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INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS ON HILBERT SPACES: A SURVEY OF RECENT RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Dedicated to Professor S. S. Dragomir for his 60th birthday.

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[1]MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.,
sever.dragomir@vu.edu.au
URL: https://rgmia.org/dragomir

[2]DST-NRF CENTRE OF EXCELLENCE IN THE MATHEMATICAL, AND STATISTICAL SCIENCES, SCHOOL OF COMPUTER SCIENCE, & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND,, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA.

ABSTRACT. The main aim of this survey is to present recent results concerning inequalities for continuous functions of selfadjoint operators on complex Hilbert spaces. It is intended for use by both researchers in various fields of Linear Operator Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

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Contents

Foreword

Linear Operator Theory in Hilbert spaces plays a central role in contemporary mathematics with numerous applications for Partial Differential Equations, in Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory & Statistics and other fields.

The main aim of this survey is to present recent results concerning inequalities for continuous functions of bounded selfadjoint operators on complex Hilbert spaces.

The survey is intended for use by both researchers in various fields of Linear Operator Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

In the first chapter we recall some fundamental facts concerning bounded selfadjoint operators on complex Hilbert spaces. The generalized Schwarz's inequality for positive selfadjoint operators as well as some results for the spectrum of this class of operators are presented. Then we introduce and explore the fundamental results for polynomials in a linear operator, continuous functions of selfadjoint operators as well as the step functions of selfadjoint operators. By the use of these results we then introduce the spectral decomposition of selfadjoint operators (the *Spectral Representation Theorem*) that will play a central role in the rest of the survey. This result is used as a key tool in obtaining various new inequalities for continuous functions of selfadjoint operators, functions which are of bounded variation, Lipschitzian, monotonic or absolutely continuous. Another tool that will greatly simplify the error bounds provided in the survey is the *Total Variation Schwarz's Inequality* for which a simple proof is offered.

The chapter is concluded with some well known operator inequalities of Jensen's type for convex and operator convex functions. Finally, some Grüss' type inequalities obtained in 1993 by Mond & Pečarić are also presented.

Jensen's type inequalities in their various settings ranging from discrete to continuous case play an important role in different branches of Modern Mathematics. A simple search in the *MathSciNet* database of the American Mathematical Society with the key words "jensen" and "inequality" in the title reveals more than 300 items intimately devoted to this famous result. However, the number of papers where this inequality is applied is a lot larger and far more difficult to find.

In the second chapter we present some recent results obtained by the author that deal with different aspects of this well research inequality than those recently reported in the survey [**20**]. They include but are not restricted to the operator version of the Dragomir-Ionescu inequality, Slater's type inequalities for operators and its inverses, Jensen's inequality for twice differentiable functions whose second derivatives satisfy some upper and lower bounds conditions, Jensen's type inequalities for log-convex functions and for differentiable log-convex functions and their applications to Ky Fan's inequality. Finally, some Hermite-Hadamard's type inequalities for convex functions and Hermite-Hadamard's type inequalities for operator convex functions are presented as well.

The third chapter is devoted to Čebyšev and Grüss' type inequalities.

The *Čebyšev*, or in a different spelling - *Chebyshev, inequality* which compares the integral/discrete mean of the product with the product of the integral/discrete means is famous in the literature devoted to Mathematical Inequalities. It has been extended, generalized, refined etc...by many authors during the last century. A simple search utilizing either spellings and the key word "inequality" in the title in the comprehensive *MathSciNet* database produces more than 200 research articles devoted to this result.

The sister inequality due to Grüss which provides error bounds for the magnitude of the difference between the integral mean of the product and the product of the integral means has also attracted much interest since it has been discovered in 1935 with more than 180 papers published, as a simple search in the same database reveals. Far more publications have been devoted to the applications of these inequalities and an accurate picture of the impacted results in various fields of Modern Mathematics is difficult to provide.

In this chapter, however, we present only some recent results due to the author for the corresponding operator versions of these two famous inequalities. Applications for particular functions of selfadjoint operators such as the power, logarithmic and exponential functions are provided as well.

The next chapter is devoted to the Ostrowski's type inequalities. They provide sharp error estimates in approximating the value of a function by its integral mean and can be utilized to obtain a priory error bounds for different quadrature rules in approximating the Riemann integral by different Riemann sums. They also shows, in general, that the mid-point rule provides the best approximation in the class of all Riemann sums sampled in the interior points of a given partition.

As revealed by a simple search in *MathSciNet* with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 280 papers that can be found by performing the above search. Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In this chapter we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. Applications for mid-point inequalities and some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

From a complementary viewpoint to Ostrowski/mid-point inequalities, trapezoidal type inequality provide a priory error bounds in approximating the Riemann integral by a (generalized) trapezoidal formula.

Just like in the case of Ostrowski's inequality the development of these kind of results have registered a sharp growth in the last decade with more than 50 papers published, as one can easily asses this by performing a search with the key word "trapezoid" and "inequality" in the title of the papers reviewed by *MathSciNet*.

Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In chapter five we present some recent results obtained by the author in extending trapezoidal type inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. Applications for some elementary functions of operators are provided as well.

In approximating n-time differentiable functions around a point, perhaps the classical Taylor's expansion is one of the simplest and most convenient and elegant methods that has been employed in the development of Mathematics for the last three centuries.

In the sixth and last chapter of the survey, we present some error bounds in approximating n-time differentiable functions of selfadjoint operators by the use of operator Taylor's type expansions around a point or two points from its spectrum for which the remainder is known in an integral form. Some applications for elementary functions including the exponential and logarithmic functions are provided as well.

For the sake of completeness, all the results presented are completely proved and the original references where they have been firstly obtained are mentioned. The chapters are followed by the list of references used therein and therefore are relatively independent and can be read separately.

CHAPTER 1

Functions of Selfadjoint Operators in Hilbert Spaces

1. INTRODUCTION

In this introductory chapter we recall some fundamental facts concerning bounded selfadjoint operators on complex Hilbert spaces. Since all the operators considered in this survey are supposed to be bounded, we no longer mention this but understand it implicitly.

The generalized Schwarz's inequality for positive selfadjoint operators as well as some results for the spectrum of this class of operators are presented. Then we introduce and explore the fundamental results for polynomials in a linear operator, continuous functions of selfadjoint operators as well as the step functions of selfadjoint operators. By the use of these results we then introduce the spectral decomposition of selfadjoint operators (the *Spectral Representation Theorem*) that will play a central role in the rest of the survey. This result is used as a key tool in obtaining various new inequalities for continuous functions of selfadjoint operators which are of bounded variation, Lipschitzian, monotonic or absolutely continuous. Another tool that will greatly simplify the error bounds provided in the survey is the *Total Variation Schwarz's Inequality* for which a simple proof is offered.

The chapter is concluded with some well known operator inequalities of Jensen's type for convex and operator convex functions. More results in this spirit can be found in the recent survey [**1**].

Finally, some Grüss' type inequalities obtained in 1993 by Mond & Pečarić are also presented. They are developed extensively in a special chapter later in the survey where some applications in relation with classical power operator inequalities are provided as well.

2. BOUNDED SELFADJOINT OPERATORS

2.1. Operator Order. Let $(H;\langle .,.\rangle)$ be a Hilbert space over the complex numbers field $\mathbb{C}$.

A bounded linear operator A defined on H is *selfadjoint*, i.e., $A = A^*$ if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$ and if A is selfadjoint, then

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle| . \tag{2.1}$$

We assume in what follows that all operators are bounded on defined on the whole Hilbert space H. We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators defined on H.

DEFINITION 2.1. Let A and B be selfadjoint operators on H. Then $A \leq B$ (A is less or equal to B) or, equivalently, $B \geq A$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$. In particular, A is called positive if $A \geq 0$.

It is well known that for any operator $A \in \mathcal{B}(H)$ the composite operators A^*A and AA^* are positive selfadjoint operators on H. However, the operators A^*A and AA^* are not comparable with each other in general.

The following result concerning the operator order holds (see for instance [**2**, p. 220]):

THEOREM 2.1. *Let $A, B, C \in \mathcal{B}(H)$ be selfadjoint operators and let $\alpha, \beta \in \mathbb{R}$. Then*

(1) $A \leq A$;

(2) *If* $A \leq B$ *and* $B \leq C$*, then* $A \leq C$;
(3) *If* $A \leq B$ *and* $B \leq A$*, then* $A = B$;
(4) *If* $A \leq B$ *and* $\alpha \geq 0,$ *then*

$$A + C \leq B + C, \alpha A \leq \alpha B, -A \geq -B;$$

(5) *If* $\alpha \leq \beta$*, then* $\alpha A \leq \beta A$ *for positive operator* A.

The following *generalization of Schwarz's inequality* for positive selfadjoint operators A holds:

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle \langle Ay, y\rangle \tag{2.2}$$

for any $x, y \in H$.

The following inequality is of interest as well, see [**2**, p. 221]

THEOREM 2.2. *Let* A *be a positive selfadjoint operator on* H. *Then*

$$\|Ax\|^2 \leq \|A\| \langle Ax, x\rangle \tag{2.3}$$

for any $x \in H$.

THEOREM 2.3. *Let* $A_n, B \in \mathcal{B}(H)$ *with* $n \geq 1$ *be selfadjoint operators with the property that*

$$A_1 \leq A_2 \leq ... \leq A_n \leq ... \leq B.$$

Then there exists a bounded selfadjoint operator A *defined on* H *such that*

$$A_n \leq A \leq B \text{ for all } n \geq 1$$

and

$$\lim_{n\to\infty} A_n x = Ax \text{ for all } x \in H.$$

An analogous assertion holds if the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and bounded below.

DEFINITION 2.2. We say that a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges strongly to an operator $A \in \mathcal{B}(H)$, called the strong limit of the sequence $\{A_n\}_{n=1}^{\infty}$ and we denote this by $(s)\lim_{n\to\infty} A_n = A$, if $\lim_{n\to\infty} A_n x = Ax$ for all $x \in H$.

The convergence in norm, i.e. $\lim_{n\to\infty} \|A_n - A\| = 0$ will be called the *"uniform convergence"* as opposed to strong convergence. We denote $\lim_{n\to\infty} A_n = A$ for the convergence in norm. From the inequality

$$\|A_m x - A_n x\| \leq \|A_m - A_n\| \|x\|$$

that holds for all n, m and $x \in H$ it follows that uniform convergence of the sequence $\{A_n\}_{n=1}^{\infty}$ to A implies strong convergence of $\{A_n\}_{n=1}^{\infty}$ to A. However, the converse of this assertion is false.

It is also possible to introduce yet another concept of *"weak convergence"* in $\mathcal{B}(H)$ by defining $(w)\lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} \langle A_n x, y\rangle = \langle Ax, y\rangle$ for all $x, y \in H$.

The following result holds (see [**2**, p. 225]):

THEOREM 2.4. *Let* A *be a bounded selfadjoint operator on* H. *Then*

$$\alpha_1 \ : \ = \inf_{\|x\|=1} \langle Ax, x\rangle = \max\{\alpha \in \mathbb{R} \,|\alpha I \leq A\};$$
$$\alpha_2 \ : \ = \sup_{\|x\|=1} \langle Ax, x\rangle = \min\{\alpha \in \mathbb{R} \,|A \leq \alpha I\};$$

and

$$\|A\| = \max\{|\alpha_1|, |\alpha_2|\}.$$

Moreover, if $Sp(A)$ *denotes the spectrum of* A, *then* $\alpha_1, \alpha_2 \in Sp(A)$ *and* $Sp(A) \subset [\alpha_1, \alpha_2]$.

REMARK 2.1. We remark that, if A, α_1, α_2 are as above, then obviously

$$\begin{aligned} \alpha_1 &= \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A); \\ \alpha_2 &= \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A); \\ \|A\| &= \max\{|\lambda| \,|\lambda \in Sp(A)\}. \end{aligned}$$

We also observe that

(1) A is positive iff $\alpha_1 \geq 0$;
(2) A is positive and invertible iff $\alpha_1 > 0$;
(3) If $\alpha_1 > 0$, then A^{-1} is a positive selfadjoint operator and $\min Sp(A^{-1}) = \alpha_2^{-1}$, $\max Sp(A^{-1}) = \alpha_1^{-1}$.

3. CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS

3.1. Polynomials in a Bounded Operator. For two functions $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ we adhere to the canonical notation:

$$\begin{aligned} (\varphi + \psi)(s) &:= \varphi(s) + \psi(s), \\ (\lambda\varphi)(s) &:= \lambda\varphi(s), \\ (\varphi\psi)(s) &:= \varphi(s)\psi(s) \end{aligned}$$

for sum, scalar multiple and product of these functions. We denote by $\bar{\varphi}(s)$ the complex conjugate of $\varphi(s)$.

As a first class of functions we consider the algebra $\mathcal{P}$ of all polynomials in one variable with complex coefficients, namely

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \,|n \geq 0, \alpha_k \in \mathbb{C}, 0 \leq k \leq n \right\}.$$

THEOREM 3.1. *Let* $A \in \mathcal{B}(H)$ *and for* $\varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \in \mathcal{P}$ *define* $\varphi(A) := \sum_{k=0}^{n} \alpha_k A^k \in \mathcal{B}(H)\,(A^0 = I)$ *and* $\bar{\varphi}(A) := \sum_{k=0}^{n} \bar{\alpha}_k (A^*)^k \in \mathcal{B}(H)$. *Then the mapping* $\varphi(s) \to \varphi(A)$ *has the following properties*

a) $(\varphi + \psi)(A) = \varphi(A) + \psi(A)$;
b) $(\lambda\varphi)(A) = \lambda\varphi(A)$;
c) $(\varphi\psi)(A) = \varphi(A)\psi(A)$;
d) $[\varphi(A)]^* = \bar{\varphi}(A)$.

Note that $\varphi(A)\psi(A) = \psi(A)\varphi(A)$ and the constant polynomial $\varphi(s) = \alpha_0$ is mapped into the operator.

Recall that, a mapping $a \to a'$ of an algebra $\mathcal{U}$ into an algebra $\mathcal{U}'$ is called a *homomorphism* if it has the properties

a) $(a + b)' = a' + b'$;
b) $(\lambda\varphi)' = \lambda a'$;
c) $(ab)' = a'b'$.

With this terminology, Theorem 3.1 asserts that the mapping which associates with any polynomial $\varphi(s)$ the operator $\varphi(A)$ is a homomorphism of $\mathcal{P}$ into $\mathcal{B}(H)$ satisfying the additional property d).

The following result provides a connection between the spectrum of A and the spectrum of the operator $\varphi(A)$.

THEOREM 3.2. *If $A \in \mathcal{B}(H)$ and $\varphi \in \mathcal{P}$, then $Sp(\varphi(A)) = \varphi(Sp(A))$.*

COROLLARY 3.3. *If $A \in \mathcal{B}(H)$ is selfadjoint and the polynomial $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint and*

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{3.1}$$

REMARK 3.1. If $A \in \mathcal{B}(H)$ and $\varphi \in \mathcal{P}$, then

(1) $\varphi(A)$ is invertible iff $\varphi(\lambda) \neq 0$ for all $\lambda \in Sp(A)$;
(2) If $\varphi(A)$ is invertible, then $Sp\left(\varphi(A)^{-1}\right) = \left\{\varphi(\lambda)^{-1}, \lambda \in Sp(A)\right\}$.

3.2. Continuous Functions of Selfadjoint Operators. Assume that A is a bounded selfadjoint operator on the Hilbert space H. If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A = \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is continuous, in particular if φ is a polynomial, then the supremum is actually assumed for some points in $Sp(A)$ which is compact. Therefore the supremum may then be written as a maximum and the formula (3.1) can be written in the form $\|\varphi(A)\| = \|\varphi\|_A$.

Consider $\mathcal{C}(\mathbb{R})$ the algebra of all continuous complex valued functions defined on $\mathbb{R}$. The following fundamental result for continuous functional calculus holds, see for instance [**2**, p. 232]:

THEOREM 3.4. *If A is a bounded selfadjoint operator on the Hilbert space H and $\varphi \in \mathcal{C}(\mathbb{R})$, then there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ with the property that whenever $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{P}$ such that $\lim_{n\to\infty} \|\varphi - \varphi_n\|_A = 0$, then $\varphi(A) = \lim_{n\to\infty} \varphi_n(A)$. The mapping $\varphi \to \varphi(A)$ is a homomorphism of the algebra $\mathcal{C}(\mathbb{R})$ into $\mathcal{B}(H)$ with the additional properties $[\varphi(A)]^* = \bar{\varphi}(A)$ and $\|\varphi(A)\| \leq 2\|\varphi\|_A$. Moreover, $\varphi(A)$ is a normal operator, i.e. $[\varphi(A)]^* \varphi(A) = \varphi(A)[\varphi(A)]^*$. If φ is real-valued, then $\varphi(A)$ is selfadjoint.*

As examples we notice that, if $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = e^{is}$, $s \in \mathbb{R}$ then

$$e^{iA} = \sum_{k=0}^{\infty} \frac{1}{k!}(iA)^k.$$

Moreover, e^{iA} is a unitary operator and its inverse is the operator

$$\left(e^{iA}\right)^* = e^{-iA} = \sum_{k=0}^{\infty} \frac{1}{k!}(-iA)^k.$$

Now, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = \frac{1}{s-\lambda} \in \mathcal{C}(\mathbb{R})$, then $\varphi(A) = (A - \lambda I)^{-1}$.

If the selfadjoint operator $A \in \mathcal{B}(H)$ and the functions $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ are given, then we obtain the commutativity property $\varphi(A)\psi(A) = \psi(A)\varphi(A)$. This property can be extended for another operator as follows, see for instance [**2**, p. 235]:

THEOREM 3.5. *Assume that $A \in \mathcal{B}(H)$ and the function $\varphi \in \mathcal{C}(\mathbb{R})$ are given. If $B \in \mathcal{B}(H)$ is such that $AB = BA$, then $\varphi(A)B = B\varphi(A)$.*

The next result extends Theorem 3.2 to the case of continuous functions, see for instance [**2**, p. 235]:

THEOREM 3.6. *If A is abounded selfadjoint operator on the Hilbert space H and φ is continuous, then $Sp(\varphi(A)) = \varphi(Sp(A))$.*

As a consequence of this result we have:

COROLLARY 3.7. *With the assumptions in Theorem 3.6 we have:*

a) *The operator* $\varphi(A)$ *is selfadjoint iff* $\varphi(\lambda) \in \mathbb{R}$ *for all* $\lambda \in Sp(A)$;
b) *The operator* $\varphi(A)$ *is unitary iff* $|\varphi(\lambda)| = 1$ *for all* $\lambda \in Sp(A)$;
c) *The operator* $\varphi(A)$ *is invertible iff* $\varphi(\lambda) \neq 0$ *for all* $\lambda \in Sp(A)$;
d) *If* $\varphi(A)$ *is selfadjoint, then* $\|\varphi(A)\| = \|\varphi\|_A$.

In order to develop inequalities for functions of selfadjoint operators we need the following result, see for instance [**2**, p. 240]:

THEOREM 3.8. *Let* A *be a bounded selfadjoint operator on the Hilbert space* H. *The homomorphism* $\varphi \to \varphi(A)$ *of* $\mathcal{C}(\mathbb{R})$ *into* $\mathcal{B}(H)$ *is order preserving, meaning that, if* $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ *are real valued on* $Sp(A)$ *and* $\varphi(\lambda) \geq \psi(\lambda)$ *for any* $\lambda \in Sp(A)$, *then*

$$\varphi(A) \geq \psi(A) \text{ in the operator order of } \mathcal{B}(H). \tag{P}$$

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [**2**, p. 240]:

THEOREM 3.9. *If the operator* $A \in \mathcal{B}(H)$ *is selfadjoint and positive, then there exists a unique positive selfadjoint operator* $B := \sqrt{A} \in \mathcal{B}(H)$ *such that* $B^2 = A$. *If* A *is invertible, then so is* B.

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

Analogously to the familiar factorization of a complex number

$$\xi = |\xi| \, e^{i \arg \xi}$$

a bounded normal operator on H may be written as a commutative product of a positive selfadjoint operator, representing its absolute value, and a unitary operator, representing the factor of absolute value one.

In fact, the following more general result holds, see for instance [**2**, p. 241]:

THEOREM 3.10. *For every bounded linear operator* A *on* H, *there exists a positive selfadjoint operator* $B = |A| \in \mathcal{B}(H)$ *and an isometric operator* C *with the domain* $\mathcal{D}_C = \overline{B(H)}$ *and range* $\mathcal{R}_C = C(\mathcal{D}_C) = \overline{A(H)}$ *such that* $A = CB$.

In particular, we have:

COROLLARY 3.11. *If the operator* $A \in \mathcal{B}(H)$ *is normal, then there exists a positive selfadjoint operator* $B = |A| \in \mathcal{B}(H)$ *and a unitary operator* C *such that* $A = BC = CB$. *Moreover, if* A *is invertible, then* B *and* C *are uniquely determined by these requirements.*

REMARK 3.2. Now, suppose that $A = CB$ where $B \in \mathcal{B}(H)$ is a positive selfadjoint operator and C is an isometric operator. Then

a) $B = \sqrt{A^*A}$; consequently B is uniquely determined by the stated requirements;
b) C is uniquely determined by the stated requirements iff A is one-to-one.

