \forall \tilde{\alpha} \in L_1.
$$

(3.13)

In (3.13) and (3.17), we use an abstract existence and unique result which may be found in [21, Lemma 4.2.]. Next, we consider the form $G : W \times W \to \mathbb{R}$,

$$
G(\xi, \phi) = \sum_{\ell=1}^{2} \left( G^\ell \nabla \xi^\ell, \nabla \phi^\ell \right)_{H^\ell} \; \forall \xi, \phi \in W.
$$

(3.18)

We use (2.27), (2.28), (2.36) and (3.18) to show that the form $G$ is bilinear continuous, symmetric and coercive on $W$, moreover using (2.43) and the Riesz representation theorem we may define an element $w_{\lambda\mu} : [0, T] \to W$ such that

$$
(w_{\lambda\mu}(t), \phi)_W = (q(t), \phi)_W + \sum_{\ell=1}^{2} \left( \mathcal{E}^\ell \varepsilon(\mathbf{u}_{\lambda\mu}^\ell(t)), \nabla \phi^\ell \right)_{H^\ell} \; \forall \phi \in W, t \in (0, T).
$$

(3.17)
We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\xi_{\lambda\mu\eta}(t) \in W$ such that

$$G(\xi_{\lambda\mu\eta}(t), \phi) = (w_{\lambda\mu\eta}(t), \phi)_W \quad \forall \phi \in W.$$  

(3.19)

It follows from (3.19) that $\xi_{\lambda\mu\eta}$ is a solution of the equation (3.14). Let $t_1, t_2 \in [0, T]$, it follows from (3.14) that

$$\|\xi_{\lambda\mu\eta}(t_1) - \xi_{\lambda\mu\eta}(t_2)\|_W \leq C\|u_{\lambda\mu\eta}(t_1) - u_{\lambda\mu\eta}(t_2)\|_V + \|q(t_1) - q(t_2)\|_W.$$  

(3.20)

Now, from (2.39), (3.20) and $u_{\lambda\mu\eta} \in C^1(0, T; V)$, we obtain that $\xi_{\lambda\mu\eta} \in C(0, T; W)$.

On the other hand, we consider the mapping $H_{\lambda\mu\eta} : [0, T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$H_{\lambda\mu\eta}(t, \zeta) = H_{ad}((\zeta(t), \alpha\zeta, R_{\nu}([u_{\lambda\mu\eta}(t)]), R_{\tau}([u_{\lambda\mu\eta}(t)])),$$  

for all $t \in [0, T]$ and $\zeta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator $R_{\nu}$ and $R_{\tau}$ that $H_{\lambda\mu\eta}$ is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\zeta \in L^2(\Gamma_3)$, the mapping $t \to H_{\lambda\mu\eta}(t, \zeta)$ belongs to $L^\infty(0, T; L^2(\Gamma_3))$. Thus using the Cauchy-Lipschitz Theorem (see [26], p.48), we deduce that there exists a unique function $\zeta_{\lambda\mu\eta} \in W^{1,\infty}(0, T; L^2(\Gamma_3))$ solution of the equation (3.15). Also, the arguments used in Remark 2.1 show that $0 \leq \zeta_{\lambda\mu\eta}(t) \leq 1$ for all $t \in [0, T]$; a.e. on $\Gamma_3$. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\zeta_{\lambda\mu\eta} \in \mathcal{Z}$. This completes the proof. \]

In the third step, let us consider the element

$$\Pi(\eta, \lambda, \mu)(t) = (\Pi^1(\eta, \lambda, \mu)(t), \Pi^2(\eta, \lambda, \mu)(t), \Pi^3(\eta, \lambda, \mu)(t)) \in V \times L_0 \times L_0,$$

defined by the equations

$$\Pi^1(\eta, \lambda, \mu)(t, v)_V = -\sum_{\ell=1}^{2}((\mathcal{E}^{\ell})^*E^{\ell}(\varepsilon_{\lambda\mu\eta}^\ell), \varepsilon(v^\ell))_{\mathcal{H}^\ell} + j_{ad}(\zeta_{\lambda\mu\eta}(t), u_{\lambda\mu\eta}(t), v)$$  

(3.22)

$$+ \sum_{\ell=1}^{2}\left(\int_0^t \mathcal{Q}^\ell(t-s, v(u_{\lambda\mu\eta}^\ell(s)), \tau_{\lambda}^\ell(s), \varepsilon_{\mu}^\ell(s)) \, ds, \varepsilon(v^\ell)\right)_{\mathcal{H}^\ell}, \forall v \in V,$$

