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APPLICATIONS OF VON NEUMANN ALGEBRAS TO RIGIDITY PROBLEMS OF (2-STEP) RIEMANNIAN (NIL-)MANIFOLDS

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ABSTRACT. In this paper, basic notions of von Neumann algebra and its direct analogues in the realm of groupoids and measure spaces have been considered. By recovering the action of a locally compact Lie group from a crossed product of a von Neumann algebra, other proof of one of a geometric propositions of O'Neil and an extension of it has been proposed. Also, using the advanced exploration of nilmanifolds in measure spaces and their corresponding automorphisms (Lie algebraic derivations) a different proof of an analytic theorem of Gordon and Mao has been attained. These two propositions are of the most important ones for rigidity problems of Riemannian manifolds especially 2-step nilmanifolds.

Key words and phrases: Von Neumann algebras; 2-step nilmanifolds; Free and ergodic actions; Derivations; Automorphisms.

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1. INTRODUCTION

Motivation. Let M is a simply connected 2-step nilpotent Lie group with a left invariant metric and Γ is a cocompact discrete subgroup of isometries of M. In the literature, one of the most important rigidity problems of geodesic flows for compact nilmanifolds is the following problem:

Problem. Whether two compact 2-step nilmanifolds $\frac{M}{\Gamma}$ and $\frac{M'}{\Gamma'}$ are isometric or not, if they have conjugated geodesic flows?

This problem has been studied well through the works of Eberlein, Gordon and Mao, (e.g., [3, 6, 7]). We have already considered these in [4] by an Algebraic-Geometric approach, especially in the category of Lie groupoids. Also, we studied a result of Gordon, Mao and Schueth about compact 2-step nilmanifolds with symplectically conjugate flows, [7]. Then, via Poisson cohomology and other respective notions, we presented a proof of their result which extends not only symplectic concepts to Poisson geometry, but also 2-step nilmanifolds to manifolds with extensible momentum maps, [5].

On the other hand, many objects in Poisson geometry and of course, in groupoids, which we used them in [4, 5], such as dual pairs, bimodules, tensor products, and Morita equivalence have direct analogues in the realm of von Neumann algebras. Also, the theory of von Neumann algebras replaces ordinary measure theory when one has to deal with noncommutative spaces which naturally arise in geometry or noncommutative geometry, specially through the papers of Connes, [2].

These links do not seem to exist with C^*-algebras on any types of analytic algebras. For examples, for a subset $\mathcal{A} \subset B(\mathcal{H})$, we define the commutant $\mathcal{A}'$ to be $\{L \in B(\mathcal{H}) : \ \forall a \in \mathcal{A}, La = aL\}$. Similarly, if $\mathcal{B}$ is a subset of a Poisson algebra P, then its commutant is $\mathcal{B}' = \{f \in P : \ \{f, \mathcal{B}\} = 0\}$. On the analytic side, a dual pair $(\mathcal{A}, \mathcal{A}')$ is a pair of unital *-subalgebras $\mathcal{A}$ and $\mathcal{A}'$ of $B(\mathcal{H})$ that are the mutual commutants of each other. The Double Commutant Theorem of von Neumann implies that all von Neumann algebras satisfy this condition, [2, 8].

Structure. After some preliminaries about von Neumann algebras, by recovering the action of a locally compact Lie group from a crossed product of a von Neumann algebra, we reach to a direct proof of one of the well-known proposition of O'Neil. This is about the properly discontinuous group of isometries, Γ, acting on a simply connected Riemannian manifold M. This gives the characterization of the isometry group of $\frac{M}{\Gamma}$ by normalizers $N(\Gamma)$ and it is usually used to solve problems of rigidity, [4, 5]. The exposed proof leads to an extension of it to the ergodic actions of the countable discrete infinite groups on a σ-finite measure space. More details can be found in Theorem 3.1.

Lastly, advanced exploration of nilmanifolds in measure spaces via special measurable functionals and suitable actions of Lie groups on simply connected manifolds leads to study those works using concepts of von Neumann algebras. This gives us the other proof of the analytic proposition of Gordon and Mao which exposed in Theorem 3.2.

It is to be noted that proofs provided, although long and completely different from the standard proof used in existing resources have a general approach to its structure. The general approach introduced is such that the given proposition is a special case of it. For this reason, while providing a link between some geometric and analytic concepts (apparently unrelated), it can include many results in each notion.

2. MAIN CONCEPTS

2.1. **Preliminaries about von Neumann Algebras.** In this section we review some notions about von Neumann algebras. We assume that the reader know the main concepts of C^*-algebras and von Neumann algebras which can found in [8, 10].

