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SEVERAL APPLICATIONS OF A LOCAL NON-CONVEX YOUNG-TYPE INEQUALITY

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ABSTRACT. A local version of the Young inequality for positive numbers is used in order to deduce some inequalities about determinants and norms for real quadratic matrices and norms of positive operators on complex Hilbert spaces.

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1. INTRODUCTION

The classical inequality of W. H. Young is

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

where a and b are distinct positive real numbers and $0 < \nu < 1$, see [25], being also an inequality between arithmetic and geometric mean.

There are many generalizations and refinements of Young's inequality, see for example [1], [2], [14], [13], [17], [4], [3], [5], [8] and references therein. Among those there is the following local version of Young inequality, presented by the authors in [16], which will be used to enunciate several applications for determinant, norm and trace inequalities.

Proposition 1.1. *(*[16]*) (a) For any* $\alpha + \beta > 1,$ *and* $\alpha \in (0,1)$*, there is* $r > 0$ *such that for any* $x, y \in (1-r, 1+r)$*, it is true the inequality*

$$\alpha x + \beta y > x^{\alpha}y^{\beta} + \alpha + \beta - 1.$$

(b) For any $\alpha + \beta < 1$, $\alpha \in (0,1)$ *and* $\beta < 0$ *there is* $q > 0$ *such that for any* $x, y \in (1-q, 1+q)$*, it is true the same inequality.*

Corollary 1.2. *(*[16]*) For any* $\alpha + \beta > 1$ *and* $\alpha \in (0,1)$*, there is* $r > 0$ *such that for any* $x, y \in (1-r, 1+r)$ *it is true that*

$$\alpha x + \beta y > x^{\alpha}y^{\beta}.$$

It is necessary to recall some basic and well-known things about the functional calculus with continuous functions on spectrum. As in [9], we recall that for selfadjoint operators $A, B \in B(\mathcal{H})$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in \mathcal{H}$, or $B - A$ is a positive operator. Firstly, we will consider A as a selfadjoint linear operator on a complex Hilbert space $(\mathcal{H}; < .,. >)$. A fundamental role will be played by *Gelfand map* $\Phi : C(Sp(A)) \to C^*(A)$, which is a $*$- isometrically isomorphism between the set of all *continuous functions* defined on the *spectrum* of A and the C^*- algebra generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$. Φ has the following properties: for any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

The *continuous functional calculus* for a selfadjoint operator A is based on the notation

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)),$$

and on the following remark:

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on $\mathcal{H}$. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{1.1}$$

in the operator order of $B(\mathcal{H})$.

We also will need some basic, well-known properties for trace of operators. The main properties of the trace can be found in [5] and the references therein, but we mention just what we need. For any orthonormal basis $\{e_i\}_{i\in I}$ of a separable Hilbert space $\mathcal{H}$, the operator $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$||A||_1 = \sum_{i\in I} < |A|e_i, e_i >< \infty.$$

The definition of $||A||_1$ is independent of the choice of the orthogonal basis $\{e_i\}_{i\in I}$. We denote the set of trace class operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}_1(\mathcal{H})$.

We will need some of the well-known properties of trace:

(a) $||A||_1 = ||A^*||_1$, for any $A \in \mathcal{B}_1(\mathcal{H})$
(b) $(\mathcal{B}_1(\mathcal{H}), ||.||_1)$ is a Banach space.
(c) $\mathcal{B}(\mathcal{H})\mathcal{B}_1(\mathcal{H})\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H})$ that is $\mathcal{B}_1(\mathcal{H})$ is a bilateral operator ideal in $\mathcal{B}(\mathcal{H})$.

We consider, following [5], the *trace* of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ to be

$$tr(A) = \sum_{i\in I} < Ae_i, e_i >,$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of $\mathcal{H}$. The previous series absolutely converges and it is basis independent. The definition is a an extension of the usual definition of the trace if $\mathcal{H}$ is finite dimensional.