4. STEP FUNCTIONS OF SELFADJOINT OPERATORS

Let A be a bonded selfadjoint operator on the Hilbert space H. We intend to extend the order preserving homomorphism $\varphi \to \varphi(A)$ of the algebra $\mathcal{C}(\mathbb{R})$ of continuous functions φ defined on $\mathbb{R}$ into $\mathcal{B}(H)$, restricted now to real-valued functions, to a larger domain, namely an algebra of functions containing the "step functions" φ_λ, $\lambda \in \mathbb{R}$, defined by

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \le \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Observe that $\overline{\varphi}_\lambda(s) = \varphi_\lambda(s)$ and $\varphi_\lambda^2(s) = \varphi_\lambda(s)$ which will imply that $[\varphi_\lambda(A)]^* = \varphi_\lambda(A)$ and $[\varphi_\lambda(A)]^2 = \varphi_\lambda(A)$, i.e. $\varphi_\lambda(A)$ will then be a projection. However, since the function φ_λ cannot be approximated uniformly by continuous functions on any interval containing λ, then, in general, there is no way to define an operator $\varphi_\lambda(A)$ as a uniform limit of operators $\varphi_{\lambda,n}(A)$ with $\varphi_{\lambda,n} \in \mathcal{C}(\mathbb{R})$.

The uniform limit of operators can be relaxed to the concept of strong limit of operators (see Definition 2.2) in order to define the operator $\varphi_\lambda(A)$. In order to do that, observe that the function φ_λ may be obtained as a pointwise limit of a decreasing sequence of real-valued continuous functions $\varphi_{\lambda,n}$ defined by

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \le \lambda, \\ \\ 1 - n(s-\lambda), \text{ for } \lambda \le s \le \lambda + 1/n \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

By Theorem 2.3 we observe that the sequence of corresponding selfadjoint operators $\varphi_{\lambda,n}(A)$ is nondecreasing and bounded below by zero in the operator order of $\mathcal{B}(H)$. It therefore converges strongly to some bounded selfadjoint operator $\varphi_\lambda(A)$ on H, see [**2**, p. 244].

To provide a formal presentation of the above, we need the following definition.

DEFINITION 4.1. A real-valued function φ on $\mathbb{R}$ is called upper semi-continuous if it is a pointwise limit of a non-increasing sequence of continuous real-valued functions on $\mathbb{R}$.

We observe that it can be shown that a real-valued functions φ on $\mathbb{R}$ is upper semi-continuous iff for every $s_0 \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varphi(s) < \varphi(s_0) + \varepsilon \text{ for all } s \in (s_0 - \delta, s_0 + \delta).$$

We can introduce now the operator $\varphi(A)$ as follows, see for instance [**2**, p. 245]:

THEOREM 4.1. *Let A be a bonded selfadjoint operator on the Hilbert space H and let φ be a nonnegative upper semi-continuous function on $\mathbb{R}$. Then there exists a unique positive selfadjoint operator $\varphi(A)$ such that whenever $\{\varphi_n\}_{n=1}^\infty$ is any non-increasing sequence of non-negative functions in $\mathcal{C}(\mathbb{R})$, pointwise converging to φ on $Sp(A)$, then $\varphi(A) = (s)\lim \varphi_n(A)$.*

If φ is continuous, then the operator $\varphi(A)$ defined by Theorem 3.4 coincides with the one defined by Theorem 4.1.

THEOREM 4.2. *Let $A \in \mathcal{B}(H)$ be selfadjoint, let φ and ψ be non-negative upper semi-continuous functions on $\mathbb{R}$, and let $\alpha > 0$ be given. Then the functions $\varphi + \psi$, $\alpha\varphi$ and $\varphi\psi$ are non-negative upper semi-continuous and $(\varphi + \psi)(A) = \varphi(A) + \psi(A)$, $(\alpha\varphi)(A) = \alpha\varphi(A)$ and $(\varphi\psi)(A) = \varphi(A)\psi(A)$. Moreover, if $\varphi(s) \le \psi(s)$ for all $s \in Sp(A)$ then $\varphi(A) \le \psi(A)$.*

We enlarge the class of non-negative upper semi-continuous functions to an algebra by defining $\mathcal{R}(\mathbb{R})$ as the set of all functions $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are nonnegative and upper semi-continuous functions defined on $\mathbb{R}$. It is easy to see that $\mathcal{R}(\mathbb{R})$ endowed with pointwise sum, scalar multiple and product is an algebra.

The following result concerning functions of operators $\varphi(A)$ with $\varphi \in \mathcal{R}(\mathbb{R})$ can be stated, see for instance [**2**, p. 249-p. 250]:

THEOREM 4.3. *Let $A \in \mathcal{B}(H)$ be selfadjoint and let $\varphi \in \mathcal{R}(\mathbb{R})$. Then there exists a unique selfadjoint operator $\varphi(A) \in \mathcal{B}(H)$ such that if $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are nonnegative and upper semi-continuous functions defined on $\mathbb{R}$, then $\varphi(A) = \varphi_1(A) - \varphi_2(A)$. The mapping $\varphi \to \varphi(A)$ is a homomorphism of $\mathcal{R}(\mathbb{R})$ into $\mathcal{B}(H)$ which is order preserving in the following sense: if $\varphi, \psi \in \mathcal{R}(\mathbb{R})$ with the property that $\varphi(s) \le \psi(s)$ for any $s \in Sp(A)$, then $\varphi(A) \le \psi(A)$. Moreover, if $B \in \mathcal{B}(H)$ satisfies the commutativity condition $AB = BA$, then $\varphi(A) B = B\varphi(A)$.*

5. THE SPECTRAL DECOMPOSITION OF SELFADJOINT OPERATORS

Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \le \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [**2**, p. 256]

THEOREM 5.1 (Spectral Representation Theorem). *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$*;*
c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{5.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon \tag{5.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \textit{ for } 1 \le k \le n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \textit{ for } 1 \le k \le n \end{cases} \tag{5.4}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda) \, dE_\lambda, \tag{5.5}$$

where the integral is of Riemann-Stieltjes type.

COROLLARY 5.2. *With the assumptions of Theorem 5.1 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \text{ for all } x \in H \tag{5.6}$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \text{ for all } x, y \in H. \tag{5.7}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \text{ for all } x \in H. \tag{5.8}$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \text{ for all } x \in H. \tag{5.9}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 5.1, see for instance [**2**, p. 258]:

THEOREM 5.3. *Let* A *be a bonded selfadjoint operator on the Hilbert space* H *and let* $m = \min Sp(A)$ *and* $M = \max Sp(A)$. *If* $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ *is a family of projections satisfying the requirements a), b) and c) in Theorem 5.1, then* $F_\lambda = E_\lambda$ *for all* $\lambda \in \mathbb{R}$ *where* E_λ *is defined by (5.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A. The spectral family also reflects in a direct way the properties of the operator A as follows, see [**2**, p. 263-p.266]

THEOREM 5.4. *Let* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *be the spectral family of the bounded selfadjoint operator* A. *If* B *is a bounded linear operator on* H, *then* $AB = BA$ *iff* $E_\lambda B = BE_\lambda$ *for all* $\lambda \in \mathbb{R}$. *In particular* $E_\lambda A = AE_\lambda$ *for all* $\lambda \in \mathbb{R}$.

THEOREM 5.5. *Let* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *be the spectral family of the bounded selfadjoint operator* A *and* $\mu \in \mathbb{R}$. *Then*

a) μ *is a regular value of* A,*i.e.,* $A - \mu I$ *is invertible iff there exists a* $\theta > 0$ *such that* $E_{\mu-\theta} = E_{\mu+\theta}$;
b) $\mu \in Sp(A)$ *iff* $E_{\mu-\theta} < E_{\mu+\theta}$ *for all* $\theta > 0$;
c) μ *is an eigenvalue of* A *iff* $E_{\mu-0} < E_\mu$.

The following result will play a key role in many results concerning inequalities for bounded selfadjoint operators in Hilbert spaces. Since we were not able to locate it in the literature, we will provide here a complete proof:

THEOREM 5.6 (Total Variation Schwarz's Inequality). *Let* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *be the spectral family of the bounded selfadjoint operator* A *and let* $m = \min Sp(A)$ *and* $M = \max Sp(A)$. *Then for any* $x, y \in H$ *the function* $\lambda \to \langle E_\lambda x, y\rangle$ *is of bounded variation on* $[m - \epsilon, M]$ *for any* $\epsilon > 0$ *and we have the inequality*

$$\left| \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \right| \le \|x\|\,\|y\|, \tag{TVSI}$$

where $\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ *denotes the limit* $\lim_{\epsilon\to 0+}\bigvee_{m-\epsilon}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$.

PROOF. If P is a nonnegative selfadjoint operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle, \tag{5.10}$$

for any $x,y\in H$.

Let $\epsilon>0$. Now, if $d: m-\epsilon=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m-\epsilon,M]$, then we have by Schwarz's inequality for nonnegative operators (5.10) that

$$\begin{aligned}
&\bigvee_{m-\epsilon}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&=\sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\
&\leq\sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=I.
\end{aligned}\tag{5.11}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
&I \\
&\leq\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\
&\leq\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\
&=\left[\bigvee_{m-\epsilon}^{M}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_{m-\epsilon}^{M}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2}=\|x\|\,\|y\|
\end{aligned}\tag{5.12}$$

for any $x,y\in H$.

On making use of (5.11) and (5.12) and letting $\epsilon\to 0+$ we deduce the desired result (TVSI). ∎

6. JENSEN'S TYPE INEQUALITIES

6.1. Jensen's Inequality. The following result that provides an operator version for the *Jensen inequality* is due to Mond & Pečarić [**5**] (see also [**1**, p. 5]):

THEOREM 6.1 (Mond- Pečarić, 1993, [**5**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle \tag{MP}$$

for each $x\in H$ with $\|x\|=1$.

As a special case of Theorem 6.1 we have the following *Hölder-McCarthy inequality*:

THEOREM 6.2 (Hölder-McCarthy, 1967, [**3**]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then*
(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r < 0$ *and* $x \in H$ *with* $\|x\| = 1$.

The following theorem is a multiple operator version of Theorem 6.1 (see for instance [**1**, p. 5]):

THEOREM 6.3 (Furuta-Mićić-Pečarić-Seo, 2005, [**1**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a convex function on $[m, M]$, then*

$$f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle . \tag{6.1}$$

The following particular case is of interest.

COROLLARY 6.4. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $m < M$. If $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle , \tag{6.2}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Follows from Theorem 6.3 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \dots, n\}$ where $x \in H$ with $\|x\| = 1$. ∎

REMARK 6.1. The above inequality can be used to produce some norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0, M]$. Namely, we have:

$$f\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \leq \left\|\sum_{j=1}^{n} p_j f(A_j)\right\| . \tag{6.3}$$

The inequality (6.3) reverses if the function is concave on $[0, M]$.
As particular cases we can state the following inequalities:

$$\left\|\sum_{j=1}^{n} p_j A_j\right\|^p \leq \left\|\sum_{j=1}^{n} p_j A_j^p\right\| , \tag{6.4}$$

for $p \geq 1$ and

$$\left\|\sum_{j=1}^{n} p_j A_j\right\|^p \geq \left\|\sum_{j=1}^{n} p_j A_j^p\right\| \tag{6.5}$$

for $0 < p < 1$.
If A_j are positive definite for each $j \in \{1, \dots, n\}$ then (6.4) also holds for $p < 0$.
If one uses the inequality (6.3) for the exponential function, that one obtains the inequality

$$\exp\left(\left\|\sum_{j=1}^{n} p_j A_j\right\|\right) \leq \left\|\sum_{j=1}^{n} p_j \exp(A_j)\right\| , \tag{6.6}$$

where A_j are positive operators for each $j \in \{1, \dots, n\}$.

6.2. Reverses of Jensen's Inequality. In Section 2.4 of the monograph [**1**] there are numerous interesting converses of the Jensen's type inequality (6.1) from which we would like to mention only two of the simplest.

The following result is an operator version of the well known Lah-Ribarić's reverse of the Jensen inequality for real functions of a real variable, see for instance [**1**]:

THEOREM 6.5. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a continuous convex function defined on $[m, M]$, then*

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \frac{1}{M-m} \left[f(M) \sum_{j=1}^{n} \langle (A_j - mI) x_j, x_j \rangle + f(m) \sum_{j=1}^{n} \langle (MI - A_j) x_j, x_j \rangle \right]. \tag{6.7}$$

THEOREM 6.6 (Mićić-Seo-Takahasi-Tominaga, 1999, [**4**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \dots, n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a strictly convex function twice differentiable on $[m, M]$, then for any positive real number α we have*

$$\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \alpha f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) + \beta, \tag{6.8}$$

where

$$\beta = \mu_f t_0 + \nu_f - \alpha f(t_0),$$
$$\mu_f = \frac{f(M) - f(m)}{M - m}, \quad \nu_f = \frac{M f(m) - m f(M)}{M - m}$$

and

$$t_0 = \begin{cases} f'^{-1}\left(\frac{\mu_f}{\alpha}\right) & \text{if } m < f'^{-1}\left(\frac{\mu_f}{\alpha}\right) < M \\ M & \text{if } M \leq f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \\ m & \text{if } f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \leq m. \end{cases}$$

The case of equality was also analyzed, see [**1**, p. 61] but will be not stated in here.

6.3. Operator Monotone and Operator Convex Functions. We say that a real valued continuous function f defined on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e. if A and B are bounded selfadjoint operators with $A \leq B$ and $Sp(A), Sp(A) \subset I$, then $f(A) \leq f(B)$. The function is said to be *operator convex* (*operator concave*) if for any A, B bounded selfadjoint operators with $Sp(A), Sp(A) \subset I$, we have

$$f[(1-\lambda) A + \lambda B] \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{6.9}$$

for any $\lambda \in [0, 1]$.

EXAMPLE 6.1. *The following examples are well know in the literature and can be found for instance in* [**1**, p. 7-p. 9] *where simple proofs were also provided.*

(1) *The affine function $f(t) = \alpha + \beta t$ is operator monotone on every interval for all $\alpha \in \mathbb{R}$ and $\beta \geq 0$. It is operator convex for all $\alpha, \beta \in \mathbb{R}$;*

(2) *If f,g are operator monotone, and if $\alpha,\beta\geq 0$ then the linear combination $\alpha f+\beta g$ is also operator monotone. If the functions f_n are operator monotone and $f_n(t)\to f(t)$ as $n\to\infty$, then f is also operator monotone;*
(3) *The function $f(t)=t^2$ is operator convex on every interval, however it is not operator monotone on $[0,\infty)$ even though it is monotonic nondecreasing on this interval;*
(4) *The function $f(t)=t^3$ is not operator convex on $[0,\infty)$ even though it is a convex function on this interval;*
(5) *The function $f(t)=\frac{1}{t}$ is operator convex on $(0,\infty)$ and $f(t)=-\frac{1}{t}$ is operator monotone on $(0,\infty)$;*
(6) *The function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$;*
(7) *The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$;*
(8) *The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone on any interval of $\mathbb{R}$.*

The following monotonicity property for the function $f(t)=t^r$ with $r\in[0,1]$ is well known in the literature as the *Löwner-Heinz inequality* and was established essentially in 1934:

THEOREM 6.7 (Löwner-Heinz Inequality). *Let A and B be positive operators on a Hilbert space H. If $A\geq B\geq 0$, then $A^r\geq B^r$ for all $r\in[0,1]$.*

The following characterization of operator convexity holds, see [**1**, p. 10]

THEOREM 6.8 (Jensen's Operator Inequality). *Let H and K be Hilbert spaces. Let f be a real valued continuous function on an interval J. Let A and A_j be selfadjoint operators on H with spectra contained in J, for each $j=1,2,...,k$. Then the following conditions are mutually equivalent:*

(i) *f is operator convex on J;*
(ii) *$f(C^*AC)\leq C^*f(A)C$ for every selfadjoint operator $A:H\to H$ and isometry $C:K\to H, i.e., C^*C=1_K$;*
(iii) *$f(C^*AC)\leq C^*f(A)C$ for every selfadjoint operator $A:H\to H$ and isometry $C:H\to H$;*
(iv) *$f\left(\sum_{j=1}^k C_j^*A_jC_j\right)\leq\sum_{j=1}^k C_j^*f(A_j)C_j$ for every selfadjoint operator $A_j:H\to H$ and bounded linear operators $C_j:K\to H$, with $\sum_{j=1}^k C_j^*C_j=1_K\ (j=1,...,k)$;*
(v) *$f\left(\sum_{j=1}^k C_j^*A_jC_j\right)\leq\sum_{j=1}^k C_j^*f(A_j)C_j$ for every selfadjoint operator $A_j:H\to H$ and bounded linear operators $C_j:H\to H$, with $\sum_{j=1}^k C_j^*C_j=1_H\ (j=1,...,k)$;*
(vi) *$f\left(\sum_{j=1}^k P_jA_jP_j\right)\leq\sum_{j=1}^k P_jf(A_j)P_j$ for every selfadjoint operator $A_j:H\to H$ and projection $P_j:H\to H$, with $\sum_{j=1}^k P_j=1_H\ (j=1,...,k)$.*

The following well known result due to Hansen & Pedersen also holds:

THEOREM 6.9 (Hansen-Pedersen-Jensen's Inequality). *Let J be an interval containing 0 and let f be a real valued continuous function defined on J. Let A and A_j be selfadjoint operators on H with spectra contained in J, for each $j=1,2,...,k$. Then the following conditions are mutually equivalent:*

(i) *f is operator convex on J and $f(0)\leq 0$;*
(ii) *$f(C^*AC)\leq C^*f(A)C$ for every selfadjoint operator $A:H\to H$ and contraction $C:H\to H, i.e., C^*C\leq 1_H$;*
(iii) *$f\left(\sum_{j=1}^k C_j^*A_jC_j\right)\leq\sum_{j=1}^k C_j^*f(A_j)C_j$ for every selfadjoint operator $A_j:H\to H$ and bounded linear operators $C_j:H\to H$, with $\sum_{j=1}^k C_j^*C_j\leq 1_H\ (j=1,...,k)$;*

(iv) $f(PAP) \leq Pf(A)P$ *for every selfadjoint operator* $A : H \to H$ *and projection* P.

The case of continuous and negative functions is as follows, [**1**, p. 13]:

THEOREM 6.10. *Let* f *be continuous on* $[0,\infty)$. *If* $f(t) \leq 0$ *for all* $t \in [0,\infty)$, *then each of the conditions (i)-(vi) from Theorem 6.8 is equivalent with*

(vii) $-f$ *is an operator monotone function.*

COROLLARY 6.11. *Let* f *be a real valued continuous function mapping the positive half line* $[0,\infty)$ *into itself. Then* f *is operator monotone if and only if* f *is operator concave.*

The following result may be stated as well [**1**, p. 14]:

THEOREM 6.12. *Let* f *be continuous on the interval* $[0,r)$ *with* $r \leq \infty$. *Then the following conditions are mutually equivalent:*

(i) f *is operator convex and* $f(0) \leq 0$;
(ii) *The function* $t \mapsto \frac{f(t)}{t}$ *is operator monotone on* $(0,r)$.

As a particular case of interest, we can state that [**1**, p. 15]:

COROLLARY 6.13. *Let* f *be continuous on* $[0,\infty)$ *and taking positive values. The function* f *is operator monotone if and only if the function* $t \mapsto \frac{t}{f(t)}$ *is operator monotone.*

Finally we recall the following result as well [**1**, p. 16]:

THEOREM 6.14. *Let* f *be a real valued continuous function on the interval* $J = [\alpha,\infty)$ *and bounded below, i.e., there exists* $m \in \mathbb{R}$ *such that* $m \leq f(t)$ *for all* $t \in J$. *Then the following conditions are mutually equivalent:*

(i) f *is operator concave on* J;
(ii) f *is operator monotone on* J.

As a particular case of this result we note that, the function $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$.

7. GRÜSS' TYPE INEQUALITIES

The following operator version of the Grüss inequality was obtained by Mond & Pečarić in [**6**]:

THEOREM 7.1 (Mond-Pečarić, 1993, [**6**]). *Let* C_j, $j \in \{1,\ldots,n\}$ *be selfadjoint operators on the Hilbert space* $(H,\langle .,. \rangle)$ *and such that* $m_j \cdot 1_H \leq C_j \leq M_j \cdot 1_H$ *for* $j \in \{1,\ldots,n\}$, *where* 1_H *is the identity operator on* H. *Further, let* $g_j, h_j : [m_j, Mj] \to \mathbb{R}$, $j \in \{1,\ldots,n\}$ *be functions such that*

$$\varphi \cdot 1_H \leq g_j(C_j) \leq \Phi \cdot 1_H \quad \text{and} \quad \gamma \cdot 1_H \leq h_j(C_j) \leq \Gamma \cdot 1_H \tag{7.1}$$

for each $j \in \{1,\ldots,n\}$.

If $x_j \in H$, $j \in \{1,\ldots,n\}$ *are such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then*

$$\left| \sum_{j=1}^{n} \langle g_j(C_j) h_j(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g_j(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h_j(C_j) x_j, x_j \rangle \right|$$
$$\leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma). \tag{7.2}$$

If $C_j, j \in \{1,\ldots,n\}$ are selfadjoint operators such that $Sp(C_j) \subseteq [m,M]$ for $j \in \{1,\ldots,n\}$ and for some scalars $m < M$ and if $g,h : [m,M] \longrightarrow \mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\left|\sum_{j=1}^{n} \langle g(C_j) h(C_j) x_j, x_j\rangle - \sum_{j=1}^{n} \langle g(C_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle h(C_j) x_j, x_j\rangle\right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma), \tag{7.3}$$

where $x_j \in H$, $j \in \{1,\ldots,n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and $\varphi = \min_{t\in[m,M]} g(t)$, $\Phi = \max_{t\in[m,M]} g(t)$, $\gamma = \min_{t\in[m,M]} h(t)$ and $\Gamma = \max_{t\in[m,M]} h(t)$.