Let $B(\mathcal{H})$, as usual, is the set of bounded operators on a Hilbert space $\mathcal{H}$. A von Neumann algebra is an involutive subalgebra $\mathcal{A}$ of the algebra of $B(\mathcal{H})$ that has the property of being the commutant of its commutant. Let G is a group acting by automorphisms such as u on a von Neumann algebra $\mathcal{A}$ and consider the vector space of finite formal sums $\sum_{g\in G} a_g u_g$ with $a_g \in \mathcal{A}$. We use the crossed product $\mathcal{A} \rtimes G$ which can be obtained by multiplying the sums with the rules $u_g u_h = u_{gh}$(and $u_1 = 1$) and $u_g a u_g^{-1} = g(a)$.

In the case of von Neumann algebras, there is a (strong continuous) unitary group representation $g \mapsto u_g$ with $u_g \mathcal{A} u_g^* = \mathcal{A}, \forall g \in G$. In this setting $\alpha_g(x) = u_g x u_g^*$ $(g \in G, x \in \mathcal{A})$, defines an action of G on $\mathcal{A}$. Each α_g is a *-automorphism of $\mathcal{A}$ and that the mapping $g \to \alpha_g$ is a homomorphism of G into $Aut(\mathcal{A})$. All finite linear combinations of all vector states (i.e, positive linear functionals on $\mathcal{A}$ with norm equals 1) on $\mathcal{A}$ are dense in $\mathcal{A}'$. Then, the action α is implemented by the unitary representation u_g. Finally, an inner automorphism of $\mathcal{A}$ is in the form $Adu(x) = uxu^*$ for u a unitary in $\mathcal{A}$ and an outer automorphism, if the only g in G for which α_g is inner is the identity. Also, an action G on $\mathcal{A}$ is said to be ergodic if the stabilizer $\mathcal{A}^G = \mathbf{C}id$.

We assume, $\alpha : G \to Aut(\mathcal{A})$ is an (continuous) action of the locally compact group G with (left) Haar measure dg on the von Neumann algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$. Form the Hilbert space $\mathcal{K} = L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$ and let G act on $\mathcal{K}$ by $u_g = \lambda_g \otimes 1$, λ being the left regular representation of G in the Hilbert space $L^2(G)$, i.e, $(\lambda_g \xi)(h) = \xi(g^{-1}h), \forall g, h \in G, \xi \in L^2(G)$. The action α of G on $\mathcal{A}$ is encoded by the action $\mathcal{A}$ on $\mathcal{K}$:

$$(\tilde{x}f)(g) = \alpha_{g^{-1}}(f(g)), \quad g \in G, f \in \mathcal{A}, \tag{2.1}$$

which satisfies the equivariance condition

$$\tilde{x} \circ \alpha_g(f(g)) = \lambda_g(\tilde{x}f)(g)\lambda_g^{-1}, \quad \forall g \in G, f \in \mathcal{A}, \tag{2.2}$$

particularly, $u_g \tilde{x} u_g^* = \widetilde{\alpha_g(x)}$. In this way, the crossed product $\mathcal{A} \rtimes_\alpha G$ is the von Neumann algebra on $\mathcal{K} = L^2(G) \otimes \mathcal{H}$ generated by $\{u_g : g \in G\}$ and $\{\tilde{x} : x \in \mathcal{A}\}$.

Equality (2.1) says that finite linear combinations $\sum_g \tilde{x}_g u_g$ form a dense *-subalgebra of $\mathcal{A} \rtimes_\alpha G$. Moreover the u_g's are linearly independent over $\mathcal{A}$ in the sense that $\sum_g \tilde{x}_g u_g = 0$ result to $\tilde{x}_g = 0$ for each g in the sum.

When the group G is discrete, any element of the crossed product can be uniquely written as above formal sum, where the $f = f(g)$'s are uniquely determined as matrix elements in the natural basis of $\ell^2(G)$, i.e, matrix of operators on $\mathcal{K} = \ell^2(G) \otimes \mathcal{H}$. Also, since sum converges pointwise at least on the dense set of functions of finite support from G to $\mathcal{H}$, any matrix of this form which defines a bounded operator on $\mathcal{K}$ is in $\mathcal{A} \rtimes_\alpha G$. This is no longer the case when the group G is not discrete. For more details refer to [8, 12].

2.2. **Preliminaries about nilmanifolds.** As the notions of [3, 6, 7], a Riemannian nilmanifold is a quotient $\frac{M}{\Gamma}$ of a simply connected nilpotent Lie group M by a discrete subgroup Γ; together with a Riemannian metric g whose lift to $M$is left-invariant. A nilmanifold $\frac{M}{\Gamma}$ has step size k if M is k-step nilpotent.