$\mathcal{B}_1(\mathcal{H})$ is closed for to the $*$-operation, and $tr(A^*) = \overline{tr(A)}$. If $A \in \mathcal{B}_1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ then $AT,\ TA \in \mathcal{B}_1(\mathcal{H})$ and $tr(AT) = tr(TA)$ and $|tr(AT)| \leq ||A||_1||T||$. The application $tr(.)$ is a bounded linear functional on $\mathcal{B}_1(\mathcal{H})$ with $||tr|| = 1$.

Many trace inequalities for matrices and operators can be found for example in [20], [22], [23], [5],[7], [17], [24], [18], [19] and also, references therein.

2. A LOCAL YOUNG INEQUALITY FOR DETERMINANTS

We recall that if A and B are positive invertible operators on a complex Hilbert space $(\mathcal{H}, <.,.>)$ then the following notation $A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}$ is used for the *weighted geometric mean*, where $\nu \in (0,1)$. It is necessary to mention that we use the symbol $\sharp_\nu$, as in [10] not as in [1] and [13].

Moreover, this notation will be used even when ν is not in the interval $(0,1)$.

Theorem 2.1. *If:*

(1) $A, B \in M_n$ *are positive definite matrices having the eigenvalues* $\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_n(A)$ *and* $\lambda_1(B) \leq \lambda_2(B) \leq ... \leq \lambda_n(B)$,
(2) $\alpha \in (0,1),\ \beta \in \mathbf{R}$ *with the property* $\alpha + \beta > 1$,
(3) *there is* $r \in (0,1)$ *such that* $1 - r < \frac{\lambda_1(B)}{\lambda_n(A)}$ *and* $\frac{\lambda_n(B)}{\lambda_1(A)} < 1 + r$,

then we have

$$det(\alpha A + \beta B) > det(A\sharp_\beta B),$$

and it is also true that:

$$det(\alpha A + \beta B) > det(B\sharp_\alpha A).$$

Proof. We take $x = 1$, in the inequality of Corollary 1.2: $\alpha x + \beta y > x^{\alpha} y^{\beta}$, and we get

$$\alpha + \beta y > y^{\beta}, \text{ for all } y \in (1 - r, 1 + r). \tag{2.1}$$

Now we consider the positive definite matrix $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. By a theorem of Ostrowski ([11], Theorem 4.5.9), we know that the eigenvalues of C check

$$1 - r < \frac{\lambda_1(B)}{\lambda_n(A)} \leq \lambda_i(C) \leq \frac{\lambda_n(B)}{\lambda_1(A)} < 1 + r,$$

fo all $i = 1, 2, ..., n$.

By (2.1) we obtain,

$$\alpha + \beta \lambda_i(C) > \lambda_i^{\beta}(C),$$

for all $i = 1, 2, ..., n$.

Using now that the determinant of a matrix is the product of its eigenvalues, we have,

$$det(\alpha I + \beta A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) = \prod_{i=1}^{n} (\alpha + \beta \lambda_i(C)) \geq \prod_{i=1}^{n} \lambda_i^{\beta}(C) = det(C^{\beta}).$$

Multiplying by $det(A^{\frac{1}{2}})$ we get,

$$det(A^{\frac{1}{2}}) det(\alpha I + \beta A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) det(A^{\frac{1}{2}}) \geq det(A^{\frac{1}{2}}) det(C^{\beta}) det(A^{\frac{1}{2}}),$$

or using the multiplicity of the determinant, we have:

$$det(\alpha A + \beta B) \geq det\left(A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\beta} A^{\frac{1}{2}}\right).$$