(3.23)

$$\Pi^2(\eta, \lambda, \mu) = \left(\Theta^1(\sigma_{\lambda\mu\eta}^1, \varepsilon(u_{\lambda\mu\eta}^1), \tau_{\lambda}^1, \varepsilon_{\mu}^1), \Theta^2(\sigma_{\lambda\mu\eta}^2, \varepsilon(u_{\lambda\mu\eta}^2), \tau_{\lambda}^2, \varepsilon_{\mu}^2)\right),$$  

(3.24)

where the mapping $\sigma_{\lambda\mu\eta}^\ell$ is given by

$$\sigma_{\lambda\mu\eta}^\ell = B^{\ell}\left(\varepsilon(u_{\lambda\mu\eta}^\ell), \tau_{\lambda}^\ell, \varepsilon_{\mu}^\ell\right) + \int_0^t \mathcal{Q}^\ell(t-s, v(u_{\lambda\mu\eta}^\ell(s)), \tau_{\lambda}^\ell(s), \varepsilon_{\mu}^\ell(s)) \, ds - (\mathcal{E}^{\ell})^*E^{\ell}(\varepsilon_{\lambda\mu\eta}^\ell).$$  

(3.25)

\textbf{Lemma 3.4.} The mapping $\Lambda$ has a fixed point $(\eta^0, \lambda^0, \mu^0) \in C(0, T; V \times L_0 \times L_0)$.

\textbf{Proof.} Let $(\eta_1, \lambda_1, \mu_1), (\eta_2, \lambda_2, \mu_2) \in C(0, T; V \times L_0 \times L_0)$ and denote by $\tau_i, \varepsilon_i, u_i, \xi_i, \zeta_i$ and $\sigma_i$, the functions obtained in Lemmas 3.2, 3.3 and the relation (3.25), for $(\eta, \lambda, \mu) = (\eta_1, \lambda_1, \mu_1)$,

$\Lambda(\eta_1, \lambda_1, \mu_1) = \eta_2, \lambda_2, \mu_2$.

Therefore $\Lambda$ is a contraction and has a unique fixed point $(\eta^0, \lambda^0, \mu^0) \in C(0, T; V \times L_0 \times L_0)$.
Let \( t \in [0, T] \). We use (2.35), (2.34), (2.46) and the definition of \( R_\nu, R_\tau \), we have

\[
\sum_{\ell=1}^{2} \int_0^t \left| \mathcal{Q}_{\ell}^t (t - s, \varepsilon (u_1^\ell (s)), \tau_1^\ell (s), s) - \mathcal{Q}_{\ell}^t (t - s, \varepsilon (u_2^\ell (s)), \tau_2^\ell (s), s) \right|^2 d\tau +
\int_0^t \left| \mathcal{Q}_1^t (t - s, \varepsilon (u_1^1 (s)), v_1 (s)) - \mathcal{Q}_1^t (t - s, \varepsilon (u_2^1 (s)), v_2 (s)) \right|^2 d\tau +
\sum_{\ell=1}^{2} \int_0^t \left| \mathcal{R}_{\ell}^t (t - s, \varepsilon (u_1^\ell (s)), \tau_1^\ell (s), s) - \mathcal{R}_{\ell}^t (t - s, \varepsilon (u_2^\ell (s)), \tau_2^\ell (s), s) \right|^2 d\tau +
\sum_{\ell=1}^{2} \int_0^t \left| \mathcal{R}_1^t (t - s, \varepsilon (u_1^1 (s)), v_1 (s)) - \mathcal{R}_1^t (t - s, \varepsilon (u_2^1 (s)), v_2 (s)) \right|^2 d\tau
\]

Therefore,

\[
\| \Pi^1 (\eta_1, \lambda_1, \mu_1) (t) - \Pi^1 (\eta_2, \lambda_2, \mu_2) (t) \| \leq \left( \int_0^t \| u_1 (s) - u_2 (s) \|^2 d\tau + \right.
\int_0^t \| \tau_1 (s) - \tau_2 (s) \|^2 d\tau + \int_0^t \| \zeta_1 (s) - \zeta_2 (s) \|^2 d\tau \right)
\]

(3.36)