In particular, if the selfadjoint operator C satisfy the condition $Sp(C) \subseteq [m,M]$ for some scalars $m < M$, then

$$|\langle g(C) h(C) x, x\rangle - \langle g(C) x, x\rangle \cdot \langle h(C) x, x\rangle| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma), \tag{7.4}$$

for any $x \in H$ with $\|x\| = 1$.

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CHAPTER 2

Inequalities for Convex Functions

1. Introduction

Jensen's type inequalities in their various settings ranging from discrete to continuous case play an important role in different branches of Modern Mathematics. A simple search in the *MathSciNet* database of the American Mathematical Society with the key words "jensen" and "inequality" in the title reveals more than 300 items intimately devoted to this famous result. However, the number of papers where this inequality is applied is a lot larger and far more difficult to find. It can be a good project in itself for someone to write a monograph devoted to Jensen's inequality in its different forms and its applications across Mathematics.

In the introductory chapter we have recalled a number of Jensen's type inequalities for convex and operator convex functions of selfadjoint operators in Hilbert spaces. In this chapter we present some recent results obtained by the author that deal with different aspects of this well research inequality than those recently reported in the survey [**20**]. They include but are not restricted to the operator version of the Dragomir-Ionescu inequality, Slater's type inequalities for operators and its inverses, Jensen's inequality for twice differentiable functions whose second derivatives satisfy some upper and lower bounds conditions, Jensen's type inequalities for log-convex functions and for differentiable log-convex functions and their applications to Ky Fan's inequality.

Finally, some Hermite-Hadamard's type inequalities for convex functions and Hermite-Hadamard's type inequalities for operator convex functions are presented as well.

All the above results are exemplified for some classes of elementary functions of interest such as the power function and the logarithmic function.

2. Reverses of the Jensen Inequality

2.1. An Operator Version of the Dragomir-Ionescu Inequality. The following result holds:

THEOREM 2.1 (Dragomir, 2008, [**9**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Since f is convex and differentiable, we have that

$$f(t) - f(s) \leq f'(t) \cdot (t - s)$$

for any $t, s \in [m, M]$.

Now, if we chose in this inequality $s = \langle Ax, x\rangle \in [m, M]$ for any $x \in H$ with $\|x\| = 1$ since $Sp(A) \subseteq [m, M]$, then we have

$$f(t) - f(\langle Ax, x\rangle) \leq f'(t) \cdot (t - \langle Ax, x\rangle) \tag{2.2}$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.2) and apply the property (P) then we get

$$\langle [f(A) - f(\langle Ax, x\rangle) 1_H] x, x\rangle \le \langle f'(A) \cdot (A - \langle Ax, x\rangle 1_H) x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent to the desired inequality (2.1). ∎

COROLLARY 2.2 (Dragomir, 2008, [**9**]). *Assume that f is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, \ldots, n\}$ and $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$(0 \le) \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \le \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j\rangle - \sum_{j=1}^n \langle A_j x_j, x_j\rangle \cdot \sum_{j=1}^n \langle f'(A_j) x_j, x_j\rangle. \tag{2.3}$$

PROOF. As in [**20**, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle, \qquad \left\langle \widetilde{A}\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^n \langle A_j x_j, x_j\rangle$$

and so on.

Applying Theorem 2.1 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (2.3). ∎

COROLLARY 2.3 (Dragomir, 2008, [**9**]). *Assume that f is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, \ldots, n\}$ and $p_j \ge 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$(0 \le) \left\langle \sum_{j=1}^n p_j f(A_j) x, x\right\rangle - f\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right) \le \left\langle \sum_{j=1}^n p_j f'(A_j) A_j x, x\right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle \cdot \left\langle \sum_{j=1}^n p_j f'(A_j) x, x\right\rangle. \tag{2.4}$$

for each $x \in H$ with $\|x\| = 1$.

REMARK 2.1. The inequality (2.4), in the scalar case, namely

$$(0 \le) \sum_{j=1}^n p_j f(x_j) - f\left(\sum_{j=1}^n p_j x_j\right) \le \sum_{j=1}^n p_j f'(x_j) x_j - \sum_{j=1}^n p_j x_j \cdot \sum_{j=1}^n p_j f'(x_j), \tag{2.5}$$

where $x_j \in \mathring{I}$, $j \in \{1, \ldots, n\}$, has been obtained by the first time in 1994 by Dragomir & Ionescu, see [**17**].

The following particular cases are of interest:

EXAMPLE 2.1. ***a.*** *Let A be a positive definite operator on the Hilbert space H. Then we have the following inequality:*

$$(0\le)\ln(\langle Ax,x\rangle)-\langle\ln(A)x,x\rangle\le\langle Ax,x\rangle\cdot\left\langle A^{-1}x,x\right\rangle-1, \tag{2.6}$$

for each $x\in H$ with $\|x\|=1$.

b. *If A is a selfadjoint operator on H, then we have the inequality:*

$$\begin{aligned}(0\le)&\langle\exp(A)x,x\rangle-\exp(\langle Ax,x\rangle)\\&\le\langle A\exp(A)x,x\rangle-\langle Ax,x\rangle\cdot\langle\exp(A)x,x\rangle,\end{aligned} \tag{2.7}$$

for each $x\in H$ with $\|x\|=1$.

c. *If $p\ge1$ and A is a positive operator on H, then*

$$(0\le)\langle A^px,x\rangle-\langle Ax,x\rangle^p\le p\left[\langle A^px,x\rangle-\langle Ax,x\rangle\cdot\left\langle A^{p-1}x,x\right\rangle\right], \tag{2.8}$$

for each $x\in H$ with $\|x\|=1$. If A is positive definite, then the inequality (2.8) also holds for $p<0$.

If $0<p<1$ and A is a positive definite operator then the reverse inequality also holds

$$\langle A^px,x\rangle-\langle Ax,x\rangle^p\ge p\left[\langle A^px,x\rangle-\langle Ax,x\rangle\cdot\left\langle A^{p-1}x,x\right\rangle\right]\ge0, \tag{2.9}$$

for each $x\in H$ with $\|x\|=1$.

Similar results can be stated for sequences of operators, however the details are omitted.

2.2. Further Reverses.

In applications would be perhaps more useful to find upper bounds for the quantity

$$\langle f(A)x,x\rangle-f(\langle Ax,x\rangle),\quad x\in H\quad\text{with}\quad\|x\|=1,$$

that are in terms of the spectrum margins m,M and of the function f.

The following result may be stated:

THEOREM 2.4 (Dragomir, 2008, [**9**]). *Let I be an interval and $f:I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A)\subseteq[m,M]\subset\mathring{I}$, then*

$$\begin{aligned}(0\le)&\langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\&\le\begin{cases}\frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2-\langle f'(A)x,x\rangle^2\right]^{1/2}\\ \\ \frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}\end{cases}\\&\le\frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned} \tag{2.10}$$

for any $x\in H$ with $\|x\|=1$.

We also have the inequality

$$\begin{aligned}(0\le)&\langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\&\le\frac{1}{4}(M-m)(f'(M)-f'(m))\\&-\begin{cases}\left[\langle Mx-Ax,Ax-mx\rangle\langle f'(M)x-f'(A)x,f'(A)x-f'(m)x\rangle\right]^{\frac{1}{2}},\\ \\ \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle-\frac{f'(M)+f'(m)}{2}\right|\end{cases}\\&\le\frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned} \tag{2.11}$$

for any $x \in H$ *with* $\|x\| = 1.$

Moreover, if $m > 0$ *and* $f'(m) > 0$, *then we also have*

$$(2.12) \qquad (0 \le) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \le \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)[\langle Ax, x\rangle \langle f'(A)x, x\rangle]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ *with* $\|x\| = 1.$

PROOF. We use the following Grüss' type result we obtained in [**6**]:

Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., .\rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If $h$and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} h(t)$ and $\Gamma := \max_{t\in[m,M]} h(t)$, then

$$(2.13) \qquad \begin{aligned} &|\langle h(A)g(A)x, x\rangle - \langle h(A)x, x\rangle \cdot \langle g(A)x, x\rangle| \\ &\le \frac{1}{2} \cdot (\Gamma - \gamma)\left[\|g(A)x\|^2 - \langle g(A)x, x\rangle^2\right]^{1/2} \\ &\left(\le \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)\right) \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Therefore, we can state that

$$(2.14) \qquad \begin{aligned} &\langle Af'(A)x, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \\ &\le \frac{1}{2} \cdot (M - m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ &\le \frac{1}{4}(M - m)(f'(M) - f'(m)) \end{aligned}$$

and

$$(2.15) \qquad \begin{aligned} &\langle Af'(A)x, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \\ &\le \frac{1}{2} \cdot (f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\ &\le \frac{1}{4}(M - m)(f'(M) - f'(m)) \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.10).

On making use of the inequality obtained in [**7**]:

$$(2.16) \qquad \begin{aligned} &|\langle h(A)g(A)x, x\rangle - \langle h(A)x, x\rangle \langle g(A)x, x\rangle| \\ &\le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \\ &- \begin{cases} [\langle \Gamma x - h(A)x, f(A)x - \gamma x\rangle \langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle h(A)x, x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x, x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, we can state that

$$\begin{aligned}&\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \\ &\leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \\ &- \begin{cases} [\langle Mx-Ax, Ax-mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right|. \end{cases}\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.11).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [**7**]:

If γ and δ are positive, then

$$\begin{aligned}&|\langle h(A)g(A)x,x\rangle - \langle h(A)x,x\rangle\langle g(A)x,x\rangle| \\ &\leq \begin{cases} \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\langle h(A)x,x\rangle\langle g(A)x,x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle h(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}}. \end{cases}\end{aligned} \tag{2.17}$$

for each $x \in H$ with $\|x\| = 1$.

Now, on making use of (2.17) we can state that

$$\begin{aligned}&\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \\ &\leq \begin{cases} \frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle\langle f'(A)x,x\rangle]^{\frac{1}{2}}. \end{cases}\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, which together with (2.1) provide the desired result (2.12). ∎

COROLLARY 2.5 (Dragomir, 2008, [**9**]). *Assume that f is as in the Theorem 2.4. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m,M] \subset \mathring{I}$, $j \in \{1,\dots,n\}$, then*

$$\begin{aligned}&(0\leq)\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right) \\ &\leq \begin{cases} \frac{1}{2}\cdot(M-m)\left[\sum_{j=1}^{n}\|f'(A_j)x_j\|^2 - \left(\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle\right)^2\right]^{1/2}, \\ \frac{1}{2}\cdot(f'(M)-f'(m))\left[\sum_{j=1}^{n}\|A_jx_j\|^2 - \left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)^2\right]^{1/2}, \end{cases} \\ &\leq \frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned} \tag{2.18}$$

for any $x_j \in H, j \in \{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

We also have the inequality

$$(0 \le) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \tag{2.19}$$
$$\le \frac{1}{4}(M-m)(f'(M) - f'(m))$$
$$- \begin{cases} \left[\sum_{j=1}^{n} \langle M x_j - A_j x, A_j x_j - m x_j \rangle\right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} \langle f'(M) x_j - f'(A_j) x_j, f'(A_j) x_j - f'(m) x_j \rangle\right]^{1/2}, \\ \left|\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - \frac{M+m}{2}\right| \left|\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases}$$
$$\le \frac{1}{4}(M-m)(f'(M) - f'(m)),$$

for any $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

Moreover, if $m > 0$ *and* $f'(m) > 0$*, then we also have*

$$(0 \le) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \tag{2.20}$$
$$\le \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle, \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right) \\ \times \left[\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle\right]^{\frac{1}{2}}, \end{cases}$$

for any $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

The following corollary also holds:

COROLLARY 2.6 (Dragomir, 2008, [**9**]). *Assume that* f *is as in the Theorem 2.1. If* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$*,* $j \in \{1, \ldots, n\}$ *and* $p_j \ge 0$*,* $j \in \{1, \ldots, n\}$

with $\sum_{j=1}^{n} p_j = 1$, *then*

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \tag{2.21}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M-m) \left[\sum_{j=1}^{n} p_j \|f'(A_j) x\|^2 - \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle^2 \right]^{1/2}, \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[\sum_{j=1}^{n} p_j \|A_j x\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right]^{1/2}, \end{cases}$$

$$\leq \frac{1}{4} (M-m)(f'(M) - f'(m)),$$

for any $x \in H$ *with* $\|x\| = 1$.

We also have the inequality

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \tag{2.22}$$

$$\leq \frac{1}{4} (M-m)(f'(M) - f'(m))$$

$$- \begin{cases} \left[\sum_{j=1}^{n} p_j \langle Mx - A_j x, A_j x - mx \rangle \right]^{\frac{1}{2}} \\ \times \left[\sum_{j=1}^{n} p_j \langle f'(M) x - f'(A_j) x, f'(A_j) x - f'(m) x \rangle \right]^{1/2}, \\ \\ \left| \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \frac{M+m}{2} \right| \left| \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle - \frac{f'(M)+f'(m)}{2} \right| \end{cases}$$

$$\leq \frac{1}{4} (M-m)(f'(M) - f'(m)),$$

for any $x \in H$ *with* $\|x\| = 1$.

Moreover, if $m > 0$ *and* $f'(m) > 0$, *then we also have*

$$(0 \leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \tag{2.23}$$

$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{f'(M)} - \sqrt{f'(m)} \right) \\ \times \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle \right]^{\frac{1}{2}}, \end{cases}$$

for any $x \in H$ *with* $\|x\| = 1$.

REMARK 2.2. Some of the inequalities in Corollary 2.6 can be used to produce reverse norm inequalities for the sum of positive operators in the case when the convex function f is nonnegative and monotonic nondecreasing on $[0, M]$.

For instance, if we use the inequality (2.21), then we have

$$(0 \le) \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| - f\left(\left\| \sum_{j=1}^{n} p_j A_j \right\| \right) \le \frac{1}{4}(M-m)(f'(M) - f'(m)). \tag{2.24}$$

Moreover, if we use the inequality (2.23), then we obtain

$$(0 \le) \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| - f\left(\left\| \sum_{j=1}^{n} p_j A_j \right\| \right) \le \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'(A_j) \right\|, \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right) \left[\left\| \sum_{j=1}^{n} p_j A_j \right\| \left\| \sum_{j=1}^{n} p_j f'(A_j) \right\| \right]^{\frac{1}{2}}. \end{cases} \tag{2.25}$$

2.3. Some Particular Inequalities of Interest. **1.** Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. On utilising the inequality (2.10), then for any positive definite operator A on the Hilbert space H, we have the inequality

$$(0 \le) \ln(\langle Ax, x \rangle) - \langle \ln(A) x, x \rangle \le \begin{cases} \frac{1}{2} \cdot (M-m) \left[\|A^{-1}x\|^2 - \langle A^{-1}x, x \rangle^2 \right]^{1/2} \\ \frac{1}{2} \cdot \frac{M-m}{mM} \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \left(\le \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \right) \tag{2.26}$$

for any $x \in H$ with $\|x\| = 1$.

However, if we use the inequality (2.11), then we have the following result as well

$$\begin{aligned} (0 \le) \ln(\langle Ax, x \rangle) - \langle \ln(A) x, x \rangle &\le \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \\ &- \begin{cases} \left[\langle Mx - Ax, Ax - mx \rangle \langle M^{-1}x - A^{-1}x, A^{-1}x - m^{-1}x \rangle \right]^{\frac{1}{2}}, \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle A^{-1}x, x \rangle - \frac{M+m}{2mM} \right| \end{cases} \\ &\left(\le \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \right) \end{aligned} \tag{2.27}$$

for any $x \in H$ with $\|x\| = 1$.

2. Now consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = x \ln x$. On utilising the inequality (2.10), then for any positive definite operator A on the Hilbert space H, we have the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot (M-m) \left[\|\ln (eA) x\|^2 - \langle \ln (eA) x, x \rangle^2 \right]^{1/2} \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \left(\leq \frac{1}{2} (M-m) \ln \sqrt{\frac{M}{m}} \right) \tag{2.28}$$

for any $x \in H$ with $\|x\| = 1$.

If we now apply the inequality (2.11), then we have the following result as well

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \frac{1}{2} (M-m) \ln \sqrt{\frac{M}{m}} - \begin{cases} \left[\langle Mx - Ax, Ax - mx \rangle \langle \ln (M) x - \ln (A) x, \ln (A) x - \ln (m) x \rangle \right]^{\frac{1}{2}}, \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle \ln (A) x, x \rangle - \ln \sqrt{mM} \right| \end{cases} \left(\leq \frac{1}{2} (M-m) \ln \sqrt{\frac{M}{m}} \right) \tag{2.29}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if we assume that $m > e^{-1}$, then, by utilising the inequality (2.12) we can state the inequality

$$(0 \leq) \langle A \ln (A) x, x \rangle - \langle Ax, x \rangle \ln (\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot \frac{(M-m) \ln \sqrt{\frac{M}{m}}}{\sqrt{Mm \ln (eM) \ln (em)}} \langle Ax, x \rangle \langle \ln (eA) x, x \rangle, \\ \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{\ln (eM)} - \sqrt{\ln (em)} \right) \left[\langle Ax, x \rangle \langle \ln (eA) x, x \rangle \right]^{\frac{1}{2}}, \end{cases} \tag{2.30}$$

for any $x \in H$ with $\|x\| = 1$.

3. Consider now the following convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(\alpha x)$ with $\alpha > 0$. If we apply the inequalities (2.10), (2.11) and (2.12) for $f(x) = \exp(\alpha x)$ and for a selfadjoint operator A, then we get the following results

$$(0 \leq) \langle \exp (\alpha A) x, x \rangle - \exp (\alpha \langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot \alpha (M-m) \left[\|\exp (\alpha A) x\|^2 - \langle \exp (\alpha A) x, x \rangle^2 \right]^{1/2} \\ \frac{1}{2} \cdot \alpha (\exp (\alpha M) - \exp (\alpha m)) \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \left(\leq \frac{1}{4} \alpha (M-m) (\exp (\alpha M) - \exp (\alpha m)) \right), \tag{2.31}$$

and

$$
\begin{aligned}
(2.32)\qquad & (0\le)\left\langle \exp(\alpha A)x,x\right\rangle-\exp(\alpha\left\langle Ax,x\right\rangle)\\
&\le\frac{1}{4}\alpha(M-m)(\exp(\alpha M)-\exp(\alpha m))\\
&-\alpha\begin{cases}[\left\langle Mx-Ax,Ax-mx\right\rangle]^{1/2}\\ \quad\times[\left\langle \exp(\alpha M)x-\exp(\alpha A)x,\exp(\alpha A)x-\exp(\alpha m)x\right\rangle]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle \exp(\alpha A)x,x\right\rangle-\frac{\exp(\alpha M)+\exp(\alpha m)}{2}\right|\end{cases}\\
&\left(\le\frac{1}{4}\alpha(M-m)(\exp(\alpha M)-\exp(\alpha m))\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.33)\qquad & (0\le)\left\langle \exp(\alpha A)x,x\right\rangle-\exp(\alpha\left\langle Ax,x\right\rangle)\\
&\le\alpha\times\begin{cases}\frac{1}{4}\cdot\frac{(M-m)(\exp(\alpha M)-\exp(\alpha m))}{\sqrt{Mm}\exp\left[\frac{\alpha(M+m)}{2}\right]}\left\langle Ax,x\right\rangle\left\langle \exp(\alpha A)x,x\right\rangle,\\ \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\exp\left(\frac{\alpha M}{2}\right)-\exp\left(\frac{\alpha m}{2}\right)\right)\\ \qquad\times[\left\langle Ax,x\right\rangle\left\langle \exp(\alpha A)x,x\right\rangle]^{\frac{1}{2}}\end{cases}
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$, respectively.

Now, consider the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(-\beta x)$ with $\beta>0$. If we apply the inequalities (2.10) and (2.11) for $f(x)=\exp(-\beta x)$ and for a selfadjoint operator A, then we get the following results

$$
\begin{aligned}
(2.34)\qquad & (0\le)\left\langle \exp(-\beta A)x,x\right\rangle-\exp(-\beta\left\langle Ax,x\right\rangle)\\
&\le\beta\times\begin{cases}\frac{1}{2}\cdot(M-m)\left[\|\exp(-\beta A)x\|^2-\left\langle \exp(-\beta A)x,x\right\rangle^2\right]^{1/2}\\ \\ \frac{1}{2}\cdot(\exp(-\beta m)-\exp(-\beta M))\left[\|Ax\|^2-\left\langle Ax,x\right\rangle^2\right]^{1/2}\end{cases}\\
&\left(\le\frac{1}{4}\beta(M-m)(\exp(-\beta m)-\exp(-\beta M))\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.35)\qquad & (0\le)\left\langle \exp(-\beta A)x,x\right\rangle-\exp(-\beta\left\langle Ax,x\right\rangle)\\
&\le\frac{1}{4}\beta(M-m)(\exp(-\beta m)-\exp(-\beta M))\\
&-\beta\begin{cases}[\left\langle Mx-Ax,Ax-mx\right\rangle]^{1/2}\\ \times[\left\langle \exp(-\beta M)x-\exp(-\beta A)x,\exp(-\beta A)x-\exp(-\beta m)x\right\rangle]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle \exp(-\beta A)x,x\right\rangle-\frac{\exp(-\beta M)+\exp(-\beta m)}{2}\right|\end{cases}\\
&\left(\le\frac{1}{4}\beta(M-m)(\exp(-\beta m)-\exp(-\beta M))\right)
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$, respectively.