So we obtain:

$$det(\alpha A + \beta B) \geq det(A \sharp_{\beta} B),$$

or analogously, if we use the inequality $\alpha \lambda_i(C) + \beta > \lambda_i^{\alpha}(C)$,(i.e. we take $y = 1$ in inequality $\alpha x + \beta y > x^{\alpha} y^{\beta}$)

we get

$$det(\alpha A + \beta B) > det(B \sharp_{\alpha} A).$$

∎

3. A LOCAL YOUNG INEQUALITY FOR NORMS

Theorem 3.1. *If:*

(1) $A, B, X \in M_n$ *and* A, B *are positive definite matrices having the eigenvalues* $\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_n(A)$ *and* $\lambda_1(B) \leq \lambda_2(B) \leq ... \leq \lambda_n(B)$,

(2) $\alpha \in (0, 1),\ \beta \in \mathbf{R}$ *with the property* $\alpha + \beta > 1$,

(3) *there is* $r \in (0, 1)$ *such that* $1 - r < \lambda_1(A), \lambda_1(B)$ *and* $\lambda_n(A), \lambda_n(B) < 1 + r$

then we have:

$$||\alpha AX + \beta XB||_2 \geq ||A^{\alpha} X B^{\beta}||_2,$$

where $||.||_2$ *is the Hilbert-Schmidt norm.*

Proof. Using the Spectral Theorem for the unitary matrices U and V we have,
$A = Udiag(\lambda_1(A), ..., \lambda_n(A))U^*$ and $B = Vdiag(\lambda_1(B), ..., \lambda_n(B))V^*$.
If we set $Y = U^*XV = (y_{ij})$, we obtain $\alpha AX + \beta XB = U((\alpha\lambda_i(A) + \beta\lambda_j(B))y_{ij})V^*$ and $A^\alpha XB^\beta = U(\lambda_i(A)^\alpha\lambda_j(B)^\beta y_{ij})V^*$.
We have the following inequalities by using Corollary 1.2:

$$||\alpha AX+\beta XB||_2^2 = \sum_{i,j=1}^{n}(\alpha\lambda_i(A)+\beta\lambda_j(B))^2|y_{ij}|^2 \geq \sum_{i,j=1}^{n}(\lambda_i(A)^\alpha\lambda_j(B)^\beta)^2|y_{ij}|^2 = ||A^\alpha XB^\beta||_2^2$$

or

$$||\alpha AX + \beta XB||_2 \geq ||A^\alpha XB^\beta||_2.$$

∎

4. A LOCAL YOUNG INEQUALITY FOR TRACE OF OPERATORS

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, < ., >)$ and $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators. All the properties concerning the trace of operators will be considered on separable Hilbert spaces.

Theorem 4.1. *If:*
(a) $A, B \in \mathcal{B}(\mathcal{H})$ *are positive operators*
(b) $\alpha \in (0, 1)$*,* $\beta \in \mathbf{R}$ *with the property* $\alpha + \beta > 1$,
(c) there is $r \in (0, 1)$ *such that* $(1 - r)I \leq A, B \leq (1 + r)I$,
then the following inequalities hold:

$$\alpha||A^{\frac{1}{2}}x||^2||y||^2 + \beta||B^{\frac{1}{2}}y||^2||x||^2 >< A^\alpha x, x >< B^\beta y, y > +(\alpha + \beta - 1)||x||^2||y||^2, \tag{4.1}$$

for all $x, y \in \mathcal{H}$, *and*

$$\alpha||A^{\frac{1}{2}}x||^2 + \beta||B^{\frac{1}{2}}y||^2 >< A^\alpha x, x >< B^\beta y, y > +(\alpha + \beta - 1), \tag{4.2}$$

when $||x|| = ||y|| = 1$.