By similar arguments, from (3.23), (3.25) and (2.32) it follows that

\[
\| \Pi^2 (\eta_1, \lambda_1, \mu_1) (t) - \Pi^2 (\eta_2, \lambda_2, \mu_2) (t) \| \leq C \left( \| u_1 (t) - u_2 (t) \|^2 + \right.
\int_0^t \| u_1 (s) - u_2 (s) \|^2 d\tau + \int_0^t \| \tau_1 (s) - \tau_2 (s) \|^2 d\tau + \int_0^t \| \zeta_1 (s) - \zeta_2 (s) \|^2 d\tau \right)
\]

(3.27)

Similarly, using (2.35) implies

\[
\| \Pi^3 (\eta_1, \lambda_1, \mu_1) (t) - \Pi^3 (\eta_2, \lambda_2, \mu_2) (t) \| \leq C \left( \| u_1 (t) - u_2 (t) \|^2 + \right.
\int_0^t \| u_1 (s) - u_2 (s) \|^2 d\tau + \int_0^t \| \tau_1 (s) - \tau_2 (s) \|^2 d\tau + \int_0^t \| \zeta_1 (s) - \zeta_2 (s) \|^2 d\tau \right)
\]

(3.28)

It follows now from (3.26), (3.27) and (3.28) that

\[
\| \Pi (\eta_1, \lambda_1, \mu_1) (t) - \Pi (\eta_2, \lambda_2, \mu_2) (t) \| \leq C \left( \| u_1 (t) - u_2 (t) \|^2 + \right.
\int_0^t \| u_1 (s) - u_2 (s) \|^2 d\tau + \int_0^t \| \tau_1 (s) - \tau_2 (s) \|^2 d\tau + \int_0^t \| \zeta_1 (s) - \zeta_2 (s) \|^2 d\tau \right)
\]

(3.29)
Also, since
\[ u_1^\ell(t) = \int_0^t u_1^\ell(s)ds + u_0^\ell(t), \quad t \in [0, T], \quad \ell = 1, 2, \]
we have
\[ \|u_1(t) - u_2(t)\|_V \leq \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds \]
and using this inequality in (3.17) yields
\[
(3.30) \quad \|u_1(t) - u_2(t)\|_V \leq C \left( \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds + \int_0^t \|u_1(s) - u_2(s)\|_V ds \right). 
\]
Next, we apply Gronwall’s inequality to deduce
\[
(3.31) \quad \|u_1(t) - u_2(t)\|_V \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \quad \forall t \in [0, T].
\]
On the other hand, from the Cauchy problem (3.15) we can write
\[ \zeta_i(t) = \zeta_0 - \int_0^t R_{ad}(\zeta_i(s), \alpha_{\zeta_i}(s), R_{\nu}([u_{\nu i}]), R_{\tau}([u_{\tau i}])) ds \]
and employing (2.23) and (2.33) we obtain that
\[
\left\| \zeta_1(t) - \zeta_2(t) \right\|_{L^2(\Gamma_3)} \leq C \int_0^t \left\| \zeta_1(s) - \zeta_2(s) \right\|_{L^2(\Gamma_3)} ds + C \int_0^t \left\| R_{\nu}([u_{\nu 1}]) - R_{\nu}([u_{\nu 2}]) \right\|_{L^2(\Gamma_3)} ds + C \int_0^t \left\| R_{\tau}([u_{\tau 1}]) - R_{\tau}([u_{\tau 2}]) \right\|_{L^2(\Gamma_3)} ds.
\]
Using the definition of \( R_{\nu} \) and \( R_{\tau} \) and writing \( \zeta_1 = \zeta_1 - \zeta_2 + \zeta_2 \), we get
\[
\left\| \zeta_1(t) - \zeta_2(t) \right\|_{L^2(\Gamma_3)} \leq C \left( \int_0^t \left\| \zeta_1(s) - \zeta_2(s) \right\|_{L^2(\Gamma_3)} ds + \int_0^t \left\| u_1(s) - u_2(s) \right\|_{L^2(\Gamma_3)} ds \right). 
\]
Next, we apply Gronwall’s inequality and from the Sobolev trace theorem we obtain
\[
(3.33) \quad \left\| \zeta_1(t) - \zeta_2(t) \right\|_{H^1(\Gamma_3)}^2 \leq C \int_0^t \left\| u_1(s) - u_2(s) \right\|_{H^1(\Gamma_3)}^2 ds.
\]
We use now (3.14), (2.27), (2.34) and (2.36) to find
\[
(3.34) \quad \left\| \xi_1(t) - \xi_2(t) \right\|_{W}^2 \leq C \|u_1(t) - u_2(t)\|_{V}^2.
\]
From (3.8) we deduce that
\[
(\dot{\tau}_1 - \dot{\tau}_2, \tau_1 - \tau_2)_{L^0_0} + a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) + (\lambda_1 - \lambda_2, \theta_1 - \theta_2)_{L^0_0} = 0.
\]
We integrate this equality with respect to time, using the initial conditions \( \tau_1(0) = \tau_2(0) = \tau_0 \) and inequality \( a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) \geq 0 \), to find
\[
\frac{1}{2} \left\| \tau_1(t) - \tau_2(t) \right\|_{L^0_0}^2 \leq \int_0^t \left( \lambda_1(s) - \lambda_2(s), \tau_1(s) - \tau_2(s) \right)_{L^0_0} ds,
\]
which implies that
\[
\left\| \tau_1(t) - \tau_2(t) \right\|_{L^0_0}^2 \leq \int_0^t \left\| \lambda_1(s) - \lambda_2(s) \right\|_{L^0_0}^2 ds + \int_0^t \left\| \tau_1(s) - \tau_2(s) \right\|_{L^0_0}^2 ds.
\]
This inequality combined with Gronwall’s inequality leads to