4. Finally, if we consider the convex function $f:[0,\infty)\to[0,\infty)$, $f(x)=x^p$ with $p\ge1$, then on applying the inequalities (2.10) and (2.11) for the positive operator A we have the inequalities

$$(0 \le) \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le p \times \begin{cases} \frac{1}{2} \cdot (M-m) \left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot \left(M^{p-1} - m^{p-1}\right) \left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \left(\le \frac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right)\right) \tag{2.36}$$

and

$$(0 \le) \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le \frac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) - p \begin{cases} \left[\langle Mx - Ax, Ax - mx\rangle \left\langle M^{p-1}x - A^{p-1}x, A^{p-1}x - m^{p-1}x\right\rangle\right]^{\frac{1}{2}}, \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\left\langle A^{p-1}x, x\right\rangle - \frac{M^{p-1}+m^{p-1}}{2}\right| \end{cases} \left(\le \frac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right)\right) \tag{2.37}$$

for any $x \in H$ with $\|x\| = 1$, respectively.

If the operator A is positive definite $(m > 0)$ then, by utilising the inequality (2.12), we have

$$(0 \le) \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le p \times \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(M^{p-1}-m^{p-1}\right)}{M^{p/2} m^{p/2}} \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) \left[\langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle\right]^{\frac{1}{2}}, \end{cases} \tag{2.38}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we consider the convex function $f : [0, \infty) \to [0, \infty)$, $f(x) = -x^p$ with $p \in (0, 1)$, then from the inequalities (2.10) and (2.11) and for the positive definite operator A we have the inequalities

$$(0 \le) \langle Ax, x\rangle^p - \langle A^p x, x\rangle \le p \times \begin{cases} \frac{1}{2} \cdot (M-m) \left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot \left(m^{p-1} - M^{p-1}\right) \left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \left(\le \frac{1}{4} p (M-m) \left(m^{p-1} - M^{p-1}\right)\right) \tag{2.39}$$

and

$$(0\leq)\left\langle Ax,x\right\rangle^{p}-\left\langle A^{p}x,x\right\rangle \tag{2.40}$$
$$\leq\frac{1}{4}p\left(M-m\right)\left(m^{p-1}-M^{p-1}\right)$$
$$-p\begin{cases}\left[\left\langle Mx-Ax,Ax-mx\right\rangle\left\langle M^{p-1}x-A^{p-1}x,A^{p-1}x-m^{p-1}x\right\rangle\right]^{\frac{1}{2}},\\ \\ \left|\left\langle Ax,x\right\rangle-\frac{M+m}{2}\right|\left|\left\langle A^{p-1}x,x\right\rangle-\frac{M^{p-1}+m^{p-1}}{2}\right|\end{cases}$$
$$\left(\leq\frac{1}{4}p\left(M-m\right)\left(m^{p-1}-M^{p-1}\right)\right)$$

for any $x\in H$ with $\|x\|=1$, respectively.

Similar results may be stated for the convex function $f:(0,\infty)\to(0,\infty)$, $f(x)=x^{p}$ with $p<0$. However the details are left to the interested reader.

3. Some Slater Type Inequalities

3.1. Slater Type Inequalities for Functions of Real Variables. Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f:I\to\mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $f_{-}'(x)\leq f_{+}'(x)\leq f_{-}'(y)\leq f_{+}'(y)$ which shows that both f_{-}' and f_{+}' are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f:I\to\mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi:I\to[-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right)\subset\mathbb{R}$ and

$$f(x)\geq f(a)+(x-a)\varphi(a)\qquad\text{for any } x,a\in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f_{-}', $f_{+}'\in\partial f$ and if $\varphi\in\partial f$, then

$$f_{-}'(x)\leq\varphi(x)\leq f_{+}'(x)\qquad\text{for any } x\in\mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f=\{f'\}$.

The following result is well known in the literature as *the Slater inequality:*

THEOREM 3.1 (Slater, 1981, [**37**]). *If $f:I\to\mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i\in I, p_i\geq0$ with $P_n:=\sum_{i=1}^{n}p_i>0$ and $\sum_{i=1}^{n}p_i\varphi(x_i)\neq0$, where $\varphi\in\partial f$, then*

$$\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\leq f\left(\frac{\sum_{i=1}^{n}p_ix_i\varphi(x_i)}{\sum_{i=1}^{n}p_i\varphi(x_i)}\right). \tag{3.1}$$

As pointed out in [**5**, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^{n}p_ix_i\varphi(x_i)}{\sum_{i=1}^{n}p_i\varphi(x_i)}\in I, \tag{3.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

3.2. Some Slater Type Inequalities for Operators. The following result holds:

THEOREM 3.2 (Dragomir, 2008, [**10**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H then*

$$0 \leq f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \leq f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right], \tag{3.3}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \tag{3.4}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\langle f'(A) \cdot (t \cdot 1_H - A)x, x\rangle \leq \langle [f(t) \cdot 1_H - f(A)]x, x\rangle \leq \langle f'(t) \cdot (t \cdot 1_H - A)x, x\rangle \tag{3.5}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (3.5) is equivalent with

$$\begin{aligned} t\langle f'(A)x, x\rangle - \langle f'(A)Ax, x\rangle &\leq f(t) - \langle f(A)x, x\rangle \\ &\leq f'(t)t - f'(t)\langle Ax, x\rangle \end{aligned} \tag{3.6}$$

for any $t \in [m, M]$ any $x \in H$ with $\|x\| = 1$.

Now, since A is selfadjoint with $mI \leq A \leq MI$ and $f'(A)$ is positive definite, then $mf'(A) \leq Af'(A) \leq Mf'(A)$, i.e., $m\langle f'(A)x, x\rangle \leq \langle Af'(A)x, x\rangle \leq M\langle f'(A)x, x\rangle$ for any $x \in H$ with $\|x\| = 1$, which shows that

$$t_0 := \frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in [m, M] \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1.$$

Finally, if we put $t = t_0$ in the equation (3.6), then we get the desired result (3.3). ∎

REMARK 3.1. It is important to observe that, the condition that $f'(A)$ is a positive definite operator on H can be replaced with the more general assumption that

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in \mathring{I} \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1, \tag{3.7}$$

which may be easily verified for particular convex functions f.

REMARK 3.2. Now, if the functions is concave on $\mathring{I}$ and the condition (3.7) holds, then we have the inequality

$$0 \leq \langle f(A)x, x\rangle - f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) \leq f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Ax, x\rangle\langle f'(A)x, x\rangle - \langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right], \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

The following examples are of interest:

EXAMPLE 3.1. *If A is a positive definite operator on H, then*

$$(0 \leq) \langle \ln Ax, x \rangle - \ln\left(\left\langle A^{-1}x, x \right\rangle^{-1}\right) \leq \langle Ax, x \rangle \cdot \left\langle A^{-1}x, x \right\rangle - 1, \tag{3.9}$$

for any $x \in H$ with $\|x\| = 1$.

Indeed, we observe that if we consider the concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} = \frac{1}{\langle A^{-1}x, x\rangle} \in (0, \infty), \quad \text{for any } x \in H \quad \text{with} \quad \|x\| = 1$$

and by the inequality (3.8) we deduce the desired result (3.9).

The following example concerning powers of operators is of interest as well:

EXAMPLE 3.2. *If A is a positive definite operator on H, then for any $x \in H$ with $\|x\| = 1$ we have*

$$\begin{aligned} 0 &\leq \langle A^p x, x\rangle^{p-1} - \left\langle A^{p-1}x, x\right\rangle^p \\ &\leq p \langle A^p x, x\rangle^{p-2} \left[\langle A^p x, x\rangle - \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle\right] \end{aligned} \tag{3.10}$$

for $p \geq 1$,

$$\begin{aligned} 0 &\leq \left\langle A^{p-1}x, x\right\rangle^p - \langle A^p x, x\rangle^{p-1} \\ &\leq p \langle A^p x, x\rangle^{p-2} \left[\langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle - \langle A^p x, x\rangle\right] \end{aligned} \tag{3.11}$$

for $0 < p < 1$, and

$$\begin{aligned} 0 &\leq \langle A^p x, x\rangle^{p-1} - \left\langle A^{p-1}x, x\right\rangle^p \\ &\leq (-p) \langle A^p x, x\rangle^{p-2} \left[\langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle - \langle A^p x, x\rangle\right] \end{aligned} \tag{3.12}$$

for $p < 0$.

The proof follows from the inequalities (3.3) and (3.8) for the convex (concave) function $f(t) = t^p, p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$ by performing the required calculation. The details are omitted.

3.3. Further Reverses. The following results that provide perhaps more useful upper bounds for the nonnegative quantity

$$f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \quad \text{for } x \in H \quad \text{with} \quad \|x\| = 1,$$

can be stated:

THEOREM 3.3 (Dragomir, 2008, [**10**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. Assume that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive definite operator on H. If we define*

$$B(f', A; x) := \frac{1}{\langle f'(A)x, x\rangle} \cdot f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)$$

then

$$
\begin{aligned}
(3.13)\qquad &(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \\
&\leq B(f',A;x)\times\begin{cases}\frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2-\langle f'(A)x,x\rangle^2\right]^{1/2}\\[2mm] \frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}\end{cases}\\
&\leq\frac{1}{4}(M-m)(f'(M)-f'(m))B(f',A;x)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.14)\qquad &(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \\
&\leq B(f',A;x)\times\left[\frac{1}{4}(M-m)(f'(M)-f'(m))\right.\\
&\left.-\begin{cases}[\langle Mx-Ax,Ax-mx\rangle\langle f'(M)x-f'(A)x,f'(A)x-f'(m)x\rangle]^{\frac{1}{2}},\\ \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle-\frac{f'(M)+f'(m)}{2}\right|\end{cases}\right]\\
&\leq\frac{1}{4}(M-m)(f'(M)-f'(m))B(f',A;x),
\end{aligned}
$$

for any $x\in H$ *with* $\|x\|=1$, *respectively.*

Moreover, if A *is a positive definite operator, then*

$$
\begin{aligned}
(3.15)\qquad &(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle \\
&\leq B(f',A;x)\\
&\times\begin{cases}\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle,\\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle\langle f'(A)x,x\rangle]^{\frac{1}{2}},\end{cases}
\end{aligned}
$$

for any $x\in H$ *with* $\|x\|=1$.

PROOF. We use the following Grüss' type result we obtained in **[6]**:

Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If h and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}h(t)$ and $\Gamma:=\max_{t\in[m,M]}h(t)$, then

$$
\begin{aligned}
(3.16)\qquad &|\langle h(A)g(A)x,x\rangle-\langle h(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\\
&\leq\frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)x\|^2-\langle g(A)x,x\rangle^2\right]^{1/2}\\
&\left(\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right),
\end{aligned}
$$

for each $x\in H$ with $\|x\|=1$, where $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$.

Therefore, we can state that

$$\begin{aligned}\langle Af'(A)x,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle\\ \le\frac{1}{2}\cdot(M-m)\left[\|f'(A)x\|^2-\langle f'(A)x,x\rangle^2\right]^{1/2}\\ \le\frac{1}{4}(M-m)(f'(M)-f'(m))\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}\langle Af'(A)x,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle\\ \le\frac{1}{2}\cdot(f'(M)-f'(m))\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}\\ \le\frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned}\tag{3.18}$$

for each $x\in H$ with $\|x\|=1$, which together with (3.3) provide the desired result (3.13).

On making use of the inequality obtained in [**7**]

$$\begin{aligned}&|\langle h(A)g(A)x,x\rangle-\langle h(A)x,x\rangle\langle g(A)x,x\rangle|\\ &\le\frac{1}{4}\cdot(\Gamma-\gamma)(\Delta-\delta)\\ &-\begin{cases}[\langle\Gamma x-h(A)x,f(A)x-\gamma x\rangle\langle\Delta x-g(A)x,g(A)x-\delta x\rangle]^{\frac{1}{2}},\\[2mm] \left|\langle h(A)x,x\rangle-\frac{\Gamma+\gamma}{2}\right|\left|\langle g(A)x,x\rangle-\frac{\Delta+\delta}{2}\right|,\end{cases}\end{aligned}\tag{3.19}$$

for each $x\in H$ with $\|x\|=1$, we can state that

$$\begin{aligned}&\langle Af'(A)x,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle\\ &\le\frac{1}{4}(M-m)(f'(M)-f'(m))\\ &-\begin{cases}[\langle Mx-Ax,Ax-mx\rangle\langle f'(M)x-f'(A)x,f'(A)x-f'(m)x\rangle]^{\frac{1}{2}},\\[2mm] \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle-\frac{f'(M)+f'(m)}{2}\right|,\end{cases}\end{aligned}$$

for each $x\in H$ with $\|x\|=1$, which together with (3.3) provide the desired result (3.14).

Further, in order to prove the third inequality, we make use of the following result of Grüss type obtained in [**7**]:

If γ and δ are positive, then

$$\begin{aligned}&|\langle h(A)g(A)x,x\rangle-\langle h(A)x,x\rangle\langle g(A)x,x\rangle|\\ &\le\begin{cases}\frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\langle h(A)x,x\rangle\langle g(A)x,x\rangle,\\[2mm] \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle h(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}},\end{cases}\end{aligned}\tag{3.20}$$

for each $x\in H$ with $\|x\|=1$.

Now, on making use of (3.20) we can state that

$$\langle Af'(A)x,x\rangle - \langle Ax,x\rangle \cdot \langle f'(A)x,x\rangle \leq \begin{cases} \frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}\langle Ax,x\rangle\langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)\left[\langle Ax,x\rangle\langle f'(A)x,x\rangle\right]^{\frac{1}{2}}, \end{cases} \tag{3.21}$$

for each $x\in H$ with $\|x\|=1,$ which together with (3.3) provide the desired result (3.15). ∎

REMARK 3.3. We observe, from the first inequality in (3.15), that

$$(1\leq)\,\frac{\langle Af'(A)x,x\rangle}{\langle Ax,x\rangle\langle f'(A)x,x\rangle} \leq \frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1$$

which implies that

$$f'\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right) \leq f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right),$$

for each $x\in H$ with $\|x\|=1,$ since f' is monotonic nondecreasing and A is positive definite.

Now, the first inequality in (3.15) implies the following result

$$\begin{aligned}(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle &\leq \frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \\ &\times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)\langle Ax,x\rangle,\end{aligned} \tag{3.22}$$

for each $x\in H$ with $\|x\|=1.$

From the second inequality in (3.15) we also have

$$\begin{aligned}(0\leq)f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle &\leq \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right) \\ &\times f'\left(\left[\frac{1}{4}\cdot\frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}}+1\right]\langle Ax,x\rangle\right)\left[\frac{\langle Ax,x\rangle}{\langle f'(A)x,x\rangle}\right]^{\frac{1}{2}},\end{aligned} \tag{3.23}$$

for each $x\in H$ with $\|x\|=1.$

REMARK 3.4. If the condition that $f'(A)$ is a positive definite operator on H from the Theorem 3.3 is replaced by the condition (3.7), then the inequalities (3.13) and (3.16) will still hold. Similar inequalities for concave functions can be stated. However, the details are not provided here.

3.4. Multivariate Versions. The following result for sequences of operators can be stated.

THEOREM 3.4 (Dragomir, 2008, [**10**]). *Let I be an interval and $f: I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If*

$A_j, j \in \{1, \dots, n\}$ *are selfadjoint operators on the Hilbert space* H *with* $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$ *and*

$$\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle} \in \mathring{I} \tag{3.24}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then*

$$\begin{aligned} 0 &\le f\left(\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right) - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \\ &\le f'\left(\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right) \\ &\quad \times \left[\frac{\sum_{j=1}^{n} \langle A_j f'(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle}\right], \end{aligned} \tag{3.25}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

PROOF. Follows from Theorem 3.2. The details are omitted. ∎

The following particular case is of interest

COROLLARY 3.5 (Dragomir, 2008, [**10**]). *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I) *whose derivative* f' *is continuous on* $\mathring{I}$. *If* $A_j, j \in \{1, \dots, n\}$ *are selfadjoint operators on the Hilbert space* H *with* $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$ *and for* $p_j \ge 0$ *with* $\sum_{j=1}^{n} p_j = 1$ *if we also assume that*

$$\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle} \in \mathring{I} \tag{3.26}$$

for each $x \in H$ *with* $\|x\| = 1$, *then*

$$\begin{aligned} 0 &\le f\left(\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle}\right) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \\ &\le f'\left(\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle}\right) \\ &\quad \times \left[\frac{\left\langle \sum_{j=1}^{n} p_j A_j f'(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j f'(A_j) x, x \right\rangle}\right], \end{aligned} \tag{3.27}$$

for each $x \in H$ *with* $\|x\| = 1$.

PROOF. Follows from Theorem 3.4 on choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \dots, n\}$, where $p_j \ge 0, j \in \{1, \dots, n\}, \sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

The following examples are interesting in themselves:

EXAMPLE 3.3. *If A_j, $j \in \{1, \dots, n\}$ are positive definite operators on H, then*

$$(0 \le) \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \ln \left[\left(\sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \right)^{-1} \right] \le \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle - 1, \tag{3.28}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

If $p_j \ge 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1,$ then we also have the inequality

$$(0 \le) \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle - \ln \left[\left(\left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle \right)^{-1} \right] \le \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle - 1, \tag{3.29}$$

for each $x \in H$ with $\|x\| = 1.$

4. OTHER INEQUALITIES FOR CONVEX FUNCTIONS

4.1. Some Inequalities for Two Operators. The following result holds:

THEOREM 4.1 (Dragomir, 2008, [**11**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned} &\langle f'(A) x, x \rangle \langle By, y \rangle - \langle f'(A) Ax, x \rangle \\ &\le \langle f(B) y, y \rangle - \langle f(A) x, x \rangle \\ &\le \langle f'(B) By, y \rangle - \langle Ax, x \rangle \langle f'(B) y, y \rangle \end{aligned} \tag{4.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

In particular, we have

$$\begin{aligned} &\langle f'(A) x, x \rangle \langle Ay, y \rangle - \langle f'(A) Ax, x \rangle \\ &\le \langle f(A) y, y \rangle - \langle f(A) x, x \rangle \\ &\le \langle f'(A) Ay, y \rangle - \langle Ax, x \rangle \langle f'(A) y, y \rangle \end{aligned} \tag{4.2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\begin{aligned} &\langle f'(A) x, x \rangle \langle Bx, x \rangle - \langle f'(A) Ax, x \rangle \\ &\le \langle f(B) x, x \rangle - \langle f(A) x, x \rangle \\ &\le \langle f'(B) Bx, x \rangle - \langle Ax, x \rangle \langle f'(B) x, x \rangle \end{aligned} \tag{4.3}$$

for any $x \in H$ with $\|x\| = 1.$

PROOF. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \le f(t) - f(s) \le f'(t) \cdot (t - s) \tag{4.4}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\langle f'(A) \cdot (t \cdot 1_H - A) x, x \rangle \leq \langle [f(t) \cdot 1_H - f(A)] x, x \rangle \leq \langle f'(t) \cdot (t \cdot 1_H - A) x, x \rangle \tag{4.5}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (4.5) is equivalent with

$$\begin{aligned} t \langle f'(A) x, x \rangle - \langle f'(A) Ax, x \rangle &\leq f(t) - \langle f(A) x, x \rangle \\ &\leq f'(t) t - f'(t) \langle Ax, x \rangle \end{aligned} \tag{4.6}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (4.6) and apply the property (P) for the operator B, then we get

$$\begin{aligned} &\langle [\langle f'(A) x, x \rangle B - \langle f'(A) Ax, x \rangle 1_H] y, y \rangle \\ &\leq \langle [f(B) - \langle f(A) x, x \rangle 1_H] y, y \rangle \\ &\leq \langle [f'(B) B - \langle Ax, x \rangle f'(B)] y, y \rangle \end{aligned}$$

for each $y \in H$ with $\|y\| = 1$, which is clearly equivalent to the desired inequality (4.1). ∎

REMARK 4.1. If we fix $x \in H$ with $\|x\| = 1$ and choose $B = \langle Ax, x \rangle \cdot 1_H$, then we obtain from the first inequality in (4.1) the reverse of the Mond-Pečarić inequality obtained by the author in [**9**]. The second inequality will provide the Mond-Pečarić inequality for convex functions whose derivatives are continuous.

The following corollary is of interest:

COROLLARY 4.2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. Also, suppose that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$. If g is nonincreasing and continuous on $[m, M]$ and*

$$f'(A) [g(A) - A] \geq 0 \tag{4.7}$$

in the operator order of $B(H)$, then

$$(f \circ g)(A) \geq f(A) \tag{4.8}$$

in the operator order of $B(H)$.

PROOF. If we apply the first inequality from (4.3) for $B = g(A)$ we have

$$\begin{aligned} &\langle f'(A) x, x \rangle \langle g(A) x, x \rangle - \langle f'(A) Ax, x \rangle \\ &\leq \langle f(g(A)) x, x \rangle - \langle f(A) x, x \rangle \end{aligned} \tag{4.9}$$

any $x \in H$ with $\|x\| = 1$.

We use the following Čebyšev type inequality for functions of operators established by the author in [**8**]:

Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $h, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and *synchronous (asynchronous)* on $[m, M]$, then

$$\langle h(A) g(A) x, x \rangle \geq (\leq) \langle h(A) x, x \rangle \cdot \langle g(A) x, x \rangle \tag{4.10}$$

for any $x \in H$ with $\|x\| = 1$.