Proof. We will use the same method as in [9]. Firstly, we use the Functional Calculus with continuous functions on spectrum for the operator A, for all $b \in (1 - r, 1 + r)$ and $x \in \mathcal{H}$. Secondly, we use the Functional Calculus with continous functions on spectrum for the operator B, obtaining succesively,

$$\alpha < Ax, x > I + \beta B < x, x >>< A^\alpha x, x > B^\beta + (\alpha + \beta - 1) < x, x > I,$$

and then

$$\alpha < Ax, x >< y, y > +\beta < By, y >< x, x >>$$
$$>< A^\alpha x, x >< B^\beta y, y > +(\alpha + \beta - 1) < x, x >< y, y >$$

for all $x, y \in \mathcal{H}$.
Using the norm, we get,

$$\alpha||A^{\frac{1}{2}}x||^2||y||^2 + \beta||B^{\frac{1}{2}}y||^2||x||^2 >< A^\alpha x, x >< B^\beta y, y > +(\alpha + \beta - 1)||x||^2||y||^2$$

for all $x, y \in \mathcal{H}$. By (4.1) we obtain immediately (4.2).

∎

Corollary 4.2. *In the same hypothesis as in Theorem 4.1 the following inequality holds:*

$$\alpha||A^{\frac{1}{2}}x||^2+\beta||B^{\frac{1}{2}}y||^2 >< A^{\alpha}x,x>< B^{\beta}y,y>,$$

when $||x||=||y||=1.$

The following result is obtained as an application of Corollary 1.2 and Theorem 3.1 for the trace of an operator.

Theorem 4.3. *If:*

(1) $A,B\in\mathcal{B}(\mathcal{H})$ *are positive operators and* $P,\ Q\in\mathcal{B}_1(\mathcal{H})$ *with* $P,\ Q>0.$
(2) $\alpha\in(0,1)$*,* $\beta\in\mathbf{R}$ *with the property* $\alpha+\beta>1,$
(3) *there is* $r\in(0,1)$ *such that* $(1-r)I\le A,B\le(1+r)I,$

then the following inequality takes place:

$$\alpha\cdot tr(PA)trQ+\beta\cdot tr(QB)trP>tr(PA^{\alpha})tr(QB^{\beta})+(\alpha+\beta-1)trPtrQ,$$

and in particular,

$$\alpha\cdot tr(PA)trQ+\beta\cdot tr(QB)trP>tr(PA^{\alpha})tr(QB^{\beta}).$$

Proof. If we take $x=P^{\frac{1}{2}}e,\ y=Q^{\frac{1}{2}}f$ where $e,f\in\mathcal{H}$ in the proof of Theorem 4.1, we will find,

$$\alpha< AP^{\frac{1}{2}}e,P^{\frac{1}{2}}e>< Q^{\frac{1}{2}}f,Q^{\frac{1}{2}}f>+\beta< BQ^{\frac{1}{2}}f,Q^{\frac{1}{2}}f>< P^{\frac{1}{2}}e,P^{\frac{1}{2}}e>>$$
$$>< AP^{\frac{1}{2}}e,P^{\frac{1}{2}}e>< BQ^{\frac{1}{2}}f,Q^{\frac{1}{2}}f>+(\alpha+\beta-1)< P^{\frac{1}{2}}e,P^{\frac{1}{2}}e>< Q^{\frac{1}{2}}f,Q^{\frac{1}{2}}f>,$$

for all $e,f\in\mathcal{H}$.

Let $\{e_i\}_{i\in I}$, and $\{f_j\}_{j\in J}$ be two orthonormal bases of $\mathcal{H}$. We take in previous inequality $e=e_i,\ i\in I$ and $f=f_j,\ j\in J$ and then summing over $i\in I$ and $j\in J$, we get the following double inequality:

$$\alpha\sum_{i\in I}< P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i,e_i>\sum_{j\in J}< Qf_j,f_j>+\beta\sum_{j\in J}< Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j,f_j>\sum_{i\in I}< Pe_i,e_i>>$$
$$>\sum_{i\in I}< P^{\frac{1}{2}}A^{\alpha}P^{\frac{1}{2}}e_i,e_i>\sum_{j\in J}< Q^{\frac{1}{2}}B^{\beta}Q^{\frac{1}{2}}f_j,f_j>+$$
$$+(\alpha+\beta-1)\sum_{i\in I}< Pe_i,e_i>\sum_{j\in J}< Qf_j,f_j>.$$

Now, using the well-known properties of trace, we obtain the desired inequality.
∎

Next three results are several applications of Theorem 4.3.

