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THE JACOBSON DENSITY THEOREM FOR NON-COMMUTATIVE ORDERED BANACH ALGEBRAS

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ABSTRACT. The Jacobson density theorem for general non-commutative Banach algebras states as follows: Let π be a continuous, irreducible representation of a non-commutative Banach algebra A on a Banach space X. If $x_1, x_2, \ldots, x_n$ are linearly independent in X and if $y_1, y_2, \ldots, y_n$ are in X, then there exists an $a \in A$ such that $\pi(a)x_i = y_i$ for $i = 1, 2, \ldots, n$. By considering ordered Banach algebras A and ordered Banach spaces X, we shall establish an order-theoretic version of the Jacobson density theorem.

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1. INTRODUCTION

Throughout A or B will be a complex Banach algebra with unity $\mathbf{1}$ and X will be a complex Banach space. The *spectrum* and *spectral radius* of an element in A will be denoted by $\sigma(a)$ and $r(a)$ respectively. All ideals F will be assumed to be two sided. The *Jacobson radical* of A will be denoted by Rad(A) and A is said to be *semisimple* if $\text{Rad}(A) = \{0\}$. The *center* of A will be denoted by $Z(A)$. The term *homomorphism* from A to B will mean an algebra homomorphism from A to B, and the *canonical homomorphism* from A to a quotient algebra A/F will be the homomorphism $a \mapsto a + F$. The set of all non-negative real numbers will be denoted by $\mathbb{R}^*$. A Banach space X ordered by a positive cone P will be denoted by (X, P), and the abbreviation OBS will stand for ordered Banach space.

Every Banach algebra A can be ordered by an *algebra cone*, which is a subset C of A such that $\mathbf{1} \in C$ and C is closed under addition, multiplication and multiplication by scalars in $\mathbb{R}^*$. Then A is called an *ordered Banach algebra* (OBA) and the elements of C are called *positive*. The algebra cone C is said to be *closed* if it is topologically closed in A, *proper* if $C \cap -C = \{0\}$, *inverse closed* if $a \in C$ and a is invertible imply $a^{-1} \in C$, *normal* if there is a fixed scalar $\alpha > 0$ such that $||a|| \leq \alpha ||b||$ whenever $0 \leq a \leq b$, and the spectral radius is *monotone* relative to C if $r(a) \leq r(b)$ whenever $0 \leq a \leq b$. A representation π of an OBA (A, C) on an on OBS (X, P) will be called *positive* if $\pi(a)$ is a positive operator on X whenever $a \in C$.

Spectral theory in OBAs first appeared in [14], where fundamental properties of algebra cones were established, and key spectral theoretic results such as the OBA version of the Perron-Frobenius theorem were obtained (See, [[14], Theorem 5.2]). Since then, OBAs have recieved a considerable amount of attention and various aspects of the theory have been developed. For instance, some domination theory is developed in [5] and [12]. In [7]and [13] asymptotic properties of positive elements in OBAs are studied. In the year 2012, a groundbreaking paper on the irreducibility in OBAs was published by Aleckno (see, [1]). The results of this paper have found wide applicability in the theory of OBAs. For instance, they have played an important role in the development of Fredholm theory in OBAs (see, [6]) and the references given there. However, not much has been done in the area of representation theory of OBAs. This work seeks to begin to address that gap, by establishing in the OBA setting one of the main results in the representation theory of non-commutative Banach algebras (see, [[3], Theorem 4.2.5]), namely the Jacobson density theorem.

The original Jacobson density theorem first appeared in [9] and is a result on the structure of simple rings. This result has very important consequences in ring theory and is related to other important results in algebra, such as the Von Neumann bicommutant theorem and the Keplansky density theorem (see, [2],[8],[10],[15]). The Banach algebra version of the Jacobson density theorem is established in ([3], Theorem 4.2.5).

2. POSITIVE HOMOMORPHISMS

We introduce positive homomorphisms between Banach algebras, which will play a central role in the work. Let (A, C_A) and (B, C_B) be OBAs. A map $\phi : A \to B$ is called a *positive homomorphism* if ϕ is an algebra homomorphism and $\phi(c) \in C_B$ for all $c \in C_A$. Positive homomorphisms on ordered Banach algebras are implicitely discussed in [11].