Now, since f' and g are continuous and are asynchronous on $[m, M]$, then by (4.10) we have the inequality

$$\langle f'(A) g(A) x, x\rangle \leq \langle f'(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{4.11}$$

for any $x \in H$ with $\|x\| = 1.$

Subtracting from both sides of (4.11) the quantity $\langle f'(A) Ax, x\rangle$ and taking into account, by (4.7), that $\langle f'(A)[g(A) - A] x, x\rangle \geq 0$ for any $x \in H$ with $\|x\| = 1$, we then have

$$\begin{aligned} 0 &\leq \langle f'(A)[g(A) - A] x, x\rangle \\ &= \langle f'(A) g(A) x, x\rangle - \langle f'(A) Ax, x\rangle \\ &\leq \langle f'(A) x, x\rangle \cdot \langle g(A) x, x\rangle - \langle f'(A) Ax, x\rangle \end{aligned}$$

which together with (4.9) will produce the desired result (4.8). ∎

We provide now some particular inequalities of interest that can be derived from Theorem 4.1:

EXAMPLE 4.1. ***a.*** *Let A, B two positive definite operators on H. Then we have the inequalities*

$$\begin{aligned} 1 - \left\langle A^{-1}x, x\right\rangle \langle By, y\rangle &\leq \langle \ln Ax, x\rangle - \langle \ln By, y\rangle \\ &\leq \langle Ax, x\rangle \left\langle B^{-1}y, y\right\rangle - 1 \end{aligned} \tag{4.12}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

In particular, we have

$$\begin{aligned} 1 - \left\langle A^{-1}x, x\right\rangle \langle Ay, y\rangle &\leq \langle \ln Ax, x\rangle - \langle \ln Ay, y\rangle \\ &\leq \langle Ax, x\rangle \left\langle A^{-1}y, y\right\rangle - 1 \end{aligned} \tag{4.13}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\begin{aligned} 1 - \left\langle A^{-1}x, x\right\rangle \langle Bx, x\rangle &\leq \langle \ln Ax, x\rangle - \langle \ln Bx, x\rangle \\ &\leq \langle Ax, x\rangle \left\langle B^{-1}x, x\right\rangle - 1 \end{aligned} \tag{4.14}$$

for any $x \in H$ with $\|x\| = 1.$

b. *With the same assumption for A and B we have the inequalities*

$$\langle By, y\rangle - \langle Ax, x\rangle \leq \langle B \ln By, y\rangle - \langle \ln Ax, x\rangle \langle By, y\rangle \tag{4.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

In particular, we have

$$\langle Ay, y\rangle - \langle Ax, x\rangle \leq \langle A \ln Ay, y\rangle - \langle \ln Ax, x\rangle \langle Ay, y\rangle \tag{4.16}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\langle Bx, x\rangle - \langle Ax, x\rangle \leq \langle B \ln Bx, x\rangle - \langle \ln Ax, x\rangle \langle Bx, x\rangle \tag{4.17}$$

for any $x \in H$ with $\|x\| = 1.$

The proof of Example **a** follows from Theorem 4.1 for the convex function $f(x) = -\ln x$ while the proof of the second example follows by the same theorem applied for the convex function $f(x) = x \ln x$ and performing the required calculations. The details are omitted.

The following result may be stated as well:

THEOREM 4.3 (Dragomir, 2008, [**11**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle By, y\rangle - \langle Ax, x\rangle) &\\ \leq \langle f(B) y, y\rangle - f(\langle Ax, x\rangle) &\\ \leq \langle f'(B) By, y\rangle - \langle Ax, x\rangle \langle f'(B) y, y\rangle & \end{aligned} \tag{4.18}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle Ay, y\rangle - \langle Ax, x\rangle) &\\ \leq \langle f(A) y, y\rangle - f(\langle Ax, x\rangle) &\\ \leq \langle f'(A) Ay, y\rangle - \langle Ax, x\rangle \langle f'(A) y, y\rangle & \end{aligned} \tag{4.19}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle Bx, x\rangle - \langle Ax, x\rangle) &\\ \leq \langle f(B) x, x\rangle - f(\langle Ax, x\rangle) &\\ \leq \langle f'(B) Bx, x\rangle - \langle Ax, x\rangle \langle f'(B) x, x\rangle & \end{aligned} \tag{4.20}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \tag{4.21}$$

for any $t, s \in [m, M]$.

If we choose $s = \langle Ax, x\rangle \in [m, M]$, with a fix $x \in H$ with $\|x\| = 1$, then we have

$$f'(\langle Ax, x\rangle) \cdot (t - \langle Ax, x\rangle) \leq f(t) - f(\langle Ax, x\rangle) \leq f'(t) \cdot (t - \langle Ax, x\rangle) \tag{4.22}$$

for any $t \in [m, M]$.

Now, if we apply the property (P) to the inequality (4.22) and the operator B, then we get

$$\begin{aligned} \langle f'(\langle Ax, x\rangle) \cdot (B - \langle Ax, x\rangle \cdot 1_H) y, y\rangle &\\ \leq \langle [f(B) - f(\langle Ax, x\rangle) \cdot 1_H] y, y\rangle &\\ \leq \langle f'(B) \cdot (B - \langle Ax, x\rangle \cdot 1_H) y, y\rangle & \end{aligned} \tag{4.23}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is equivalent with the desired result (4.18). ∎

REMARK 4.2. We observe that if we choose $B = A$ in (4.20) or $y = x$ in (4.19) then we recapture the Mond-Pečarić inequality and its reverse from (2.1).

The following particular case of interest follows from Theorem 4.3

COROLLARY 4.4 (Dragomir, 2008, [**11**]). *Assume that f, A and B are as in Theorem 4.3. If, either f is increasing on $[m, M]$ and $B \geq A$ in the operator order of $B(H)$ or f is decreasing and $B \leq A$, then we have the Jensen's type inequality*

$$\langle f(B) x, x\rangle \geq f(\langle Ax, x\rangle) \tag{4.24}$$

for any $x \in H$ with $\|x\| = 1$.

The proof is obvious by the first inequality in (4.20) and the details are omitted.

We provide now some particular inequalities of interest that can be derived from Theorem 4.3:

EXAMPLE 4.2. ***a.*** *Let A, B be two positive definite operators on H. Then we have the inequalities*

$$1-\langle Ax,x\rangle^{-1}\langle By,y\rangle\leq\ln\left(\langle Ax,x\rangle\right)-\langle\ln By,y\rangle\leq\langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1 \tag{4.25}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$1-\langle Ax,x\rangle^{-1}\langle Ay,y\rangle\leq\ln\left(\langle Ax,x\rangle\right)-\langle\ln Ay,y\rangle\leq\langle Ax,x\rangle\left\langle A^{-1}y,y\right\rangle-1 \tag{4.26}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$1-\langle Ax,x\rangle^{-1}\langle Bx,x\rangle\leq\ln\left(\langle Ax,x\rangle\right)-\langle\ln Bx,x\rangle\leq\langle Ax,x\rangle\left\langle B^{-1}x,x\right\rangle-1 \tag{4.27}$$

for any $x \in H$ with $\|x\| = 1$.

b. *With the same assumption for A and B, we have the inequalities*

$$\langle By,y\rangle-\langle Ax,x\rangle\leq\langle B\ln By,y\rangle-\langle By,y\rangle\ln\left(\langle Ax,x\rangle\right) \tag{4.28}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\langle Ay,y\rangle-\langle Ax,x\rangle\leq\langle A\ln Ay,y\rangle-\langle Ay,y\rangle\ln\left(\langle Ax,x\rangle\right) \tag{4.29}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\langle Bx,x\rangle-\langle Ax,x\rangle\leq\langle B\ln Bx,x\rangle-\langle Bx,x\rangle\ln\left(\langle Ax,x\rangle\right) \tag{4.30}$$

for any $x \in H$ with $\|x\| = 1$.

4.2. Inequalities for Two Sequences of Operators. The following result may be stated:

THEOREM 4.5 (Dragomir, 2008, [**11**]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, \ldots, n\}$, then*

$$\begin{aligned}&\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle\sum_{j=1}^{n}\langle B_jy_j,y_j\rangle-\sum_{j=1}^{n}\langle f'(A_j)A_jx_j,x_j\rangle\\&\leq\sum_{j=1}^{n}\langle f(B_j)y_j,y_j\rangle-\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle\\&\leq\sum_{j=1}^{n}\langle f'(B_j)B_jy_j,y_j\rangle-\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\sum_{j=1}^{n}\langle f'(B_j)y_j,y_j\rangle\end{aligned} \tag{4.31}$$

for any $x_j, y_j \in H$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = \sum_{j=1}^{n}\|y_j\|^2 = 1$.

In particular, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\left\langle f'\left(A_j\right)x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle A_jy_j,y_j\right\rangle-\sum_{j=1}^{n}\left\langle f'\left(A_j\right)A_jx_j,x_j\right\rangle \\
&\le\sum_{j=1}^{n}\left\langle f\left(A_j\right)y_j,y_j\right\rangle-\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \\
&\le\sum_{j=1}^{n}\left\langle f'\left(A_j\right)A_jy_j,y_j\right\rangle-\sum_{j=1}^{n}\left\langle A_jx_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle f'\left(A_j\right)y_j,y_j\right\rangle
\end{aligned}\tag{4.32}$$

for any $x_j,y_j\in H$, $j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=\sum_{j=1}^{n}\|y_j\|^2=1$ *and*

$$\begin{aligned}
&\sum_{j=1}^{n}\left\langle f'\left(A_j\right)x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle B_jx_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle f'\left(A_j\right)A_jx_j,x_j\right\rangle \\
&\le\sum_{j=1}^{n}\left\langle f\left(B_j\right)x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \\
&\le\sum_{j=1}^{n}\left\langle f'\left(B_j\right)B_jx_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_jx_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle f'\left(B_j\right)x_j,x_j\right\rangle
\end{aligned}\tag{4.33}$$

for any $x_j\in H$, $j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1.$

PROOF. Follows from Theorem 4.1 and the details are omitted. ∎

The following particular case may be of interest:

COROLLARY 4.6 (Dragomir, 2008, [**11**]). *Let* I *be an interval and* $f:I\to\mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$*. If* A_j *and* B_j *are selfadjoint operators on the Hilbert space* H *with* $Sp\left(A_j\right),Sp\left(B_j\right)\subseteq[m,M]\subset\mathring{I}$ *for any* $j\in\{1,\dots,n\}$*, then for any* $p_j,q_j\ge 0$ *with* $\sum_{j=1}^{n}p_j=\sum_{j=1}^{n}q_j=1$*, we have the inequalities*

$$\begin{aligned}
&\left\langle\sum_{j=1}^{n}p_jf'\left(A_j\right)x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jB_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf'\left(A_j\right)A_jx,x\right\rangle \\
&\le\left\langle\sum_{j=1}^{n}q_jf\left(B_j\right)y,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf\left(A_j\right)x,x\right\rangle \\
&\le\left\langle\sum_{j=1}^{n}q_jf'\left(B_j\right)B_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\left\langle\sum_{j=1}^{n}q_jf'\left(B_j\right)y,y\right\rangle
\end{aligned}\tag{4.34}$$

for any $x,y\in H$ *with* $\|x\|=\|y\|=1.$

In particular, we have

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j)\, x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j A_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j)\, A_j x, x \right\rangle \le \left\langle \sum_{j=1}^{n} q_j f(A_j)\, y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j)\, x, x \right\rangle \le \left\langle \sum_{j=1}^{n} q_j f'(A_j)\, B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j f'(A_j)\, y, y \right\rangle \tag{4.35}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j)\, x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j)\, A_j x, x \right\rangle \le \left\langle \sum_{j=1}^{n} p_j f(B_j)\, x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j)\, x, x \right\rangle \le \left\langle \sum_{j=1}^{n} p_j f'(B_j)\, B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(B_j)\, x, x \right\rangle \tag{4.36}$$

for any $x \in H$ *with* $\|x\| = 1.$

PROOF. Follows from Theorem 4.5 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{q_j} \cdot y$, $j \in \{1, \dots, n\}$, where $p_j, q_j \ge 0$, $j \in \{1, \dots, n\}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1.$ The details are omitted. ∎

EXAMPLE 4.3. ***a.*** *Let* A_j, B_j, $j \in \{1, \dots, n\}$, *be two sequences of positive definite operators on* H. *Then we have the inequalities*

$$1 - \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle \le \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle \ln B_j y_j, y_j \rangle \le \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \left\langle B_j^{-1} y_j, y_j \right\rangle - 1 \tag{4.37}$$

for any $x_j, y_j \in H$, $j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1.$

b. *With the same assumption for* A_j *and* B_j *we have the inequalities*

$$\sum_{j=1}^{n} \langle B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \le \sum_{j=1}^{n} \langle B_j \ln B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle \tag{4.38}$$

for any $x_j, y_j \in H$, $j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1.$

Finally, we have

EXAMPLE 4.4. ***a.*** *Let A_j, B_j, $j \in \{1, \dots, n\}$, be two sequences of positive definite operators on H. Then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$, we have the inequalities*

$$1 - \left\langle \sum_{j=1}^n p_j A_j^{-1} x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle \leq \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^n q_j \ln B_j y, y \right\rangle \leq \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j^{-1} y, y \right\rangle - 1 \tag{4.39}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

b. *With the same assumption for A_j, B_j, p_j and q_j, we have the inequalities*

$$\left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \leq \left\langle \sum_{j=1}^n q_j B_j \ln B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle \tag{4.40}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

REMARK 4.3. We observe that all the other inequalities for two operators obtained in Subsection 3.1 can be extended for two sequences of operators in a similar way. However, the details are left to the interested reader.

5. SOME JENSEN TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

5.1. Jensen's Inequality for Twice Differentiable Functions. The following result may be stated:

THEOREM 5.1 (Dragomir, 2008, [**12**]). *Let A be a positive definite operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a twice differentiable function on (m, M) and for $p \in (-\infty, 0) \cup (1, \infty)$ we have for some $\gamma < \Gamma$ that*

$$\gamma \leq \frac{t^{2-p}}{p(p-1)} \cdot f''(t) \leq \Gamma \quad \text{for any} \quad t \in (m, M), \tag{5.1}$$

then

$$\gamma\left(\langle A^p x, x\rangle - \langle Ax, x\rangle^p\right) \leq \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \leq \Gamma\left(\langle A^p x, x\rangle - \langle Ax, x\rangle^p\right) \tag{5.2}$$

for each $x \in H$ with $\|x\| = 1$.

If

$$\delta \leq \frac{t^{2-p}}{p(1-p)} \cdot f''(t) \leq \Delta \quad \text{for any} \quad t \in (m, M) \tag{5.3}$$

and for some $\delta < \Delta$, where $p \in (0, 1)$, then

$$\delta\left(\langle Ax, x\rangle^p - \langle A^p x, x\rangle\right) \leq \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \leq \Delta\left(\langle Ax, x\rangle^p - \langle A^p x, x\rangle\right) \tag{5.4}$$

for each $x \in H$ *with* $\|x\| = 1.$

PROOF. Consider the function $g_{\gamma,p} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,p}(t) = f(t) - \gamma t^p$ where $p \in (-\infty, 0) \cup (1, \infty)$. The function $g_{\gamma,p}$ is twice differentiable,

$$g''_{\gamma,p}(t) = f''(t) - \gamma p (p-1) t^{p-2}$$

for any $t \in (m, M)$ and by (5.1) we deduce that $g_{\gamma,p}$ is convex on (m, M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,p}$ we have

$$\begin{aligned} 0 &\le \langle (f(A) - \gamma A^p) x, x\rangle - [f(\langle Ax, x\rangle) - \gamma \langle Ax, x\rangle^p] \\ &= \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) - \gamma [\langle A^p x, x\rangle - \langle Ax, x\rangle^p] \end{aligned}$$

which is equivalent with the first inequality in (5.2).

By defining the function $g_{\Gamma,p} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,p}(t) = \Gamma t^p - f(t)$ and applying the same argument we deduce the second part of (5.2).

The rest goes likewise and the details are omitted. ∎

REMARK 5.1. We observe that if f is a twice differentiable function on (m, M) and $\varphi := \inf_{t\in(m,M)} f''(t)$, $\Phi := \sup_{t\in(m,M)} f''(t)$, then by (5.2) we get the inequality

$$\begin{aligned} \frac{1}{2}\varphi \left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right] &\le \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \\ &\le \frac{1}{2}\Phi \left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right] \end{aligned} \tag{5.5}$$

for each $x \in H$ with $\|x\| = 1.$

We observe that the inequality (5.5) holds for selfadjoint operators that are not necessarily positive.

The following version for sequences of operators can be stated:

COROLLARY 5.2 (Dragomir, 2008, [**11**]). *Let* A_j *be positive definite operators with* $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \dots, n\}$. *If* f *is a twice differentiable function on* (m, M) *and for* $p \in (-\infty, 0) \cup (1, \infty)$ *we have the condition (5.1), then*

$$\begin{aligned} &\gamma \left[\sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p\right] \\ &\le \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \\ &\le \Gamma \left[\sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p\right] \end{aligned} \tag{5.6}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1.$

If we have the condition (5.3) for $p \in (0,1)$, then

$$\delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p}-\sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right] \tag{5.7}$$
$$\leq \sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)$$
$$\leq \Delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p}-\sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right]$$

for each $x_j \in H, j \in \{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

PROOF. Follows from Theorem 5.1. ∎

COROLLARY 5.3 (Dragomir, 2008, [**11**]). *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m,M] \subset (0,\infty)$ $j \in \{1,\dots,n\}$ and $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^{n} p_j = 1$. If f is a twice differentiable function on (m,M) and for $p \in (-\infty,0)\cup(1,\infty)$ we have the condition (5.1), then*

$$\gamma\left[\left\langle\sum_{j=1}^{n} p_j A_j^{p} x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{p}\right] \tag{5.8}$$
$$\leq \left\langle\sum_{j=1}^{n} p_j f(A_j)x, x\right\rangle - f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)$$
$$\leq \Gamma\left[\left\langle\sum_{j=1}^{n} p_j A_j^{p} x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{p}\right]$$

for each $x \in H$ with $\|x\| = 1$.

If we have the condition (5.3) for $p \in (0,1)$, then

$$\delta\left[\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{p} - \left\langle\sum_{j=1}^{n} p_j A_j^{p} x, x\right\rangle\right] \tag{5.9}$$
$$\leq \left\langle\sum_{j=1}^{n} p_j f(A_j)x, x\right\rangle - f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)$$
$$\leq \Delta\left[\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{p} - \left\langle\sum_{j=1}^{n} p_j A_j^{p} x, x\right\rangle\right]$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Follows from Corollary 5.2 on choosing $x_j = \sqrt{p_j}\cdot x,\ j \in \{1,\dots,n\}$, where $p_j \geq 0, j \in \{1,\dots,n\}, \sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

REMARK 5.2. We observe that if f is a twice differentiable function on (m,M) with $-\infty < m < M < \infty$, $Sp(A_j) \subset [m,M], j \in \{1,\dots,n\}$ and $\varphi := \inf_{t\in(m,M)} f''(t), \Phi :=$

$\sup_{t\in(m,M)} f''(t)$, then

$$\varphi\left[\sum_{j=1}^{n}\left\langle A_j^2 x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^2\right] \tag{5.10}$$
$$\leq \sum_{j=1}^{n}\left\langle f(A_j) x_j, x_j\right\rangle - f\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)$$
$$\leq \Phi\left[\sum_{j=1}^{n}\left\langle A_j^2 x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^2\right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Also, if $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$, then

$$\varphi\left[\left\langle\sum_{j=1}^{n} p_j A_j^2 x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^2\right] \tag{5.11}$$
$$\leq \left\langle\sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle - f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)$$
$$\leq \Phi\left[\left\langle\sum_{j=1}^{n} p_j A_j^2 x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^2\right]$$

The next result provides some inequalities for the function f which replace the cases $p = 0$ and $p = 1$ that were not allowed in Theorem 5.1:

THEOREM 5.4 (Dragomir, 2008, [**11**]). *Let A be a positive definite operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a twice differentiable function on (m, M) and we have for some $\gamma < \Gamma$ that*

$$\gamma \leq t^2 \cdot f''(t) \leq \Gamma \quad \textit{for any} \quad t \in (m, M), \tag{5.12}$$

then

$$\gamma(\ln(\langle Ax, x\rangle) - \langle \ln Ax, x\rangle) \leq \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \tag{5.13}$$
$$\leq \Gamma(\ln(\langle Ax, x\rangle) - \langle \ln Ax, x\rangle)$$

for each $x \in H$ with $\|x\| = 1$.

If

$$\delta \leq t \cdot f''(t) \leq \Delta \quad \textit{for any} \quad t \in (m, M) \tag{5.14}$$

for some $\delta < \Delta$, then

$$\delta(\langle A \ln Ax, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle)) \tag{5.15}$$
$$\leq \langle f(A) x, x\rangle - f(\langle Ax, x\rangle)$$
$$\leq \Delta(\langle A \ln Ax, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle))$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Consider the function $g_{\gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,0}(t) = f(t) + \gamma \ln t$. The function $g_{\gamma,0}$ is twice differentiable,

$$g''_{\gamma,p}(t) = f''(t) - \gamma t^{-2}$$

for any $t \in (m, M)$ and by (5.12) we deduce that $g_{\gamma,0}$ is convex on (m, M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,0}$ we have

$$\begin{aligned} 0 &\leq \langle (f(A) + \gamma \ln A) x, x \rangle - [f(\langle Ax, x \rangle) + \gamma \ln (\langle Ax, x \rangle)] \\ &= \langle f(A) x, x \rangle - f(\langle Ax, x \rangle) - \gamma [\ln (\langle Ax, x \rangle) - \langle \ln Ax, x \rangle] \end{aligned}$$

which is equivalent with the first inequality in (5.13).

By defining the function $g_{\Gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,0}(t) = -\Gamma \ln t - f(t)$ and applying the same argument we deduce the second part of (5.13).