\begin{equation}
\|\tau_1(t) - \tau_2(t)\|_{L^0_0}^2 \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{E_0}^2 ds \quad \forall t \in [0, T].
\end{equation}

Moreover, from \(3.9\) we deduce that \(a.e. \ t \in (0, T)\),

\[
(\xi_1 - \xi_2, \xi_1 - \xi_2)_L + a(\xi_1 - \xi_2, \xi_1 - \xi_2) \leq (\mu_1 - \mu_2, \xi_1 - \xi_2)_L.
\]

Integrating the previous inequality with respect to time, using the initial conditions \(\xi_1(0) = \xi_2(0) = \xi_0\) and inequality \(a(\xi_1 - \xi_2, \xi_1 - \xi_2) \geq 0\), to find

\[
\frac{1}{2} \|\xi_1(t) - \xi_2(t)\|_{L^0_0}^2 \leq \int_0^t \|\mu_1(s) - \mu_2(s)\|_{L^0_0}^2 ds + \int_0^t \|\xi_1(s) - \xi_2(s)\|_{L^0_0}^2 ds,
\]

which implies that

\[
\|\xi_1(t) - \xi_2(t)\|_{L^0_0}^2 \leq \int_0^t \|\mu_1(s) - \mu_2(s)\|_{L^0_0}^2 ds.
\]

This inequality combined with Gronwall’s inequality leads to

\begin{equation}
\|\xi_1(t) - \xi_2(t)\|_{L^0_0}^2 \leq C \int_0^t \|\mu_1(s) - \mu_2(s)\|_{L^0_0}^2 ds.
\end{equation}

We substitute \(3.17\), \(3.33\)-\(3.36\) in \(3.29\) to obtain

\[
\|\Pi(\eta_1, \lambda_1, \mu_1)(t) - \Pi(\eta_2, \lambda_2, \mu_2)(t)\|_{V \times L^0_0}^2\]

\[
\leq C \int_0^t \|\xi_1(s) - \xi_2(s)\|_{L^0_0}^2 ds.
\]

Reiterating this inequality \(m\) times we obtain

\[
\|\Pi^m(\eta_1, \lambda_1, \mu_1) - \Pi^m(\eta_2, \lambda_2, \mu_2)\|_{C(0,T;V \times L^0_0)}^2 \leq C^m \frac{T^m}{m!} \|\eta_1 - \eta_2, \lambda_1 - \lambda_2, \mu_1 - \mu_2\|_{C(0,T;V \times L^0_0)}^2.
\]

Thus, for \(m\) sufficiently large, \(\Pi^m\) is a contraction on the Banach space \(C(0, T; V \times L^0)\),
and so \(\Pi\) has a unique fixed point. \(\square\)