Example 2.1. *Consider the Banach algebra $A = \ell^\infty$, under componentwise addition, scalar multiplication and multiplication, and with norm $||(x_n)|| = \sup_{n \in \mathbb{N}} |x_n|$. The set $C = \{(x_n) \in A : x_n \in \mathbb{R}^*\}$ is a closed, normal algebra cone in A. The following bounded linear operators are positive homomorphisms on A; every projection P on A, the left shift operator $(x_1, x_2, ...,) \mapsto (x_2, x_3, ...)$, and the right shift operator $(x_1, x_2, ...,) \mapsto (0, x_1, x_2, ...)$.*

Example 2.2. *Let $A = C(\Omega)$ be the Banach algebra of all continuous complex-valued functions on a compact topological space Ω, under the norm $||f|| = \sup_{t\in\Omega} |f(t)|$. The set $C = \{f \in A : f(t) \geq 0$ for all $t \in \Omega\}$ is a closed, normal algebra cone in A. Clearly, C is non-empty as it contains all constant functions whose ranges are singleton sets containing non-negative real numbers. For fixed $g \in C$, the bounded linear operator $f \mapsto fg$ is a positive homomorphism on A.*

Example 2.3. *Let (A, C) be an OBA and B a closed subalgebra of A containing $\mathbf{1}$. Clearly, $C_B = B \cap C$ is an algebra cone in B. The inclusion map $i : B \to A$ is a positive homomorphism.*

Example 2.4. *Let (A, C) be an OBA and F a closed ideal in A. The canonical map π from the OBA (A, C) to the OBA $(A/F, \pi C)$ is clearly a positive homomorphism.*

If (A, C) is an OBA, we will denote by K the set of all positive bounded linear operators on A. While the set of all positive homomorphims on A is not in general a positive cone as it may not be closed under addition and positive scalar multiplication, K is an algebra cone in the algebra $B(A)$ of all bounded linear operators on A. Moreover if C is closed or inverse closed, then K is also closed or inverse closed. The same applies for normality, properness and monotonicity of the spectral radius. To verify this, we will need the following lemma.

Lemma 2.1. *If (A, C) is an OBA and $T \in B(A)$, then $||T|| = \sup_{c\in C} ||Tc||$ and $r(T) = \sup_{c\in C} r(Tc)$.*

Proof. Obviously, $\sup_{c\in C} ||Tc|| \leq ||T||$. To verify the inequality $||T|| \leq \sup_{c\in C} ||Tc||$, it suffices to show that for any $a \in A$, there is a $c \in C$ such that $||Ta|| \leq ||Tc||$. Let $c = ||Ta||\mathbf{1} \in C$. Since $T\mathbf{1}$ is an idempotent, $||T\mathbf{1}|| \geq 1$ and so $||Ta|| \leq ||Ta||||T\mathbf{1}|| = ||T(||Ta||\mathbf{1})|| = ||Tc||$. To prove the second part, we have that

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{c\in C} ||(Tc)^n||^{\frac{1}{n}} = \sup_{c\in C} \lim_{n\to\infty} ||(Tc)^n||^{\frac{1}{n}} = \sup_{c\in C} r(Tc).$$

∎

Proposition 2.2. *Let (A, C) be an OBA. If C is normal, proper or the spectral radius in (A, C) is monotone, then K has the same properties.*

Proof. Suppose that $S, T \in K$ with $0 \leq S \leq T$. Then $0 \leq Sc \leq Tc$ for all $c \in C$ and by normality of C, there is a scalar $\alpha > 0$ such that $||Sc|| \leq \alpha||Tc||$ for all $c \in C$. From Lemma 2.1, it follows that K is normal in $B(A)$. To show that the spectral radius in $(B(A), K)$ is monotone, from $0 \leq Sc \leq Tc$ for all $c \in C$ and monotonicity of the spectral radius in (A, C) we have that $r(Sc) \leq r(Tc)$ for all $c \in C$. It follows from Lemma 2.1 that $r(S) \leq r(T)$. Finally, to show that K is proper, let $T \in K \cap -K$. Then for any $c \in C$ we have that $T(c) \in C \cap -C$ but if C is a proper cone, $T(c) = 0$. From Lemma 2.1, for any $a \in A$, there is a $c' \in C$ such that $||Ta|| \leq ||Tc'|| = 0$. Hence $T = 0$. ∎