The rest goes likewise for the functions

$$g_{\delta,1}(t) = f(t) - \delta t \ln t \quad \text{and} \quad g_{\Delta,0}(t) = \Delta t \ln t - f(t)$$

and the details are omitted. ∎

COROLLARY 5.5 (Dragomir, 2008, [**11**]). *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$. If f is a twice differentiable function on (m, M) and we have the condition (5.12), then*

$$\begin{aligned} \text{(5.16)} \quad & \gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right) \\ &\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\ &\leq \Gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right) \end{aligned}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If we have the condition (5.14), then

$$\begin{aligned} \text{(5.17)} \quad & \delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right) \\ &\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\ &\leq \Delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right) \end{aligned}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following particular case also holds:

COROLLARY 5.6 (Dragomir, 2008, [**11**]). *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, \ldots, n\}$ and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If f is a twice* *differentiable function on* (m, M) *and we have the condition (5.12), then*

$$\gamma\left(\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)-\left\langle\sum_{j=1}^{n}p_j\ln A_jx,x\right\rangle\right)\tag{5.18}$$
$$\leq\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle-f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)$$
$$\leq\Gamma\left(\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)-\left\langle\sum_{j=1}^{n}p_j\ln A_jx,x\right\rangle\right)$$

for each $x\in H$ *with* $\|x\|=1$.

If we have the condition (5.14), then

$$\delta\left(\left\langle\sum_{j=1}^{n}p_jA_j\ln A_jx,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right)$$
$$\leq\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle-f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)\tag{5.19}$$
$$\leq\Delta\left(\left\langle\sum_{j=1}^{n}p_jA_j\ln A_jx,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right)$$

for each $x\in H$ *with* $\|x\|=1$.

5.2. Applications. It is clear that the results from the previous section can be applied for various particular functions which are twice differentiable and the second derivatives satisfy the boundedness conditions from the statements of the Theorems 5.1, 5.4 and the Remark 5.1.

We point out here only some simple examples that are, in our opinion, of large interest.

1. For a given $\alpha>0$, consider the function $f(t)=\exp(\alpha t), t\in\mathbb{R}$. We have $f''(t)=\alpha^2\exp(\alpha t)$ and for a selfadjoint operator A with $Sp(A)\subset[m,M]$ (for some real numbers $m<M$) we also have

$$\varphi:=\inf_{t\in(m,M)}f''(t)=\alpha^2\exp(\alpha m)\text{ and }\Phi:=\sup_{t\in(m,M)}f''(t)=\alpha^2\exp(\alpha M).$$

Utilising the inequality (5.5) we get

$$\frac{1}{2}\alpha^2\exp(\alpha m)\left[\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right]\tag{5.20}$$
$$\leq\langle\exp(\alpha A)x,x\rangle-\exp(\langle\alpha Ax,x\rangle)$$
$$\leq\frac{1}{2}\alpha^2\exp(\alpha M)\left[\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right],$$

for each $x\in H$ with $\|x\|=1$.

Now, if $\beta>0$, then we also have

$$\frac{1}{2}\beta^2\exp(-\beta M)\left[\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right]\tag{5.21}$$
$$\leq\langle\exp(-\beta A)x,x\rangle-\exp(-\langle\beta Ax,x\rangle)$$
$$\leq\frac{1}{2}\beta^2\exp(-\beta m)\left[\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right],$$

for each $x\in H$ with $\|x\|=1$.

2. Now, assume that $0 < m < M$ and the operator A satisfies the condition $m \cdot 1_H \leq A \leq M \cdot 1_H$. If we consider the function $f : (0,\infty) \to (0,\infty)$ defined by $f(t) = t^p$ with $p \in (-\infty, 0) \cup (0,1) \cup (1,\infty)$. Then $f''(t) = p(p-1)t^{p-2}$ and if we consider $\varphi := \inf_{t\in(m,M)} f''(t)$ and $\Phi := \sup_{t\in(m,M)} f''(t)$, then we have

$$\varphi = p(p-1)m^{p-2}, \Phi = p(p-1)M^{p-2} \quad \text{for } p \in [2,\infty),$$

$$\varphi = p(p-1)M^{p-2}, \Phi = p(p-1)m^{p-2} \quad \text{for } p \in (1,2),$$

$$\varphi = p(p-1)m^{p-2}, \Phi = p(p-1)M^{p-2} \quad \text{for } p \in (0,1),$$

and

$$\varphi = p(p-1)M^{p-2}, \Phi = p(p-1)m^{p-2} \quad \text{for } p \in (-\infty,0).$$

Utilising the inequality (5.5) we then get the following refinements an reverses of Hölder-McCarthy's inequalities:

$$\begin{aligned}&\frac{1}{2}p(p-1)m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\\&\leq \langle A^p x,x\rangle - \langle Ax,x\rangle^p\\&\leq \frac{1}{2}p(p-1)M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \quad \text{for } p\in[2,\infty),\end{aligned} \tag{5.22}$$

$$\begin{aligned}&\frac{1}{2}p(p-1)M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\\&\leq \langle A^p x,x\rangle - \langle Ax,x\rangle^p\\&\leq \frac{1}{2}p(p-1)m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \quad \text{for } p\in(1,2),\end{aligned} \tag{5.23}$$

$$\begin{aligned}&\frac{1}{2}p(1-p)M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\\&\leq \langle Ax,x\rangle^p - \langle A^p x,x\rangle\\&\leq \frac{1}{2}p(1-p)m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \quad \text{for } p\in(0,1)\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}&\frac{1}{2}p(p-1)M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\\&\leq \langle A^p x,x\rangle - \langle Ax,x\rangle^p\\&\leq \frac{1}{2}p(p-1)m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \quad \text{for } p\in(-\infty,0),\end{aligned} \tag{5.25}$$

for each $x \in H$ with $\|x\| = 1$.

3. Now, if we consider the function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$, then $f''(t) = t^{-2}$ which gives that $\varphi = M^{-2}$ and $\Phi = m^{-2}$. Utilising the inequality (5.5) we then deduce the bounds

$$\begin{aligned}&\frac{1}{2}M^{-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\\&\leq \ln(\langle Ax,x\rangle) - \langle \ln Ax,x\rangle\\&\leq \frac{1}{2}m^{-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\end{aligned} \tag{5.26}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover, if we consider the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t\ln t$, then $f''(t)=t^{-1}$ which gives that $\varphi=M^{-1}$ and $\Phi=m^{-1}$. Utilising the inequality (5.5) we then deduce the bounds

$$\begin{aligned}\frac{1}{2}M^{-1}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right] &\\ \leq\langle A\ln Ax,x\rangle-\langle Ax,x\rangle\ln(\langle Ax,x\rangle) &\\ \leq\frac{1}{2}m^{-1}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right] &\end{aligned}\tag{5.27}$$

for each $x\in H$ with $\|x\|=1$.

REMARK 5.3. Utilising Theorem 5.1 for the particular value of $p=-1$ we can state the inequality

$$\begin{aligned}\frac{1}{2}\psi\left(\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\right)&\leq\langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\ &\leq\frac{1}{2}\Psi\left(\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\right)\end{aligned}\tag{5.28}$$

for each $x\in H$ with $\|x\|=1$, provided that f is twice differentiable on $(m,M)\subset(0,\infty)$ and

$$\psi=\inf_{t\in(m,M)}t^{3}f''(t)\quad\text{while}\quad\Psi=\sup_{t\in(m,M)}t^{3}f''(t)$$

are assumed to be finite.

We observe that, by utilising the inequality (5.28) instead of the inequality (5.5) we may obtain similar results in terms of the quantity $\langle A^{-1}x,x\rangle-\langle Ax,x\rangle^{-1}$, $x\in H$ with $\|x\|=1$. However the details are left to the interested reader.

6. SOME JENSEN'S TYPE INEQUALITIES FOR LOG-CONVEX FUNCTIONS

6.1. Preliminary Results. The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [**32**] (see also [**20**, p. 5]):

THEOREM 6.1 (Mond-Pečarić, 1993, [**32**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle\tag{MP}$$

for each $x\in H$ with $\|x\|=1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f:I\to(0,\infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1-t)a+tb)\leq[f(a)]^{1-t}[f(b)]^{t}\tag{6.1}$$

for any $a,b\in I$ and $t\in[0,1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

As an immediate consequence of the Mond-Pečarić inequality above we can provide the following result:

THEOREM 6.2 (Dragomir, 2010, [**15**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$g(\langle Ax, x\rangle) \leq \exp\langle \ln g(A)x, x\rangle \leq \langle g(A)x, x\rangle \tag{6.2}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (MP) for f we get $\ln[g(\langle Ax, x\rangle)] \leq \langle \ln g(A)x, x\rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (6.2).

If we also use (MP) for the exponential function, we get

$$\exp\langle \ln g(A)x, x\rangle \leq \langle \exp[\ln g(A)]x, x\rangle = \langle g(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. ∎

The case of sequences of operators may be of interest and is embodied in the following corollary:

COROLLARY 6.3 (Dragomir, 2010, [**15**]). *Assume that g is as in the Theorem 6.2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then*

$$g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \leq \exp\left\langle \sum_{j=1}^{n} \ln g(A_j)x_j, x_j\right\rangle \leq \left\langle \sum_{j=1}^{n} g(A_j)x_j, x_j\right\rangle. \tag{6.3}$$

PROOF. Follows from Theorem 6.2and we omit the details. ∎

In particular we have:

COROLLARY 6.4 (Dragomir, 2010, [**15**]). *Assume that g is as in the Theorem 6.2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) \leq \left\langle \prod_{j=1}^{n} [g(A_j)]^{p_j} x, x\right\rangle \leq \left\langle \sum_{j=1}^{n} p_j g(A_j)x, x\right\rangle \tag{6.4}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Follows from Corollary 6.3 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. ∎

It is also important to observe that, as a special case of Theorem 6.1 we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

THEOREM 6.5 (Hölder-McCarthy, 1967, [**26**]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^{-r} x, x\rangle \geq \langle Ax, x\rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows:

PROPOSITION 6.6. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax, x\rangle^{-r} \le \exp\left\langle \ln\left(A^{-r}\right) x, x\right\rangle \le \left\langle A^{-r}x, x\right\rangle \tag{6.5}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [**20**, p. 57]:

THEOREM 6.7. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A)x, x\rangle \le \frac{M - \langle Ax, x\rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot f(M) \tag{6.6}$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows:

THEOREM 6.8 (Dragomir, 2010, [**15**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$\begin{aligned} \langle g(A)x, x\rangle &\le \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}}\right] x, x\right\rangle \\ &\le \frac{M - \langle Ax, x\rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot g(M) \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} g(\langle Ax, x\rangle) &\le [g(m)]^{\frac{M - \langle Ax, x\rangle}{M-m}} [g(M)]^{\frac{\langle Ax, x\rangle - m}{M-m}} \\ &\le \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}}\right] x, x\right\rangle \end{aligned} \tag{6.8}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Observe that, by the log-convexity of g, we have

$$\begin{aligned} g(t) &= g\left(\frac{M - t}{M - m} \cdot m + \frac{t - m}{M - m} \cdot M\right) \\ &\le [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \end{aligned} \tag{6.9}$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A, we have that

$$\langle g(A)x, x\rangle \le \langle \Psi(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}}$, $t \in [m, M]$. This proves the first inequality in (6.7).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have

$$[g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \le \frac{M - t}{M - m} \cdot g(m) + \frac{t - m}{M - m} \cdot g(M)$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A we deduce the second inequality in (6.7).

Further on, if we use the inequality (6.9) for $t = \langle Ax, x\rangle \in [m, M]$ then we deduce the first part of (6.8).

Now, observe that the function Ψ introduced above can be rearranged to read as

$$\Psi(t) = g(m)\left[\frac{g(M)}{g(m)}\right]^{\frac{t-m}{M-m}}, t \in [m, M]$$

showing that Ψ is a convex function on $[m, M]$.

Applying Mond-Pečarić's inequality for Ψ we deduce the second part of (6.8) and the proof is complete. ∎

The case of sequences of operators is as follows:

COROLLARY 6.9 (Dragomir, 2010, [**15**]). *Assume that g is as in the Theorem 6.2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$\begin{aligned}(6.10)\qquad &\sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle \\ &\le \left\langle \sum_{j=1}^n \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}}\right] x_j, x_j \right\rangle \\ &\le \frac{M - \sum_{j=1}^n \langle A_j x_j, x_j\rangle}{M-m}\cdot g(m) + \frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle - m}{M-m}\cdot g(M)\end{aligned}$$

and

$$\begin{aligned}(6.11)\qquad &g\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \\ &\le [g(m)]^{\frac{M - \sum_{j=1}^n \langle A_j x_j, x_j\rangle}{M-m}} [g(M)]^{\frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle - m}{M-m}} \\ &\le \left\langle \sum_{j=1}^n \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}}\right] x_j, x_j \right\rangle.\end{aligned}$$

In particular we have:

COROLLARY 6.10 (Dragomir, 2010, [**15**]). *Assume that g is as in the Theorem 6.2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset\mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$\begin{aligned}(6.12)\qquad &\left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle \\ &\le \left\langle \sum_{j=1}^n p_j [g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} x, x \right\rangle \\ &\le \frac{M - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle}{M-m}\cdot g(m) + \frac{\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle - m}{M-m}\cdot g(M)\end{aligned}$$

and

$$g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) \leq [g(m)]^{\frac{M-\left\langle \sum_{j=1}^{n} p_j A_j x,x\right\rangle}{M-m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^{n} p_j A_j x,x\right\rangle - m}{M-m}} \leq \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} x, x\right\rangle . \tag{6.13}$$

The above result from Theorem 6.8 can be utilized to produce the following reverse inequality for negative powers of operators:

PROPOSITION 6.11. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M]\,(0 < m < M)$, then*

$$\left\langle A^{-r}x, x\right\rangle \leq \left\langle \left[m^{\frac{M1_H - A}{M-m}} M^{\frac{A-m1_H}{M-m}}\right]^{-r} x, x\right\rangle \leq \frac{M - \langle Ax, x\rangle}{M-m} \cdot m^{-r} + \frac{\langle Ax, x\rangle - m}{M-m} \cdot M^{-r} \tag{6.14}$$

and

$$\langle Ax, x\rangle^{-r} \leq \left[g(m)^{\frac{M-\langle Ax,x\rangle}{M-m}} g(M)^{\frac{\langle Ax,x\rangle - m}{M-m}}\right]^{-r} \leq \left\langle \left[m^{\frac{M1_H - A}{M-m}} M^{\frac{A-m1_H}{M-m}}\right]^{-r} x, x\right\rangle \tag{6.15}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

6.2. Jensen's Inequality for Differentiable Log-convex Functions. The following result provides a reverse for the Jensen type inequality (MP):

THEOREM 6.12 (Dragomir, 2008, [**9**]). *Let J be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\mathring{J}$ (the interior of J) whose derivative f' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{6.16}$$

for any $x \in H$ with $\|x\| = 1$.

The following result may be stated:

PROPOSITION 6.13 (Dragomir, 2010, [**15**]). *Let J be an interval and $g : J \to \mathbb{R}$ be a differentiable log-convex function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(1 \leq) \frac{\exp\langle \ln g(A)x, x\rangle}{g(\langle Ax, x\rangle)} \leq \exp\left[\left\langle g'(A)[g(A)]^{-1}Ax, x\right\rangle - \langle Ax, x\rangle \cdot \left\langle g'(A)[g(A)]^{-1}x, x\right\rangle\right] \tag{6.17}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. It follows by the inequality (6.16) written for the convex function $f = \ln g$ that

$$\langle \ln g(A)x, x\rangle \le \ln g(\langle Ax, x\rangle) + \left\langle g'(A)[g(A)]^{-1}Ax, x\right\rangle - \langle Ax, x\rangle \cdot \left\langle g'(A)[g(A)]^{-1}x, x\right\rangle$$

for each $x \in H$ with $\|x\| = 1$.

Now, taking the exponential and dividing by $g(\langle Ax, x\rangle) > 0$ for each $x \in H$ with $\|x\| = 1$, we deduce the desired result (6.17). ∎

COROLLARY 6.14 (Dragomir, 2010, [**15**]). *Assume that g is as in the Proposition 6.13 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$(1 \le) \frac{\exp\left\langle \sum_{j=1}^n \ln g(A_j)x_j, x_j\right\rangle}{g\left(\sum_{j=1}^n \langle A_j x, x_j\rangle\right)} \le \exp\left[\left\langle \sum_{j=1}^n g'(A_j)[g(A_j)]^{-1}A_j x_j, x_j\right\rangle - \sum_{j=1}^n \langle A_j x_j, x_j\rangle \cdot \sum_{j=1}^n \left\langle g'(A_j)[g(A_j)]^{-1}x_j, x_j\right\rangle\right]. \tag{6.18}$$

If $p_j \ge 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$(1 \le) \frac{\left\langle \prod_{j=1}^n [g(A_j)]^{p_j} x, x\right\rangle}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)} \le \exp\left[\left\langle \sum_{j=1}^n p_j g'(A_j)[g(A_j)]^{-1}A_j x, x\right\rangle - \sum_{j=1}^n p_j \langle A_j x, x\rangle \cdot \sum_{j=1}^n p_j \left\langle g'(A_j)[g(A_j)]^{-1}x, x\right\rangle\right] \tag{6.19}$$

for each $x \in H$ with $\|x\| = 1$.

REMARK 6.1. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$(1 \le) \langle Ax, x\rangle^r \exp\left\langle \ln\left(A^{-r}\right)x, x\right\rangle \le \exp\left[r\left(\langle Ax, x\rangle \cdot \left\langle A^{-1}x, x\right\rangle - 1\right)\right] \tag{6.20}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality can be stated as well:

THEOREM 6.15 (Dragomir, 2010, [**15**]). *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint* *operators on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subset \mathring{J}$, *then*

$$1 \leq \left\langle \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(A - \langle Ax, x\rangle 1_H)\right] x, x\right\rangle \leq \frac{\langle g(A)x, x\rangle}{g(\langle Ax, x\rangle)} \leq \left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right] x, x\right\rangle \tag{6.21}$$

for each $x \in H$ *with* $\|x\| = 1$, *where* 1_H *denotes the identity operator on* H.

PROOF. It is well known that if $h : J \to \mathbb{R}$ is a convex differentiable function on $\mathring{J}$, then the following *gradient inequality* holds

$$h(t) - h(s) \geq h'(s)(t - s)$$

for any $t, s \in \mathring{J}$.

Now, if we write this inequality for the convex function $h = \ln g$, then we get

$$\ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t - s) \tag{6.22}$$

which is equivalent with

$$g(t) \geq g(s)\exp\left[\frac{g'(s)}{g(s)}(t - s)\right] \tag{6.23}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (6.23), then we get

$$g(t) \geq g(\langle Ax, x\rangle)\exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

$$\begin{aligned} &\langle g(A)y, y\rangle \\ &\geq g(\langle Ax, x\rangle)\left\langle \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(A - \langle Ax, x\rangle 1_H)\right] y, y\right\rangle \\ &\geq g(\langle Ax, x\rangle)\exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(\langle Ay, y\rangle - \langle Ax, x\rangle)\right] \end{aligned} \tag{6.24}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Further, if we put $y = x$ in (6.24), then we deduce the first and the second inequality in (6.21).

Now, if we replace s with t in (6.23) we can also write the inequality

$$g(t)\exp\left[\frac{g'(t)}{g(t)}(s - t)\right] \leq g(s)$$

which is equivalent with

$$g(t) \leq g(s)\exp\left[\frac{g'(t)}{g(t)}(t - s)\right] \tag{6.25}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (6.25), then we get

$$g(t) \le g(\langle Ax, x\rangle) \exp\left[\frac{g'(t)}{g(t)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A, then we can state the following inequality that is of interest in itself as well:

$$\text{(6.26)} \quad \begin{aligned} &\langle g(A) y, y\rangle \\ &\le g(\langle Ax, x\rangle) \left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right] y, y\right\rangle \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (6.26), then we deduce the last inequality in (6.21). ∎

The case of operator sequences is embodied in the following corollary:

COROLLARY 6.16 (Dragomir, 2010, [**15**]). *Assume that g is as in the Proposition 6.13 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$\text{(6.27)} \quad \begin{aligned} &1 \\ &\le \left\langle \sum_{j=1}^n \exp\left[\frac{g'\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)}{g\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)} \left(A_j - \sum_{j=1}^n \langle A_j x_j, x_j\rangle 1_H\right)\right] x_j, x_j \right\rangle \\ &\le \frac{\sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle}{g\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)} \\ &\le \left\langle \sum_{j=1}^n \exp\left[g'(A_j)[g(A_j)]^{-1}\left(A_j - \sum_{j=1}^n \langle A_j x_j, x_j\rangle 1_H\right)\right] x_j, x_j \right\rangle. \end{aligned}$$

If $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$\text{(6.28)} \quad \begin{aligned} 1 &\le \left\langle \sum_{j=1}^n p_j \exp\left[\frac{g'\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)}\right.\right. \\ &\quad \times \left.\left.\left(A_j - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle 1_H\right)\right] x, x\right\rangle \\ &\le \frac{\left\langle \sum_{j=1}^n p_j g(A_j) x, x\right\rangle}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)} \\ &\le \left\langle \sum_{j=1}^n p_j \exp\left[g'(A_j)[g(A_j)]^{-1}\left(A_j - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle 1_H\right)\right] x, x\right\rangle. \end{aligned}$$

REMARK 6.2. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$1 \leq \left\langle \exp\left[r\left(1_H - \langle Ax,x\rangle^{-1} A\right)\right] x, x\right\rangle \leq \left\langle A^{-r}x, x\right\rangle \langle Ax,x\rangle^r \leq \left\langle \exp\left[r\left(1_H - \langle Ax,x\rangle A^{-1}\right)\right] x, x\right\rangle \tag{6.29}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well:

THEOREM 6.17 (Dragomir, 2010, [**15**]). *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$\begin{aligned}(1 \leq) \frac{\left\langle [g(M)]^{\frac{A-m1_H}{M-m}} [g(m)]^{\frac{M1_H-A}{M-m}} x, x\right\rangle}{\langle g(A)x,x\rangle} &\leq \frac{\left\langle g(A)\exp\left[\frac{(M1_H-A)(A-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] x, x\right\rangle}{\langle g(A)x,x\rangle} \\ &\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]\end{aligned} \tag{6.30}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. Utilising the inequality (6.22) we have successively

$$\frac{g((1-\lambda)t + \lambda s)}{g(s)} \geq \exp\left[(1-\lambda)\frac{g'(s)}{g(s)}(t-s)\right] \tag{6.31}$$

and

$$\frac{g((1-\lambda)t + \lambda s)}{g(t)} \geq \exp\left[-\lambda\frac{g'(t)}{g(t)}(t-s)\right] \tag{6.32}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we take the power λ in the inequality (6.31) and the power $1-\lambda$ in (6.32) and multiply the obtained inequalities, we deduce

$$\begin{aligned}\frac{[g(t)]^{1-\lambda}[g(s)]^{\lambda}}{g((1-\lambda)t+\lambda s)} \leq \exp\left[(1-\lambda)\lambda\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t-s)\right]\end{aligned} \tag{6.33}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Further on, if we choose in (6.33) $t = M, s = m$ and $\lambda = \frac{M-u}{M-m}$, then, from (6.33) we get the inequality

$$\begin{aligned}\frac{[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}}}{g(u)} \leq \exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]\end{aligned} \tag{6.34}$$

which, together with the inequality

$$\frac{(M-u)(u-m)}{M-m} \leq \frac{1}{4}(M-m)$$

produce

$$[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}} \leq g(u)\exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] \leq g(u)\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] \tag{6.35}$$

for any $u \in [m, M]$.