Let \((\eta^*, \lambda^*, \mu^*) \in C(0, T; V \times L^0)\), be the fixed point of \(\Lambda\), and

\begin{equation}
\begin{aligned}
\xi^* &= \xi^*_{\lambda^*, \mu^*}, \\
\tau^* &= \tau^*_{\lambda^*, \mu^*}, \\
\zeta^* &= \zeta^*_{\lambda^*, \mu^*},
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\sigma^* &= \mathcal{A}\varepsilon(u^*_{\lambda^*, \mu^*}) + B^\ell(\varepsilon(u^*), \tau^*_{\lambda^*, \mu^*}, \zeta^*_{\lambda^*, \mu^*}) + \int_0^t \mathcal{Q}(t - s, \varepsilon(u^*_{\lambda^*, \mu^*}(s)), \tau^*_{\lambda^*, \mu^*}(s), \zeta^*_{\lambda^*, \mu^*}(s)) ds - (\mathcal{E}^\ell)^* E^\ell(\xi^*),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\lambda^*_{\ell}(t) &= \Theta^\ell(\sigma^*, \varepsilon(u^*_{\lambda^*, \mu^*}(t)), \tau^*_{\lambda^*, \mu^*}(t), \zeta^*_{\lambda^*, \mu^*}(t)), \quad \ell = 1, 2, \\
\mu^*_{\ell}(t) &= \Psi^\ell(\sigma^*, \varepsilon(u^*_{\lambda^*, \mu^*}(t)), \zeta^*_{\lambda^*, \mu^*}(t)), \quad \ell = 1, 2.
\end{aligned}
\end{equation}
Existence. We prove \( \{ u_s, \sigma_s, \phi, \xi_s, \zeta_s, D_s \} \) satisfies (2.49)–(2.56) and the regularity (3.1)–(3.7). Indeed, we write (3.13) for \( (\eta, \lambda, \mu) = (\eta^*, \lambda^*, \mu^*) \) and use (3.37) to find
\[
\begin{align*}
\sum_{\ell=1}^{2} (A^\ell \varepsilon(\dot{u}_s^\ell(t)), \varepsilon(v^\ell) - \varepsilon(\dot{u}_s^\ell(t)))_{\mathcal{H}_s^\ell} + \sum_{\ell=1}^{2} (B^\ell \varepsilon(u_s^\ell(t)), \tau_s^\ell, \zeta_s^\ell) + \varepsilon(v^\ell) - \varepsilon(\dot{u}_s^\ell(t)))_{\mathcal{H}_s^\ell}
\end{align*}
(3.43)
\]
\[
+ j_{vc}(u_s(t), v - \dot{u}_s(t)) + j_{fr}(u_s(t), v) - j_{fr}(u_s(t), \dot{u}_s(t))
\]
\[
+ (\eta^*(t), v - \dot{u}_s(t))_{\mathcal{V}} \geq (f(t), v - \dot{u}_s(t))_{\mathcal{V}}, \quad \forall v \in \mathcal{V}.
\]
Substitute (3.40) in (3.43) to obtain
\[
\begin{align*}
\sum_{\ell=1}^{2} (A^\ell \varepsilon(\dot{u}_s^\ell(t)), \varepsilon(v^\ell) - \varepsilon(\dot{u}_s^\ell(t)))_{\mathcal{H}_s^\ell} + \sum_{\ell=1}^{2} (B^\ell \varepsilon(u_s^\ell(t)), \tau_s^\ell, \zeta_s^\ell) + (\varepsilon(v^\ell - \varepsilon(\dot{u}_s^\ell(t))))_{\mathcal{H}_s^\ell}
\end{align*}
(3.44)
\]
\[
\geq (f(t), v - \dot{u}_s(t))_{\mathcal{V}} \quad \forall v \in \mathcal{V} \quad \text{a.e. } t \in [0, T],
\]
and we substitute (3.41) in (3.8) to have
\[
\sum_{\ell=1}^{2} (\dot{\tau}_s^\ell(t), \lambda^\ell)_{L^2(\Omega_T^\ell)} + a_0(\tau_s^\ell(t), \alpha) = \sum_{\ell=1}^{2} (\lambda^\ell(t) + \rho^\ell(t), \alpha^\ell)_{L^2(\Omega_T^\ell)},
\]
for all \( \alpha \in L_0, \text{ a.e. } t \in (0, T). \)
Next, substitute (3.42) in (2.30) to obtain \( \zeta_s(t) \in K, \) and
\[
\sum_{\ell=1}^{2} (\zeta_s^\ell(t), \alpha^\ell - \zeta_s^\ell(t))_{L^2(\Omega_T^\ell)} + a(\zeta_s(t), \alpha - \zeta_s(t)) \geq 0,
\]
(3.46)
\[
\sum_{\ell=1}^{2} (\Psi^\ell(\sigma_s^\ell(t) - A^\ell \varepsilon(\dot{u}_s^\ell(t)), \varepsilon(u_s^\ell(t)), \zeta_s^\ell(t), \alpha^\ell - \zeta_s^\ell(t))_{L^2(\Omega_T^\ell)},
\]
for all \( \alpha \in K, \text{ a.e. } t \in (0, T). \) We write now (3.15) for \( (\eta, \lambda, \mu) = (\eta^*, \lambda^*, \mu^*) \) and use (3.37) to see that
\[
\sum_{\ell=1}^{2} (G^\ell E^\ell(\zeta_s^\ell(t)), \nabla \phi^\ell)_{H^\ell} + \sum_{\ell=1}^{2} (E^\ell \varepsilon(u_s^\ell(t)), \nabla \phi^\ell)_{H^\ell} = -(q(t), \phi)_W,
\]
(3.47)
\[
\text{a.e. } t \in [0, T]. \text{ The relations (3.43)–(3.48), allow us to conclude now that } \{ u_s, \sigma_s, \phi, \xi_s, \zeta_s, D_s \}
\]
\[
\text{satisfies (2.49)–(2.55). Next, (2.56) the regularity (3.1)–(3.3) and (3.6) follow from Lemmas 3.2 and 3.5. Since } u_s, \xi_s \text{ and } \zeta_s \text{ satisfies (3.1), (3.2) and (3.6), respectively. It follows from (3.38) that}
\]
(3.49)
\[
\sigma_s \in C(0, T; \mathcal{H}).
\]
For $\ell = 1, 2$, we choose $\mathbf{v} = \mathbf{u} \pm \phi$ in (3.44), with $\phi = (\phi^1, \phi^2)$, $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$, to obtain