It would be interesting to consider positive homomorphisms from an OBA (A, C_A) to an OBA (B, C_B) such that properties of C_A carry over to C_B. Let $\phi : A \to B$ be a homomorphism. If ϕ is for instance bijective and $\phi(C_A) = \phi(C_B)$ it can easily be shown that closedness or properness of C_A will imply that C_B also has the same properties. If ϕ is a positive isometry and if C_A is closed, normal, proper or the spectral radius is monotone w.r.t. C, then C_B will have the same properties. If ϕ is positive and spectrum-preserving, then monotonicity of the spectral radius in (A, C_A) implies monotonicity of the spectral radius in (B, C_B). If in addition A and B are semisimple and ϕ is onto, then ([4], Proposition 2.1) implies C_B will be closed, normal, proper or inverse closed if C_A has these properties.

3. JACOBSON DENSITY THEOREM FOR NON-COMMUTATIVE ORDERED BANACH ALGEBRAS

Here we will present the Jacobson density theorem for OBAs, which is Theorem 3.4. Its proof will rely on Lemma 3.3, which in turn relies on Lemma 3.2, and we require Lemma 3.1 to prove Lemma 3.2. To obtain the results in this section, we shall co-opt some spectral theoretic methods and follow along the lines of the development in [3].

Lemma 3.1. *Let (A, C) be an OBA with C closed and normal, and π a continuous, irreducible representation on an OBS (X, P). Let K be the set of all positive operators on X and $E(X) = \{T \in B(X) : T\pi(a) = \pi(a)T \text{ for all } a \in A\}$ a subset of $B(X)$. Then $E(X)$ is a closed subalgebra of $B(X)$ containing I and is isomorphic to $\mathbb{C}$, $K' = E(X) \cap K$ is a positive cone in $E(X)$ and $K' \subset \{\lambda I : \lambda \in \mathbb{R}^*\}$.*

Proof. That $E(X)$ is a closed subalgebra of $B(X)$ containing I and is isomorphic to $\mathbb{C}$ is obtained from ([3], Theorem 4.2.2). Therefore $E(X)$ is one-dimensional. We define the map $\phi : E(X) \to \mathbb{C}$ by $\phi(T) = \lambda$, where λ is the scalar such that $T = \lambda I$. Obviously ϕ is an injective homomorphism. In addition, ϕ is positive. To verify this suppose that $\phi(T) = \lambda$, where $T \in K$. Because K is closed and normal, we have that $r(T, E(X)) \in \sigma(T, E(X))$ by ([14], Theorem 5.2). This means that $r(T, E(X)) - T$ is not invertible in $E(X)$. Since $E(X)$ is isomorphic to $\mathbb{C}$, it follows that $T = r(T, E(X))I$, so that $\lambda = r(T, E(X))$. Hence $K' \subset \{\lambda I : \lambda \in \mathbb{R}^*\}$. ∎

To discuss the next result, we shall introduce the notion of real linear independence. If V is a real or complex vector space, we say that $v_1, v_2, ..., v_n \in V$ are *real linearly independent* if the only real solution to the equation $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ (where $\alpha_1, \alpha_2, ..., \alpha_n$ are scalars) is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. From real linear independence we define real linear dependence in the usual way. If V is real, linear independence (dependence) and real linear independence (dependence) are equivalent, whereas it may not be so if V is complex. The motivation for real linear independance is that it is associated with the scalar multiplication property of algebra cones, which will play a significant role in the proof of some results leading to the main theorem.

Lemma 3.2. *Let (A, C) be an OBA with C closed and normal and π be a continuous, irreducible, positive representation on an OBS (X, P). If $x_1, x_2 \in P$ are real linearly independent, then there exists an $a \in Z(A) \cap C$ such that $\pi(a)x_1 = 0$ and $\pi(a)x_2 \neq 0$.*

Proof. Suppose that $\pi(a)x_1 = 0$ if and only if $\pi(a)x_2 = 0$ for all $a \in Z(A) \cap C$. This implies that $\pi(b)x_1 = 0$ if and only if $\pi(b)x_2 = 0$ for all $b \in B = Z(A) \cap$ Span(C). Note that B is a closed subalgebra of A containing $\mathbf{1}$. Now consider the subsets $F_i = \{b \in B : \pi(b)x_i = 0 \text{ for } i = 1, 2\}$ of B. These are two sided closed ideals of B, and because $\pi(b)x_1 = 0$ if and only if $\pi(b)x_2 = 0$ for all $b \in B$, we have that $F_1 = F_2 = F$. We define linear maps $T_i : B/F \to X$ by $T_i(b+F) = \pi(b)x_i$ ($i = 1, 2$). By continuity of π, the maps T_i are bounded, and by positivity of π, they are positive. In addition, it is easy to show by direct calculation that they are injective.