If we apply the property (P) to the inequality (6.35) and for the operator A we deduce the desired result. ∎

COROLLARY 6.18 (Dragomir, 2010, [**15**]). *Assume that g is as in the Theorem 6.17 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$(1 \leq) \frac{\sum_{j=1}^n \left\langle [g(M)]^{\frac{A_j-m1_H}{M-m}}[g(m)]^{\frac{M1_H-A_j}{M-m}} x_j, x_j\right\rangle}{\sum_{j=1}^n \langle g(A_j)x_j, x_j\rangle} \leq \frac{\sum_{j=1}^n \left\langle g(A_j)\exp\left[\frac{(M1_H-A_j)(A_j-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] x_j, x_j\right\rangle}{\sum_{j=1}^n \langle g(A_j)x_j, x_j\rangle} \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]. \tag{6.36}$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$(1 \leq) \frac{\left\langle \sum_{j=1}^n p_j [g(M)]^{\frac{A_j-m1_H}{M-m}}[g(m)]^{\frac{M1_H-A_j}{M-m}} x, x\right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j)x, x\right\rangle} \leq \frac{\left\langle \sum_{j=1}^n p_j g(A_j)\exp\left[\frac{(M1_H-A_j)(A_j-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] x, x\right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j)x, x\right\rangle} \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]. \tag{6.37}$$

REMARK 6.3. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M]$ $(0 < m < M)$, then

$$(1 \leq) \frac{\left\langle [g(M)]^{\frac{r(m1_H-A)}{M-m}}[g(m)]^{\frac{r(A-M1_H)}{M-m}} x, x\right\rangle}{\langle A^{-r}x, x\rangle} \leq \frac{\left\langle A^{-r}\exp\left[\frac{r(M1_H-A)(A-m1_H)}{Mm}\right] x, x\right\rangle}{\langle A^{-r}x, x\rangle} \leq \exp\left[\frac{1}{4}r\frac{(M-m)^2}{mM}\right] \tag{6.38}$$

6.3. Applications for Ky Fan's Inequality. Consider the function $g:(0,1)\to\mathbb{R}$, $g(t)=\left(\frac{1-t}{t}\right)^{r}, r>0$. Observe that for the new function $f:(0,1)\to\mathbb{R}$, $f(t)=\ln g(t)$ we have

$$f'(t)=\frac{-r}{t(1-t)} \text{ and } f''(t)=\frac{2r\left(\frac{1}{2}-t\right)}{t^{2}(1-t)^{2}} \text{ for } t\in(0,1)$$

showing that the function g is log-convex on the interval $\left(0,\frac{1}{2}\right)$.

If $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$ and $t_i\in\left(0,\frac{1}{2}\right)$ for $i\in\{1,...,n\}$, then by applying the Jensen inequality for the convex function f (with $r=1$) on the interval $\left(0,\frac{1}{2}\right)$ we get

$$\frac{\sum_{i=1}^{n}p_it_i}{1-\sum_{i=1}^{n}p_it_i}\geq\prod_{i=1}^{n}\left(\frac{t_i}{1-t_i}\right)^{p_i}, \tag{6.39}$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [**3**, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^{n}\left(\frac{1-t_i}{t_i}\right)^{p_i}\geq\frac{1-\sum_{i=1}^{n}p_it_i}{\sum_{i=1}^{n}p_it_i},$$

where $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$ and $t_i\in\left(0,\frac{1}{2}\right)$ for $i\in\{1,...,n\}$.

By the weighted arithmetic mean - geometric mean inequality we also have that

$$\sum_{i=1}^{n}p_i(1-t_i)t_i^{-1}\geq\prod_{i=1}^{n}\left(\frac{1-t_i}{t_i}\right)^{p_i}$$

giving the double inequality

$$\begin{aligned}\sum_{i=1}^{n}p_i(1-t_i)t_i^{-1}&\geq\prod_{i=1}^{n}\left((1-t_i)t_i^{-1}\right)^{p_i}\\&\geq\sum_{i=1}^{n}p_i(1-t_i)\left(\sum_{i=1}^{n}p_it_i\right)^{-1}.\end{aligned} \tag{6.40}$$

The following operator inequalities generalizing (6.40) may be stated:

PROPOSITION 6.19. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A)\subset\left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}\left\langle\left(A^{-1}(1_H-A)\right)^{r}x,x\right\rangle&\geq\exp\left\langle\ln\left(A^{-1}(1_H-A)\right)^{r}x,x\right\rangle\\&\geq\left(\langle(1_H-A)x,x\rangle\langle Ax,x\rangle^{-1}\right)^{r}\end{aligned} \tag{6.41}$$

for each $x\in H$ with $\|x\|=1$ and $r>0$.

In particular,

$$\begin{aligned}\left\langle A^{-1}(1_H-A)x,x\right\rangle&\geq\exp\left\langle\ln\left(A^{-1}(1_H-A)\right)x,x\right\rangle\\&\geq\langle(1_H-A)x,x\rangle\langle Ax,x\rangle^{-1}\end{aligned} \tag{6.42}$$

for each $x\in H$ with $\|x\|=1$.

The proof follows by Theorem 6.2 applied for the log-convex function $g(t)=\left(\frac{1-t}{t}\right)^{r}, r>0, t\in\left(0,\frac{1}{2}\right)$.

PROPOSITION 6.20. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m,M] \subset \left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}(6.43)\quad & \left\langle \left((1_H - A)A^{-1}\right)^r x, x\right\rangle \\ & \leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A-m1_H)}{M-m}}\right] x, x\right\rangle \\ & \leq \frac{M - \langle Ax,x\rangle}{M-m}\cdot\left(\frac{1-m}{m}\right)^r + \frac{\langle Ax,x\rangle - m}{M-m}\cdot\left(\frac{1-M}{M}\right)^r\end{aligned}$$

and

$$\begin{aligned}(6.44)\quad & \left(\frac{1-\langle Ax,x\rangle}{\langle Ax,x\rangle}\right)^r \\ & \leq \left(\frac{1-m}{m}\right)^{\frac{r(M-\langle Ax,x\rangle)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(\langle Ax,x\rangle - m)}{M-m}} \\ & \leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A-m1_H)}{M-m}}\right] x, x\right\rangle\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Theorem 6.8 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0,\frac{1}{2}\right)$.

Finally we have:

PROPOSITION 6.21. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}(6.45)\quad & (1 \leq) \frac{\exp\left\langle \ln\left((1_H - A)A^{-1}\right)^r x, x\right\rangle}{\left((1-\langle Ax,x\rangle)\langle Ax,x\rangle^{-1}\right)^r} \\ & \leq \exp\left[r\left(\langle Ax,x\rangle\cdot\left\langle A^{-1}(1_H - A)^{-1}x,x\right\rangle - \left\langle (1_H - A)^{-1}x,x\right\rangle\right)\right]\end{aligned}$$

and

$$\begin{aligned}(6.46)\quad 1 & \leq \left\langle \exp\left[r(1-\langle Ax,x\rangle)^{-1}\left(1_H - \langle Ax,x\rangle^{-1}A\right)\right]x,x\right\rangle \\ & \leq \frac{\left\langle\left((1_H - A)A^{-1}\right)^r x,x\right\rangle}{\left((1-\langle Ax,x\rangle)\langle Ax,x\rangle^{-1}\right)^r} \\ & \leq \left\langle \exp\left[r(1_H - A)^{-1}\left(\langle Ax,x\rangle A^{-1} - 1_H\right)\right]x,x\right\rangle\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Proposition 6.13 and Theorem 6.15 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0,\frac{1}{2}\right)$. The details are omitted.

6.4. More Inequalities for Differentiable Log-convex Functions. The following results providing companion inequalities for the Jensen inequality for differentiable log-convex functions obtained above hold:

THEOREM 6.22 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If g :*

$J \to (0,\infty)$ *is a differentiable log-convex function with the derivative continuous on* $\mathring{J}$ *and* $[m,M] \subset \mathring{J}$*, then*

$$\exp\left[\frac{\langle g'(A)Ax,x\rangle}{\langle g(A)x,x\rangle} - \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\cdot\frac{\langle g'(A)x,x\rangle}{\langle g(A)x,x\rangle}\right] \geq \frac{\exp\left[\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\right]}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)} \geq 1 \tag{6.47}$$

for each $x \in H$ *with* $\|x\| = 1.$

If

$$\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} \in \mathring{J} \text{ for each } x \in H \text{ with } \|x\| = 1, \tag{C}$$

then

$$\exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} - \frac{\langle Ag(A)x,x\rangle}{\langle g(A)x,x\rangle}\right)\right] \geq \frac{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{\exp\left(\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\right)} \geq 1, \tag{6.48}$$

for each $x \in H$ *with* $\|x\| = 1.$

PROOF. By the gradient inequality for the convex function $\ln g$ we have

$$\frac{g'(t)}{g(t)}(t-s) \geq \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t-s) \tag{6.49}$$

for any $t, s \in \mathring{J}$, which by multiplication with $g(t) > 0$ is equivalent with

$$g'(t)(t-s) \geq g(t)\ln g(t) - g(t)\ln g(s) \geq \frac{g'(s)}{g(s)}(tg(t) - sg(t)) \tag{6.50}$$

for any $t, s \in \mathring{J}$.

Fix $s \in \mathring{J}$ and apply the property (P) to get that

$$\begin{aligned} &\langle g'(A)Ax,x\rangle - s\langle g'(A)x,x\rangle \\ &\geq \langle g(A)\ln g(A)x,x\rangle - \langle g(A)x,x\rangle \ln g(s) \\ &\geq \frac{g'(s)}{g(s)}(\langle Ag(A)x,x\rangle - s\langle g(A)x,x\rangle) \end{aligned} \tag{6.51}$$

for any $x \in H$ with $\|x\| = 1,$ which is an inequality of interest in itself as well.

Since

$$\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} \in [m,M] \text{ for any } x \in H \text{ with } \|x\| = 1$$

then on choosing $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}$ in (6.51) we get

$$\begin{aligned} &\langle g'(A)Ax,x\rangle - \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\langle g'(A)x,x\rangle \\ &\geq \langle g(A)\ln g(A)x,x\rangle - \langle g(A)x,x\rangle \ln g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right) \geq 0, \end{aligned}$$

which, by division with $\langle g(A)x,x\rangle > 0$, produces

$$\frac{\langle g'(A)Ax,x\rangle}{\langle g(A)x,x\rangle} - \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\cdot\frac{\langle g'(A)x,x\rangle}{\langle g(A)x,x\rangle} \geq \frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle} - \ln g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right) \geq 0 \tag{6.52}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the exponential in (6.52) we deduce the desired inequality (6.47).

Now, assuming that the condition (C) holds, then by choosing $s := \frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}$ in (6.51) we get

$$0 \geq \langle g(A)\ln g(A)x,x\rangle - \langle g(A)x,x\rangle \ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)$$
$$\geq \frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\langle Ag(A)x,x\rangle - \frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\langle g(A)x,x\rangle\right)$$

which, by dividing with $\langle g(A)x,x\rangle > 0$ and rearranging, is equivalent with

$$\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} - \frac{\langle Ag(A)x,x\rangle}{\langle g(A)x,x\rangle}\right) \geq \ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right) - \frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle} \geq 0 \tag{6.53}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, on taking the exponential in (6.53) we deduce the desired inequality (6.48). ∎

REMARK 6.4. We observe that a sufficient condition for (C) to hold is that either $g'(A)$ or $-g'(A)$ is a positive definite operator on H.

COROLLARY 6.23 (Dragomir, 2010, [**16**]). *Assume that A and g are as in Theorem 6.22. If the condition (C) holds, then we have the double inequality*

$$\ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right) \geq \frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle} \geq \ln g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right), \tag{6.54}$$

for any $x \in H$ with $\|x\| = 1$.

REMARK 6.5. Assume that A is a positive definite operator on H. Since for $r > 0$ the function $g(t) = t^{-r}$ is log-convex on $(0,\infty)$ and

$$\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} = \frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle} > 0$$

for any $x \in H$ with $\|x\| = 1$, then on applying the inequality (6.54) we deduce the following interesting result

$$\ln\left(\frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle}\right) \leq \frac{\langle A^{-r}\ln Ax,x\rangle}{\langle A^{-r}x,x\rangle} \leq \ln\left(\frac{\langle A^{-r+1}x,x\rangle}{\langle A^{-r}x,x\rangle}\right) \tag{6.55}$$

for any $x \in H$ with $\|x\| = 1$.

The details of the proof are left to the interested reader.

The case of sequences of operators is embodied in the following corollary:

COROLLARY 6.24 (Dragomir, 2010, [**16**]). *Let A_j, $j \in \{1,...,n\}$ be selfadjoint operators on the Hilbert space H and assume that $Sp(A_j) \subseteq [m,M]$ for some scalars m,M with $m<M$ and each $j \in \{1,...,n\}$. If $g : J \to (0,\infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m,M] \subset \mathring{J}$, then*

$$\exp\left[\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle} - \frac{\sum_{j=1}^{n}\langle g(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\cdot\frac{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right] \geq \frac{\exp\left[\frac{\sum_{j=1}^{n}\langle g(A_j)\ln g(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right]}{g\left(\frac{\sum_{j=1}^{n}\langle g(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)} \geq 1 \tag{6.56}$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If

$$\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle} \in \mathring{J} \tag{6.57}$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$, then

$$\exp\left[\frac{g'\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)}{g\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)} \times\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle} - \frac{\sum_{j=1}^{n}\langle A_jg(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)\right] \geq \frac{g\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)}{\exp\left(\frac{\sum_{j=1}^{n}\langle g(A_j)\ln g(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)} \geq 1, \tag{6.58}$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

The following particular case for sequences of operators also holds:

COROLLARY 6.25 (Dragomir, 2010, [**16**]). *With the assumptions of Corollary 6.24 and if* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\exp\left[\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle} - \frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle} \cdot \frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right] \geq \frac{\exp\left[\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right]}{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)} \geq 1 \tag{6.59}$$

for each $x \in H$, *with* $\|x\| = 1$.

If

$$\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle} \in \mathring{J} \tag{6.60}$$

for each $x \in H$, *with* $\|x\| = 1$, *then*

$$\exp\left[\frac{g'\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)} \times \left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle} - \frac{\left\langle \sum_{j=1}^{n} p_j A_j g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)\right] \geq \frac{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{\exp\left(\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)} \geq 1, \tag{6.61}$$

for each $x \in H$, *with* $\|x\| = 1$.

PROOF. Follows from Corollary 6.24 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. ∎

The following result providing different inequalities also holds:

THEOREM 6.26 (Dragomir, 2010, [**16**]). *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* $g : J \to (0, \infty)$ *is a differentiable log-convex function with the derivative continuous on* $\mathring{J}$ *and*

$[m, M] \subset \mathring{J}$, *then*

$$\left\langle \exp\left[g'(A)\left(A - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}1_H\right)\right]x, x\right\rangle \tag{6.62}$$
$$\geq \left\langle \left(\frac{g(A)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}\right)^{g(A)} x, x\right\rangle$$
$$\geq \left\langle \exp\left[\frac{g'\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}\left(Ag(A) - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}g(A)\right)\right]x, x\right\rangle \geq 1$$

for each $x \in H$ *with* $\|x\| = 1$.

If the condition (C) from Theorem 6.22 holds, then

$$\left\langle \exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}g(A) - Ag(A)\right)\right]x, x\right\rangle \tag{6.63}$$
$$\geq \left\langle \left(g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)[g(A)]^{-1}\right)^{g(A)} x, x\right\rangle$$
$$\geq \left\langle \exp\left[g'(A)\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}1_H - A\right)\right]x, x\right\rangle \geq 1$$

for each $x \in H$ *with* $\|x\| = 1$.

PROOF. By taking the exponential in (6.50) we have the following inequality

$$\exp[g'(t)(t-s)] \geq \left(\frac{g(t)}{g(s)}\right)^{g(t)} \geq \exp\left[\frac{g'(s)}{g(s)}(tg(t) - sg(t))\right] \tag{6.64}$$

for any $t, s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (6.64), we deduce

$$\langle \exp[g'(A)(A - s1_H)]x, x\rangle \tag{6.65}$$
$$\geq \left\langle \left(\frac{g(A)}{g(s)}\right)^{g(A)} x, x\right\rangle$$
$$\geq \left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A) - sg(A))\right]x, x\right\rangle$$

for each $x \in H$ with $\|x\| = 1$, where 1_H is the identity operator on H.

By Mond-Pečarić's inequality applied for the convex function $\exp$ we also have

$$\left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A) - sg(A))\right]x, x\right\rangle \tag{6.66}$$
$$\geq \exp\left(\frac{g'(s)}{g(s)}(\langle Ag(A)x, x\rangle - s\langle g(A)x, x\rangle)\right)$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Now, if we choose $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} \in [m, M]$ in (6.65) and (6.66) we deduce the desired result (6.62).

Observe that, the inequality (6.64) is equivalent with

$$\exp\left[\frac{g'(s)}{g(s)}(sg(t)-tg(t))\right]\geq\left(\frac{g(s)}{g(t)}\right)^{g(t)}\geq\exp\left[g'(t)(s-t)\right] \tag{6.67}$$

for any $t,s\in\mathring{J}$.

If we fix $s\in\mathring{J}$ and apply the property (P) to the inequality (6.67) we deduce

$$\begin{aligned}&\left\langle\exp\left[\frac{g'(s)}{g(s)}(sg(A)-Ag(A))\right]x,x\right\rangle\\&\geq\left\langle\left(g(s)\left[g(A)\right]^{-1}\right)^{g(A)}x,x\right\rangle\\&\geq\left\langle\exp\left[g'(A)(s1_H-A)\right]x,x\right\rangle\end{aligned} \tag{6.68}$$

for each $x\in H$ with $\|x\|=1$.

By Mond-Pečarić's inequality we also have

$$\langle\exp\left[g'(A)(s1_H-A)\right]x,x\rangle\geq\exp\left[s\langle g'(A)x,x\rangle-\langle g'(A)Ax,x\rangle\right] \tag{6.69}$$

for each $s\in\mathring{J}$ and $x\in H$ with $\|x\|=1$.

Taking into account that the condition (C) is valid, then we can choose in (6.68) and (6.69) $s:=\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}$ to get the desired result (6.63). ∎

REMARK 6.6. If we apply, for instance, the inequality (6.62) for the log-convex function $g(t)=t^{-1}, t>0,$ then, after simple calculations, we get the inequality

$$\begin{aligned}&\left\langle\exp\left(\frac{A^{-2}-\langle A^{-1}x,x\rangle A^{-1}}{A^{-2}-\langle A^{-1}x,x\rangle}\right)x,x\right\rangle\\&\geq\left\langle\left(\left\langle A^{-1}x,x\right\rangle A^{-1}\right)^{A^{-1}}x,x\right\rangle\\&\geq\left\langle\exp\left(\frac{A^{-1}-\langle A^{-1}x,x\rangle 1_H}{\langle A^{-1}x,x\rangle^2}\right)x,x\right\rangle\\&\geq 1\end{aligned} \tag{6.70}$$

for each $x\in H$ with $\|x\|=1$.

Other similar results can be obtained from the inequality (6.63), however the details are left to the interested reader.