Corollary 4.4. *If:*

(1) $A,B\in\mathcal{B}(\mathcal{H})$ *are positive operators and* $P\in\mathcal{B}_1(\mathcal{H})$ *with* $P>0.$
(2) $\alpha\in(0,1)$*,* $\beta\in\mathbf{R}$ *with the property* $\alpha+\beta>1,$
(3) *there is* $r\in(0,1)$ *such that* $(1-r)I\le A,B\le(1+r)I,$

then the following inequality takes place:

$$trP\left[\alpha\cdot tr(PA)+\beta\cdot tr(PB)\right]>tr(PA^{\alpha})tr(PB^{\beta})+(\alpha+\beta-1)(trP)^2,$$

and in particular,

$$trP\left[\alpha\cdot tr(PA)+\beta\cdot tr(PB)\right]>tr(PA^{\alpha})tr(PB^{\beta}).$$

Proof. We take in Theorem 4.3, $P = Q$.
∎

Corollary 4.5. *If:*

(1) $P,\ Q,\ S,\ V$ *be invertible positive operators on* $\mathcal{H}$ *and* $P,\ Q\ \ S,\ V \in \mathcal{B}_1(\mathcal{H})$.
(2) $\alpha \in (0,1)$, $\beta \in \mathbf{R}$ *with the property* $\alpha + \beta > 1$,
(3) *there is* $r \in (0,1)$ *such that* $(1-r)Q \leq V \leq (1+r)Q$, *and* $(1-r)P \leq S \leq (1+r)P$

then the following inequality takes place:

$$trP[\alpha \cdot trS + \beta\, trV] > tr(P\sharp_\alpha S)tr(Q\sharp_\beta V) + (\alpha+\beta-1)trPtrQ,$$

and in particular,

$$trP[\alpha \cdot trS + \beta\, trV] > tr(P\sharp_\alpha S)tr(Q\sharp_\beta V).$$

Proof. Taking into account our hypothesis, we see that for positive operators $A = P^{-\frac{1}{2}}SP^{-\frac{1}{2}}$ and $B = Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}}$, we have

$$(1-r)I \leq A \leq (1+r)I \quad \text{and} \quad (1-r)I \leq B \leq (1+r)I.$$

From Theorem 4.3, we get

$$\alpha \cdot tr(PA)trQ + \beta \cdot tr(QB)trP > tr(PA^\alpha)tr(QB^\beta) + (\alpha+\beta-1)trPtrQ,$$

and in particular,

$$\alpha \cdot tr(PA)trQ + \beta \cdot tr(QB)trP > tr(PA^\alpha)tr(QB^\beta).$$

Replacing here A and B above, we have,

$$trP(\alpha trS + \beta trV) > tr(P\sharp_\alpha S)tr(Q\sharp_\beta V) + (\alpha+\beta-1)trPtrQ,$$

and in particular,

$$trP(\alpha trS + \beta trV) > tr(P\sharp_\alpha S)tr(Q\sharp_\beta V),$$

i.e. the desired inequalities.
∎

Corollary 4.6. *If we consider in the above corollary only two invertible positive operators* P, $S \in \mathcal{B}_1(\mathcal{H})$ *then we have,*

$$trPtrS(\alpha+\beta) > tr(P\sharp_\alpha S)tr(P\sharp_\beta S) + (\alpha+\beta-1)(trP)^2,$$

and in particular,

$$trPtrS(\alpha+\beta) > tr(P\sharp_\alpha S)tr(P\sharp_\beta S).$$

Proof. If we take in Corollary 4.5, $P = Q$ and $S = V$ we will obtain the desired inequality. ∎

REFERENCES

[1] H. ALZER, C. M. FONSECA and A. KOVACEC, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra* **63**, 3, (2015), pp. 622-635.