Now we define a linear map $S = T_2 T_1^{-1}$ from $\mathrm{Im}(T_1)$ onto the corresponding subspace Y of X. Then obviously S is continuous, positive and bijective. Let $y \in Y$ such that $y = \pi(b)x_1$ for some $b \in B$. Then for any $c \in B$ we have that

$$\pi(c)Sy = \pi(c)T_2T_1^{-1}y = \pi(c)T_2(b + F) = \pi(c)\pi(b)x_2 = \pi(cb)x_2.$$

On the other hand,

$$S\pi(c)y = T_2T_1^{-1}\pi(c)y = T_2T_1^{-1}\pi(c)\pi(b)x_1 = T_2T_1^{-1}\pi(cb)x_1 = T_2(cb + F) = \pi(cb)x_2.$$

Hence $\pi(c)S = S\pi(c)$ and by Lemma 3.1, there is a positive real number λ such that $S = \lambda I$. Hence $T_2 = \lambda T_2$ and so $T_2(1 + F) = \lambda T_1(1 + F)$. This implies that $x_2 = \lambda x_1$, which is a contradiction. ∎

Lemma 3.3. *Let π be a continuous, irreducible and positive representation of an OBA (A, C) on an OBS (X, P), where C is closed and normal. If $x_1, x_2, ..., x_n \in P$ are real linearly independent, then there exists an $a \in Z(A) \cap C$ such that $\pi(a)x_i = 0$ for $1 \leq i \leq n - 1$ and $\pi(a)x_n \neq 0$.*

Proof. We prove by mathematical induction. For $n = 2$, the statement is true by Lemma 3.2. We now assume $n > 2$ and that the statement is true for $n - 1$ real linearly independent vectors in X. Then there is an $a_1 \in Z(A) \cap C$ such that $\pi(a_1)x_1 \neq 0$ and $\pi(a_1)x_i = 0$ for $2 \leq i \leq n - 1$. If $\pi(a_1)x_n = 0$, then we are done. Suppose that $\pi(a_1)x_n \neq 0$. We claim that $\pi(a_1)x_1$ and $\pi(a_1)x_n$ are real linearly independent. Suppose to the contrary that they are real linearly dependent. Let λ be the real number such that $\pi(a_1)x_n = \lambda\pi(a_1)x_1$. Then $x_n - \lambda x_1 \in Y = \text{Ker}(\pi(a_1))$. Because $\pi(a_1) \neq 0$, we have that $Y = \text{Ker}(\pi(a_1)) \neq X$. We claim that $Y = \{0\}$. Since π is irreducible, it suffices to verify that Y is invariant under $\pi(a)$ for all $a \in A$. Let $y \in Y$and $a \in A$, and suppose that $\pi(a)y = z$. Since $a_1 \in Z(A)$, we have that $0 = \pi(a_1)\pi(a)y = \pi(a)\pi(a_1)y = \pi(z)$, so that $z \in Y$. Thus Y is invariant under $\pi(a)$ for all $a \in A$. Consequently, $x_n = \lambda x_1$, which contradicts the assumption that x_n and x_1 are real linearly independent. Hence $\pi(a_1)x_1$ and $\pi(a_1)x_n$ are real linearly independent, and by Lemma 3.2 there is an $a_2 \in Z(A) \cap C$ such that $\pi(a_2)\pi(a_1)x_1 \neq 0$ but $\pi(a_2)\pi(a_1)x_n = 0$. Taking $a = a_2a_1$, we see that $\pi(a)x_1 \neq 0$ and $\pi(a)x_i = 0$ for all $2 \leq i \leq n$. ∎

We now present the main theorem of the paper.