Especially, a Lie group M is said to be two-step nilpotent if its Lie algebra $\mathcal{M}$ satisfies $[\mathcal{M}, [\mathcal{M}, \mathcal{M}]] = 0$ equivalently $[\mathcal{M}, \mathcal{M}]$ is central in $\mathcal{M}$. Let g be a left-invariant Riemannian metric on 2-step nilpotent Lie group M. Then g defines an inner product $\langle ., . \rangle$ on the Lie algebra $\mathcal{M}$ of M. Let $\mathcal{Z} = [\mathcal{M}, \mathcal{M}]$ and let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{M}$ relative to $\langle ., . \rangle$. Note that while $\mathcal{Z}$ is contained in the center of $\mathcal{M}$, it does not necessarily coincide with the full center. For $z \in \mathcal{Z}$, a skew symmetric linear transformation $J(z) : \mathcal{V} \to \mathcal{V}$ can be defined by $J(z)x = (ad(x))^* z$ for $x \in \mathcal{V}$, where $ad(x))^*$ denotes the adjoint of $ad(x)$. Equivalently,

$$\langle J(z)x, y \rangle = \langle [x, y], z \rangle, for\ x, y \in \mathcal{V}, z \in \mathcal{Z}, \tag{2.3}$$

and this process is reversible.

An authomorphism Φ of M is said to be Γ-almost inner if $\Phi(\gamma)$ is conjugate to γ for all $\gamma \in \Gamma$. The automorphism is said to be almost inner if $\Phi(x)$ is conjugate to x for all $x \in M$. A derivation φ of the Lie algebra $\mathcal{M}$ is said to be Γ-almost inner, respectively almost inner, if $\varphi(X) \in [\mathcal{M}, X]$ for all $X \in log\Gamma$, respectively, for all $X \in \mathcal{M}$.

3. APPLICATIONS

Theorem 3.1. *(*[9]*) Let Γ be a properly discontinuous group of isometries of a simply connected Riemannian manifold M. Then group $I(\frac{M}{\Gamma})$ of isometries of $\frac{M}{\Gamma}$ is isomorphic to $\frac{N(\Gamma)}{\Gamma}$, where $N(\Gamma)$ is the normalizer of Γ in $I(M)$.*

Proof. The First Approach: Let $\mathcal{A}$ is a factor whose center is $\mathbf{C}1$. For $u = \sum_g a_g u_g$ in the normalizer $N(\mathcal{A}) = \{u \text{ unitary in } \mathcal{A} \rtimes G |\ u\mathcal{A}u^* = \mathcal{A}\}$, there is a $\beta \in Aut(\mathcal{A})$ so that $ux = \beta(x)u$, $\forall x \in \mathcal{A}$. Then by Lemma 11.2.6 of [8], there can be only one g for which a_g is different from 0 and for g, a_g is unitary. Therefore, the quotient $N(\mathcal{A})/U(\mathcal{A})$ is in fact G itself where $U(\mathcal{A})$ is the unitary group as a normal subgroup. So we recover G and its action (up to inner automorphisms) on $\mathcal{A}$.

As Radon-Nikodym Theorem, [1], the only remaining case is $\mathcal{A} = L^\infty(X, \mu)$, where (X, μ) is a localizable measure space. Similar structure concludes that on the support of the transformation $a_g \in \mathcal{A}$, we have $a_g \alpha_g(x) = \beta(x)$, for all L^∞-functions x and $\beta \in Aut(\mathcal{A})$. Then, by Proposition 11.2.10 of [8], there is a partition of X into measurable subsets, one each of which the transformation of X agrees with some element of G and such a transformation is implemented by a unitary in $N(L^\infty(X), \mu)$.

Consider the group von Neumann algebra of Γ. Then, the expressed structure can be applied to $\frac{M}{\Gamma}$ which acting freely on simply connected Riemannian manifold M, (because of properly discontinuous action of Γ on it). Lastly, a group von Neumann algebra of a discrete group Γ, $vN(\Gamma)$, is the special case of the crossed product when $\mathcal{A} = \mathbf{C}$ and the action is trivial.

The Second Approach: An Extension. Consider (G, X, μ) which G denoted a countable discrete infinite group and (X, μ) be a standard σ-finite measure space on which G acts ergodically. Let $[g]$ denote the group of all Borel automorphisms α of X such that $(\alpha(x), x) \in g$ for every $x \in X$. For an ergodic transformation group (G, X, μ), the normalizer $N[g]$ is the group of all non-singular transformations T on $\{X, \mu\}$ such that $TGx = GTx$ for almost every $x \in X$. As in the relation (2.1) of before section, each $\alpha \in N[g]$ will be identified with the automorphism of $\mathcal{A} = L^\infty(X, \mu)$ defined by $(\alpha f)(x) = f(\alpha^{-1}x)$.