[2] T. ANDO, Matrix Young inequalities, *Oper. Theory Adv. Appl.*, **75**, (1995), pp. 33ÂŰ38.

[3] S. S. DRAGOMIR, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.*, **18**, (2015), pp. 12.

[4] S. S. DRAGOMIR, Some asymmeetric reverses of Young's scalar and operator inequalities with applications, *RGMIA Res. Rep. Coll.*, **19**, (2016), Art. 44, pp. 16.

[5] S. S. DRAGOMIR, Trace inequalities for positive operators via recent refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **19**, (2015), Art. 157, pp. 16. [Online: https://rgmia.org/papers/v18/v18a157.pdf].

[6] S. S. DRAGOMIR, A note on new refunements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 131. [Online: https://rgmia.org/papers/v18/v18a131.pdf].

[7] S. S. DRAGOMIR, Trace inequalities of Jensen type seldadjoint operators in Hilbert spaces: A survey of recent results, *RGMIA Res. Rep. Coll.*, **20**, (2017), Art. 20, pp. 52. [Online: https://rgmia.org/papers/v20/v20a20.pdf].

[8] S. S. DRAGOMIR, A note on new refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **18**, (2015), Art. 131.

[9] S. S. DRAGOMIR, Cebysev's type inequalities for functions on selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.*, (2009).

[10] S. FURUICHI, N. MINCULETE, Alternative reverse inequalities for Young's inequality, *J. Math. Inequalities*, 2011;5:595-600.

[11] R. A. HORN, C. R. JOHNSON, *Matrix analysis*, 2nd ed. New York : Cambridge University Press; 2013.

[12] J. I. FUJII, E. KAMEI, Uhlmann's interpolational method for operators means, *Math. Japon,*. **34** (1989), no. 4, pp. 541-547.

[13] F. KITTANEH, Y. MANASRAH, Reverse Young and Heinz inequalities for matrices, *Linear and Multilinear Algebra*, **59**, 9 (2011), pp. 1031–1037.

[14] F. KITTANEH, Y. MANASRAH, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, **361** (2010), 262–269.

[15] L. LIU, A trace class operator inequality, *J. Math. Anal. Appl.*, 328 (2007), pp. 1484-1486.

[16] S. LUGOJAN, L. CIURDARIU, A generalization of Young's theorem and some applications, *Bull. St. al UPT,* **63** (77), Issue 2, (2018).

[17] S. MANJEGANI, Holder and Young inequalities for the trace of operators, *Positivity*, **11**, (2007), pp. 239-250.

[18] H. NEUDECKER, A matrix trace inequality, *J. Math. Anal. Appl.*, **166** (1992), pp. 302-303.

[19] M. B. RUSKAI, Inequalities for trace on von Neumann algebras, *Commun. Math. Phis.*, **26** (1972), pp. 280-289.

[20] K. SHEBRAWI, H. ABADAWI, Trace inequalities for matrices, *Bull. Aust. Math. Soc.*, **87**, (2013), pp. 139-148.

[21] B. SIMON, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.

[22] Z. ULUKOK, R. TURKMEN, On some matrix trace inequalities, *J. Inequal. Appl. (2010),* Art ID 201486, pp. 8.

[23] X. M. YANG, X. Q. YANG and K. L. TEO, A matrix trace inequality, *J. Math. Anal. Appl.*, **263** (2001), pp. 327-331.

[24] X. YANG, A matrix trace inequality, *J. Math. Anal. Appl.*, **250**, (2000), pp. 372-374.

[25] WH. YOUNG, On classes of summable functions and their Fourier series, *Proc. Royal Soc. London*, Ser. A., **87**, (1912), pp. 225-229.