Theorem 3.4. *Let π be a continuous, irreducible and positive representation of an OBA (A, C) on an OBS (X, P), where C is closed and normal. Suppose that $x_1, x_2, ..., x_n \in P$ are real linearly independent. If $y_1, y_2, ..., y_n \in P$ are any elements, there exists an $a \in Z(A) \cap C$ such that $\pi(a)x_i = y_i$ for $i = 1, 2, ..., n$.*

Proof. By Lemma 3.3 there exist $b_j \in Z(A) \cap C$ such that $\pi(b_j)x_i = 0$ if $i \neq j$ and $\pi(b_j)x_j \neq 0$. We show that there exist $c_j \in Z(A) \cap C$ such that $\pi(c_j)\pi(b_j)x_j = y_j$. Suppose that $y_j = 0$. By Lemma 3.3, there exists $c_j \in Z(A) \cap C$ such that $\pi(c_j)x_j = 0$, so that $\pi(c_j)\pi(b_j)x_j = 0$. Now suppose that $y_j \neq 0$. If $\pi(c)\pi(b_j)x_j \neq y_j$ for all $c \in Z(A) \cap C$, then $\pi(c)x_j \neq 0$ for all $c \in Z(A) \cap C$, which contradicts Lemma 3.3. Hence there is a $c_j \in Z(A) \cap C$ such that $\pi(c_j)\pi(b_j)x_j = y_j$. Let $a = \sum_{j=1}^{n} b_jc_j$. Then clearly $a \in Z(A) \cap C$ and $\pi(a)x_j = y_j$ for each $j = 1, 2, ..., n$. ∎

An application of Theorem 3.4 is to the representation of an OBA that is induced by a quotient algebra. This is the next corollary.

Corollary 3.5. *Let (A, C) be an OBA where C is closed and normal, and let F be a maximal ideal in A. Suppose that $a_1 + F, a_2 + F, ..., a_n + F \in \pi C$ are real linearly independent, where π is the canonical homomorphism. If $b_1 + F, b_2 + F, ..., b_n + F \in \pi C$ there exists an $a \in Z(A) \cap C$ such that $\pi(a)(a_i + F) = b_i + F$ for all $i = 1, 2, ..., n$.*

Proof. Since F is maximal, it is closed, so that A/F is a Banach algebra. Now, π induces a continuous, irreducible representation of the OBA (A, C) on the OBS $(A/F, \pi C)$ be means of $\pi(a)(b + F) = ab + F$ (cf. [3], p.80). Obviously, this representation is positive and we apply Theorem 3.4 to deduce the result. ∎

We next apply Theorem 3.4 to obtain the following version of the Sinclair corollary ([3], Corollary 4.2.6).

Corollary 3.6. *Let (A, C) be an OBA with C closed and normal, and let π be a continuous, irreducible, positive representation of (A, C) on an OBS (X, P). Suppose that $x_1, x_2, ..., x_n \in P$ are real linearly independent. If $y_1, y_2, ..., y_n \in P$ are real linearly independent, then there is an $a \in A^{-1} \cap Z(A) \cap C$ such that $\pi(a)x_i = y_i$ for $i = 1, 2, ..., n$.*

Proof. We consider the $2n$-dimensional vector subspace Y spanned by the set

$$\{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$$

and define the map $T : Y \to Y$ by $Tx_i = y_i$ for $i = 1, 2, , , , n$. Clearly, T is bijective. Since it is finite dimensional, the algebra $B(Y)$ is isomorphic to a matrix algebra $M_k(\mathbb{C})$ for $k \leq 2n$. Thus $B(Y)$ is a Banach algebra and by ([3], Theorem 3.3.6), there is an $R \in B(Y)$ such that $T = e^R$. Let B be a basis for Y containing $x_1, x_2, ..., x_n$. By Theorem 3.4 there exists an $a \in Z(A) \cap C$ such that $\pi(a)x = Rx$ for all $x \in B$. Thus $\pi(a)$ and R coincide on Y, and so are $\pi(a)^k$ and R^k for any integer $k \geq 1$. Continuity of π implies that $\pi(e^a)x_i = e^R x_i = Tx_i = y_i$ for $i = 1, 2, ..., n$ and since C is closed, $e^a \in A^{-1} \cap Z(A) \cap C$. ∎

We end by noting that Theorem 3.4 is possibly obtainable in an ordered normed ring, and it may have interesting implications in relation to the known consequences of the classical Jacobson density theorem in ring theory.

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