$$\text{Div} \, \sigma^\ell_s(t) = -\mathbf{f}_0^\ell(t) \quad \forall t \in [0, T], \quad \ell = 1, 2,$$

where $D(\Omega^\ell)$ is the space of infinitely differentiable real functions with a compact support in $\Omega^\ell$. The regularity (3.4) follows from (2.39), (3.49) and (3.50). Let now $t_1, t_2 \in [0, T]$, from (2.27), (2.34), (2.36) and (3.39), we conclude that there exists a positive constant $C > 0$ verifying

$$\|D_s(t_1) - D_s(t_2)\|_H \leq C (\|\xi_s(t_1) - \xi_s(t_2)\|_W + \|\mathbf{u}_s(t_1) - \mathbf{u}_s(t_2)\|_V).$$

The regularity of $\mathbf{u}_s$ and $\xi_s$ given by (3.1) and (3.2) implies

$$D_s \in C(0, T; H).$$

For $\ell = 1, 2$, we choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (3.47) and using (2.43) we find

$$\text{div} \, D^\ell_s(t) = q_0^\ell(t) \quad \forall t \in [0, T], \quad \ell = 1, 2.$$

Property (3.7) follows from (2.39), (3.51) and (3.52).

Finally we conclude that the weak solution $\{\mathbf{u}_s, \sigma_s, \xi_s, \tau_s, \varsigma_s, \mathbf{D}_s\}$ of the problem PV has the regularity (3.1)–(3.7), which concludes the existence part of Theorem 3.1.

**Uniqueness.** The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator $\Pi(\ldots, \ldots)$ defined by (3.22)-(3.23) and the unique solvability of the Problems PV$_{(\lambda, \mu)}$ and PV$_{(\lambda, \mu, \eta)}$.

### 4. Conclusion

We presented a model for the quasi-static process of frictional contact between two thermo-electro-viscoelastic bodies with damage. The contact was modeled with the normal compliance condition and the associated Coulomb’s law of dry friction. The new feature in the model was the normal compliance and the friction law depend on the adhesion as presented in the differential equation (2.14). The difficulty of solving this type of problem lies not only in the coupling of viscoelastic, electrical and thermal aspects, but also in the nonlinearity of the boundary conditions modeling this type of physical phenomena (contact and friction conditions), which gives us a quasi-variational inequalities and type of nonlinear, parabolic variational equalities. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities, parabolic inequalities and fixed point theorem

### References