In this case, by using the relations (2.1) and (2.2) of before section, the normalizer $N(\mathcal{A})$ of $\mathcal{A}$ is precisely the image $\{\lambda_g : g \in [g]\}$. Then, an automorphism $\alpha \in Aut(\mathcal{A})$ can be extended to an element of $Aut(\mathcal{A} \rtimes_\alpha G)$ if and only if $\alpha \in N[g]$. Specially for a properly discontinuous group of isometries of a simply connected Riemannian manifold M we have the isomorphism $\frac{N(\Gamma)}{\Gamma} \cong I(\frac{M}{\Gamma})$ of isometries of $\frac{M}{\Gamma}$, where $N(\Gamma)$ is the normalizer of Γ in $I(M)$.

Finally, the definition of von Neumann algebras based on a commutant assumption lead us to the probably another proof of some results about the commutative assumptions. In this way, we give the other proof of a proposition of Gordon and Mao which has been used in some rigidity problems for 2-step nilmanifolds and we used it in [4, 5].

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Theorem 3.2. *([6]) Let $\mathcal{M}$ be a 2-step nilpotent Lie algebra with an inner product $\langle \cdot, \cdot \rangle$ and φ be an almost inner derivation of continuous type on $\mathcal{M}$ say $\varphi(x) = [\sigma(x), x]$ with σ continuous on $\mathcal{M}\backslash\{0\}$. Let $z \in \mathrm{Z}(\mathcal{M})$ and $y \in ker(J(z))$. Then*

$$\langle \varphi(x), z \rangle = \langle [\sigma(y), x], z \rangle, \ \ \forall x \in \mathcal{M},$$

where, $J(z) : \mathcal{V} \to \mathcal{V}$, defined by equation (2.3)*, is a skew symmetric linear transformation defined by $J(z)x = (ad(x))^* z$ for $x \in \mathcal{V}$.*
In special case, if the center of $\mathcal{M}$ properly contains the derived algebra, then every almost inner derivation of continuous type on $\mathcal{M}$ is inner.

Proof. It is to be noted that Γ-almost inner derivations of $\mathcal{M}$, $AID(\mathcal{M})$, endowing with the topology of pointwise norm convergence is a von Neumann algebra. Also, since $\mathcal{M}$ is 2-step nilpotent, $AID(\mathcal{M})$ will be 1-step nilpotent, i.e, its commutant (as a derivation, too) is abelian. If consider $AID(\mathcal{M}) \subseteq Der(\mathcal{M})$ as a von Neumann algebra, which its commutant is abelian, then as an result of [11] (Theorem 2.5.3), any derivation implemented by an (fixed) element. This automatically result to the innerness of derivations.

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REFERENCES

[1] C. D. ALIPRANTIS, *Principles of Real Analysis*, Academic Press, 1990.

[2] A. CONNES, *Noncommutative Geometry*, Academic Press, 2000.

[3] P. EBERLEIN, Geometry of two-step nilpotent groups with a left invariant metric, *Ann. Sci. École Norm. Sup.*, **27** (1994) pp. 611–660.

[4] H. R. FANAI and A. HASAN-ZADEH, An application of Lie groupoids to a rigidity problem of 2-step nilmanifolds, *Mathematica Bohemica*, **144** (2019) pp. 1–12.

[5] H. R. FANAI and A. HASAN-ZADEH, A symplectic rigidity problem for 2-step nilmanifolds, *Houston J. Math.*, **2** (2017) pp. 363–374.

[6] C. S. GORDON and Y. MAO, Geodesic conjugacy of two-step nilmanifolds, *Mich. Math. J.*, **45** (1998) pp. 451–481.

[7] C. S. GORDON, Y. MAO and D. SCHUETH, Symplectic rigidity of geodesic flows on two-step nilmanifolds, *Ann. Sci. École Norm. Sup.*, **30** (1997) pp. 417–427.

[8] V. F. R. JONES, *Von Neumann Algebras*, UC Berkely Mathematics, 2009.

[9] B. O'NEILL, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[10] W. RUDIN, *Functional Analysis*, McGraw-Hill, Inc Publisher, 1982.

[11] S. SAKAI, *Operator Algebras in Dynamical Systems*, Cambridge Univ. Press, 1991.

[12] J. A. WOLF, Infinite Dimensional Multiplicity Free Spaces II: Limits of Commutative Nilmanifolds, New developments in Lie theory and geometry, *Contemp. Math.*, **491** (2009) pp. 179–208.