6.5. A Reverse Inequality. The following reverse inequality is also of interest:

THEOREM 6.27 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If $g:J\to(0,\infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m,M]\subset\mathring{J}$, then*

$$\begin{aligned}(1\leq)\,&\frac{[g(m)]^{\frac{M-\langle Ax,x\rangle}{M-m}}[g(M)]^{\frac{\langle Ax,x\rangle-m}{M-m}}}{\exp\langle\ln g(A)x,x\rangle}\\&\leq\exp\left[\frac{\langle(M1_H-A)(A-m1_H)x,x\rangle}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]\\&\leq\exp\left[\frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]\\&\leq\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]\end{aligned} \tag{6.71}$$

for each $x \in H$ *with* $\|x\| = 1.$

PROOF. Utilising the inequality (6.49) we have successively

$$\ln g((1-\lambda)t+\lambda s) - \ln g(s) \geq (1-\lambda)\frac{g'(s)}{g(s)}(t-s) \tag{6.72}$$

and

$$\ln g((1-\lambda)t+\lambda s) - \ln g(t) \geq -\lambda\frac{g'(t)}{g(t)}(t-s) \tag{6.73}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we multiply (6.72) by λ and (6.73) by $1-\lambda$ and sum the obtained inequalities, we deduce

$$\begin{aligned} &(1-\lambda)\ln g(t) + \lambda \ln g(s) - \ln g((1-\lambda)t+\lambda s) \\ &\leq (1-\lambda)\lambda\left[\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t-s)\right] \end{aligned} \tag{6.74}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we choose $\lambda := \frac{M-u}{M-m}$, $s := m$ and $t := M$ in (6.74) then we get the inequality

$$\begin{aligned} &\frac{u-m}{M-m}\ln g(M) + \frac{M-u}{M-m}\ln g(m) - \ln g(u) \\ &\leq \left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \end{aligned} \tag{6.75}$$

for any $u \in [m, M]$.

If we use the property (P) for the operator A we get

$$\begin{aligned} &\frac{\langle Ax,x\rangle - m}{M-m}\ln g(M) + \frac{M-\langle Ax,x\rangle}{M-m}\ln g(m) - \langle \ln g(A)x,x\rangle \\ &\leq \left[\frac{\langle (M1_H - A)(A-m1_H)x,x\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \end{aligned} \tag{6.76}$$

for each $x \in H$ with $\|x\| = 1.$

Taking the exponential in (6.76) we deduce the first inequality in (6.71).

Now, consider the function $h : [m, M] \to \mathbb{R}$, $h(t) = (M-t)(t-m)$. This function is concave in $[m, M]$ and by Mond-Pečarić's inequality we have

$$\langle (M1_H - A)(A-m1_H)x,x\rangle \leq (M - \langle Ax,x\rangle)(\langle Ax,x\rangle - m)$$

for each $x \in H$ with $\|x\| = 1$, which proves the second inequality in (6.71).

For the last inequality, we observe that

$$(M - \langle Ax,x\rangle)(\langle Ax,x\rangle - m) \leq \frac{1}{4}(M-m)^2,$$

and the proof is complete. ∎

COROLLARY 6.28 (Dragomir, 2010, [**16**]). *Assume that* g *is as in Theorem 6.27 and* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.

If and $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$, *then*

$$(1 \leq) \frac{[g(m)]^{\frac{M-\sum_{j=1}^n \left\langle A_j x_j, x_j \right\rangle}{M-m}} [g(M)]^{\frac{\sum_{j=1}^n \left\langle A_j x_j, x_j \right\rangle - m}{M-m}}}{\exp\left(\sum_j^n \langle \ln g(A_j) x_j, x_j \rangle\right)} \tag{6.77}$$
$$\leq \exp\left[\frac{\sum_{j=1}^n \langle (M1_H - A_j)(A_j - m1_H) x_j, x_j \rangle}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$
$$\leq \exp\left[\frac{\left(M - \sum_{j=1}^n \langle A_j x_j, x_j \rangle\right)\left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle - m\right)}{M-m} \right.$$
$$\left. \times \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$
$$\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right].$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n p_j = 1$, *then*

$$(1 \leq) \frac{[g(m)]^{\frac{M-\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle}{M-m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle - m}{M-m}}}{\left\langle \prod_{j=1}^n [g(A_j)]^{p_j} x, x \right\rangle} \tag{6.78}$$
$$\leq \exp\left[\frac{\sum_{j=1}^n p_j \langle (M1_H - A_j)(A_j - m1_H) x_j, x_j \rangle}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$
$$\leq \exp\left[\frac{\left(M - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle\right)\left(\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle - m\right)}{M-m} \right.$$
$$\left. \times \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$
$$\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

for each $x \in H$ *with* $\|x\| = 1$.

REMARK 6.7. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$(1 \leq) \frac{m^{\frac{\langle Ax, x \rangle - M}{M-m}} M^{\frac{m - \langle Ax, x \rangle}{M-m}}}{\exp \langle \ln A^{-1} x, x \rangle} \tag{6.79}$$
$$\leq \exp\left[\frac{\langle (M1_H - A)(A - m1_H) x, x \rangle}{Mm}\right]$$
$$\leq \exp\left[\frac{(M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m)}{Mm}\right]$$
$$\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right]$$

for all $x \in H$ with $\|x\| = 1$.

7. HERMITE-HADAMARD'S TYPE INEQUALITIES

7.1. Scalar Case. If $f : I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{HH}$$

This remarkable result is well known in the literature as the *Hermite-Hadamard inequality* [**29**].

For various generalizations, extensions, reverses and related inequalities, see [**1**], [**2**], [**19**], [**21**], [**24**], [**25**], [**27**], [**29**] the monograph [**18**] and the references therein.

7.2. Some Inequalities for Convex Functions. The following inequality related to the Mond-Pečarić result also holds:

THEOREM 7.1 (Dragomir, 2010, [**14**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$\begin{aligned} \frac{f(m)+f(M)}{2} &\ge \left\langle \frac{f(A)+f((m+M)1_H - A)}{2}x, x\right\rangle \\ &\ge \frac{f(\langle Ax,x\rangle)+f(m+M-\langle Ax,x\rangle)}{2} \\ &\ge f\left(\frac{m+M}{2}\right) \end{aligned} \tag{7.1}$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \neq \frac{m+M}{2}$, then also

$$\begin{aligned} &\frac{f(\langle Ax,x\rangle)+f(m+M-\langle Ax,x\rangle)}{2} \\ &\ge \frac{2}{\frac{m+M}{2} - \langle Ax,x\rangle}\int_{\langle Ax,x\rangle}^{m+M-\langle Ax,x\rangle} f(u)\,du \ge f\left(\frac{m+M}{2}\right). \end{aligned} \tag{7.2}$$

PROOF. Since f is convex on $[m, M]$ then for each $u \in [m, M]$ we have the inequalities

$$\begin{aligned} &\frac{M-u}{M-m}f(m) + \frac{u-m}{M-m}f(M) \\ &\ge f\left(\frac{M-u}{M-m}m + \frac{u-m}{M-m}M\right) = f(u) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} &\frac{M-u}{M-m}f(M) + \frac{u-m}{M-m}f(m) \\ &\ge f\left(\frac{M-u}{M-m}M + \frac{u-m}{M-m}m\right) \\ &= f(M+m-u). \end{aligned} \tag{7.4}$$

If we add these two inequalities we get

$$f(m) + f(M) \ge f(u) + f(M+m-u)$$

for any $u \in [m, M]$, which, by the property (P) applied for the operator A, produces the first inequality in (7.1).

By the Mond-Pečarić inequality we have

$$\langle f((m+M)1_H - A)x, x\rangle \geq f(m+M-\langle Ax, x\rangle),$$

which together with the same inequality produces the second inequality in (7.1).

The third part follows by the convexity of f.

In order to prove (7.2), we use the Hermite-Hadamard inequality (HH) for the convex functions f and the choices $a = \langle Ax, x\rangle$ and $b = m + M - \langle Ax, x\rangle$.

The proof is complete. ∎

REMARK 7.1. We observe that, from the inequality (7.1) we have the following inequality in the operator order of $B(H)$

$$\left[\frac{f(m)+f(M)}{2}\right]1_H \geq \frac{f(A)+f((m+M)1_H - A)}{2} \geq f\left(\frac{m+M}{2}\right)1_H, \tag{7.5}$$

where f is a convex function on $[m, M]$ and A a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.

The case of log-convex functions may be of interest for applications and therefore is stated in:

COROLLARY 7.2 (Dragomir, 2010, [**14**]). *If g is a log-convex function on $[m, M]$, then*

$$\begin{aligned}\sqrt{g(m)\,g(M)} &\geq \exp\left\langle \ln\left[g(A)\,g((m+M)1_H - A)\right]^{1/2}x, x\right\rangle \\ &\geq \sqrt{g(\langle Ax, x\rangle)\,g(m+M-\langle Ax, x\rangle)} \\ &\geq g\left(\frac{m+M}{2}\right)\end{aligned} \tag{7.6}$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x\rangle \neq \frac{m+M}{2}$, then also

$$\begin{aligned}&\sqrt{g(\langle Ax, x\rangle)\,g(m+M-\langle Ax, x\rangle)} \\ &\geq \exp\left[\frac{2}{\frac{m+M}{2} - \langle Ax, x\rangle}\int_{\langle Ax,x\rangle}^{m+M-\langle Ax,x\rangle} \ln g(u)\,du\right] \\ &\geq g\left(\frac{m+M}{2}\right).\end{aligned} \tag{7.7}$$

The following result also holds

THEOREM 7.3 (Dragomir, 2010, [**14**]). *Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$\begin{aligned} &f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \\ &\leq \frac{1}{2}\left[f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) + f\left(t\langle Ax, x\rangle + (1-t)\langle Bx, x\rangle\right)\right] \\ &\leq \left\langle \frac{1}{2}\left[f\left((1-t)A + tB\right) + f\left(tA + (1-t)B\right)\right]x, x\right\rangle \\ &\leq \frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M-m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M-m}f(M) \end{aligned} \tag{7.8}$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1.$

Moreover, we have the Hermite-Hadamard's type inequalities:

$$\begin{aligned} &f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \\ &\leq \int_0^1 f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right)dt \\ &\leq \left\langle \left[\int_0^1 f\left((1-t)A + tB\right)dt\right]x, x\right\rangle \\ &\leq \frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M-m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M-m}f(M) \end{aligned} \tag{7.9}$$

each $x \in H$ with $\|x\| = 1.$

In addition, if we assume that $B - A$ is a positive definite operator, then

$$\begin{aligned} &f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right)\langle (B-A)x, x\rangle \\ &\leq \int_{\langle Ax, x\rangle}^{\langle Bx, x\rangle} f(u)\,du \leq \langle (B-A)x, x\rangle \left\langle \left[\int_0^1 f\left((1-t)A + tB\right)dt\right]x, x\right\rangle \\ &\leq \langle (B-A)x, x\rangle \left[\frac{M - \left\langle \frac{A+B}{2}x, x\right\rangle}{M-m}f(m) + \frac{\left\langle \frac{A+B}{2}x, x\right\rangle - m}{M-m}f(M)\right]. \end{aligned} \tag{7.10}$$

PROOF. It is obvious that for any $t \in [0, 1]$ we have

$$Sp\left((1-t)A + tB\right), Sp\left(tA + (1-t)B\right) \subseteq [m, M].$$

On making use of the Mond-Pečarić inequality we have

$$f\left((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle\right) \leq \langle f\left((1-t)A + tB\right)x, x\rangle \tag{7.11}$$

and

$$f\left(t\langle Ax, x\rangle + (1-t)\langle Bx, x\rangle\right) \leq \langle f\left(tA + (1-t)B\right)x, x\rangle \tag{7.12}$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1.$

Adding (7.11) with (7.12) and utilising the convexity of f we deduce the first two inequalities in (7.8).

By the Lah-Ribarić inequality (6.6) we also have

$$\begin{aligned}\langle f((1-t)A+tB)x,x\rangle &\le \frac{M-(1-t)\langle Ax,x\rangle - t\langle Bx,x\rangle}{M-m}\cdot f(m) \\ &+ \frac{(1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle - m}{M-m}\cdot f(M)\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}\langle f(tA+(1-t)B)x,x\rangle &\le \frac{M-t\langle Ax,x\rangle-(1-t)\langle Bx,x\rangle}{M-m}\cdot f(m) \\ &+ \frac{t\langle Ax,x\rangle+(1-t)\langle Bx,x\rangle - m}{M-m}\cdot f(M)\end{aligned} \tag{7.14}$$

for any $t\in[0,1]$ and each $x\in H$ with $\|x\|=1$.

Now, if we add the inequalities (7.13) with (7.14) and divide by two, we deduce the last part in (7.8).

Integrating the inequality over $t\in[0,1]$, utilising the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$\begin{aligned}& f\left(\left\langle \frac{A+B}{2}x,x\right\rangle\right) \\ &\le \frac{1}{2}\left[\int_0^1 f((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle)\,dt \right.\\ &\left. + \int_0^1 f(t\langle Ax,x\rangle + (1-t)\langle Bx,x\rangle)\,dt\right] \\ &\le \left\langle \frac{1}{2}\left[\int_0^1 f((1-t)A+tB)\,dt + \int_0^1 f(tA+(1-t)B)\,dt\right]x,x\right\rangle \\ &\le \frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}f(m) + \frac{\left\langle\frac{A+B}{2}x,x\right\rangle - m}{M-m}f(M).\end{aligned} \tag{7.15}$$

Since

$$\int_0^1 f((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle)\,dt = \int_0^1 f(t\langle Ax,x\rangle + (1-t)\langle Bx,x\rangle)\,dt$$

and

$$\int_0^1 f((1-t)A+tB)\,dt = \int_0^1 f(tA+(1-t)B)\,dt$$

then, by (7.15), we deduce the inequality (7.9).

The inequality (7.10) follows from (7.9) by observing that for $\langle Bx,x\rangle > \langle Ax,x\rangle$ we have

$$\int_0^1 f((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle)\,dt = \frac{1}{\langle Bx,x\rangle - \langle Ax,x\rangle}\int_{\langle Ax,x\rangle}^{\langle Bx,x\rangle} f(u)\,du$$

for each $x\in H$ with $\|x\|=1$. ∎

REMARK 7.2. We observe that, from the inequalities (7.8) and (7.9) we have the following inequalities in the operator order of $B(H)$

$$\begin{aligned}\frac{1}{2}&\left[f\left((1-t)A+tB\right)+f\left(tA+(1-t)B\right)\right]\\&\le f(m)\frac{M1_H-\frac{A+B}{2}}{M-m}+f(M)\frac{\frac{A+B}{2}-m1_H}{M-m},\end{aligned}\tag{7.16}$$

where f is a convex function on $[m,M]$ and A,B are selfadjoint operator on the Hilbert space H with $Sp(A), Sp(B)\subseteq[m,M]$ for some scalars m,M with $m<M$.

The case of log-convex functions is as follows:

COROLLARY 7.4 (Dragomir, 2010, [**14**]). *If g is a log-convex function on $[m,M]$, then*

$$\begin{aligned}&g\left(\left\langle\frac{A+B}{2}x,x\right\rangle\right)\\&\le\sqrt{g\left((1-t)\langle Ax,x\rangle+t\langle Bx,x\rangle\right)g\left(t\langle Ax,x\rangle+(1-t)\langle Bx,x\rangle\right)}\\&\le\exp\left\langle\frac{1}{2}\left[\ln g\left((1-t)A+tB\right)+\ln g\left(tA+(1-t)B\right)\right]x,x\right\rangle\\&\le g(m)^{\frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}}g(M)^{\frac{\left\langle\frac{A+B}{2}x,x\right\rangle-m}{M-m}}\end{aligned}\tag{7.17}$$

for any $t\in[0,1]$ and each $x\in H$ with $\|x\|=1$.

Moreover, we have the Hermite-Hadamard's type inequalities:

$$\begin{aligned}&g\left(\left\langle\frac{A+B}{2}x,x\right\rangle\right)\\&\le\exp\left[\int_0^1\ln g\left((1-t)\langle Ax,x\rangle+t\langle Bx,x\rangle\right)dt\right]\\&\le\exp\left\langle\left[\int_0^1\ln g\left((1-t)A+tB\right)dt\right]x,x\right\rangle\\&\le g(m)^{\frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}}g(M)^{\frac{\left\langle\frac{A+B}{2}x,x\right\rangle-m}{M-m}}\end{aligned}\tag{7.18}$$

for each $x\in H$ with $\|x\|=1$.

In addition, if we assume that $B-A$ is a positive definite operator, then

$$\begin{aligned}&g\left(\left\langle\frac{A+B}{2}x,x\right\rangle\right)^{\langle(B-A)x,x\rangle}\\&\le\exp\left[\int_{\langle Ax,x\rangle}^{\langle Bx,x\rangle}\ln g(u)\,du\right]\\&\le\exp\left[\langle(B-A)x,x\rangle\left\langle\left[\int_0^1\ln g\left((1-t)A+tB\right)dt\right]x,x\right\rangle\right]\\&\le\left[g(m)^{\frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}}g(M)^{\frac{\left\langle\frac{A+B}{2}x,x\right\rangle-m}{M-m}}\right]^{\langle(B-A)x,x\rangle}\end{aligned}\tag{7.19}$$

for each $x\in H$ with $\|x\|=1$.

From a different perspective we have the following result as well:

THEOREM 7.5 (Dragomir, 2010, [**14**]). *Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m,M]$, then*

$$f\left(\frac{\langle Ax,x\rangle + \langle By,y\rangle}{2}\right) \le \int_0^1 f((1-t)\langle Ax,x\rangle + t\langle By,y\rangle)\,dt \le \left\langle \left[\int_0^1 f((1-t)A + t\langle By,y\rangle 1_H)\,dt\right] x,x\right\rangle \le \frac{1}{2}\left[\langle f(A)x,x\rangle + f(\langle By,y\rangle)\right] \le \frac{1}{2}\left[\langle f(A)x,x\rangle + \langle f(B)y,y\rangle\right] \tag{7.20}$$

and

$$f\left(\frac{\langle Ax,x\rangle + \langle By,y\rangle}{2}\right) \le \left\langle f\left(\frac{A + \langle By,y\rangle 1_H}{2}\right)x,x\right\rangle \le \left\langle \left[\int_0^1 f((1-t)A + t\langle By,y\rangle 1_H)\,dt\right] x,x\right\rangle \tag{7.21}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. For a convex function f and any $u, v \in [m,M]$ and $t \in [0,1]$ we have the double inequality:

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{2}\left[f((1-t)u + tv) + f(tu + (1-t)v)\right] \le \frac{1}{2}\left[f(u) + f(v)\right]. \tag{7.22}$$

Utilising the second inequality in (7.22) we have

$$\frac{1}{2}\left[f((1-t)u + t\langle By,y\rangle) + f(tu + (1-t)\langle By,y\rangle)\right] \le \frac{1}{2}\left[f(u) + f(\langle By,y\rangle)\right] \tag{7.23}$$

for any $u \in [m,M]$, $t \in [0,1]$ and $y \in H$ with $\|y\| = 1$.

Now, on applying the property (P) to the inequality (7.23) for the operator A we have

$$\frac{1}{2}\left[\langle f((1-t)A + t\langle By,y\rangle)x,x\rangle + \langle f(tA + (1-t)\langle By,y\rangle)x,x\rangle\right] \le \frac{1}{2}\left[\langle f(A)x,x\rangle + f(\langle By,y\rangle)\right] \tag{7.24}$$

for any $t \in [0,1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

On applying the Mond-Pečarić inequality we also have

$$\frac{1}{2}\left[f((1-t)\langle Ax,x\rangle + t\langle By,y\rangle) + f(t\langle Ax,x\rangle + (1-t)\langle By,y\rangle)\right] \le \frac{1}{2}\left[\langle f((1-t)A + t\langle By,y\rangle 1_H)x,x\rangle + \langle f(tA + (1-t)\langle By,y\rangle 1_H)x,x\rangle\right] \tag{7.25}$$

for any $t \in [0,1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, integrating over t on $[0,1]$ the inequalities (7.24) and (7.25) and taking into account that

$$\begin{aligned}&\int_0^1 \langle f((1-t)A + t\langle By,y\rangle 1_H)x,x\rangle\, dt\\&= \int_0^1 \langle f(tA + (1-t)\langle By,y\rangle 1_H)x,x\rangle\, dt\\&= \left\langle \left[\int_0^1 f((1-t)A + t\langle By,y\rangle 1_H)\, dt\right] x,x\right\rangle\end{aligned}$$

and

$$\int_0^1 f((1-t)\langle Ax,x\rangle + t\langle By,y\rangle)\, dt = \int_0^1 f(t\langle Ax,x\rangle + (1-t)\langle By,y\rangle)\, dt,$$

we obtain the second and the third inequality in (7.20).

Further, on applying the Jensen integral inequality for the convex function f we also have

$$\begin{aligned}&\int_0^1 f((1-t)\langle Ax,x\rangle + t\langle By,y\rangle)\, dt\\&\geq f\left(\int_0^1 [(1-t)\langle Ax,x\rangle + t\langle By,y\rangle]\, dt\right)\\&= f\left(\frac{\langle Ax,x\rangle + \langle By,y\rangle}{2}\right)\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, proving the first part of (7.20).

Now, on utilising the first part of (7.22) we can also state that

$$\begin{aligned}&f\left(\frac{u + \langle By,y\rangle}{2}\right)\\&\leq \frac{1}{2}[f((1-t)u + t\langle By,y\rangle) + f(tu + (1-t)\langle By,y\rangle)]\end{aligned} \tag{7.26}$$

for any $u \in [m, M]$, $t \in [0,1]$ and $y \in H$ with $\|y\| = 1$.

Further, on applying the property (P) to the inequality (7.26) and for the operator A we get

$$\begin{aligned}&\left\langle f\left(\frac{A + \langle By,y\rangle 1_H}{2}\right)x,x\right\rangle\\&\leq \frac{1}{2}[\langle f((1-t)A + t\langle By,y\rangle 1_H)x,x\rangle + \langle f(tA + (1-t)\langle By,y\rangle 1_H)x,x\rangle]\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which, by integration over t in $[0,1]$ produces the second inequality in (7.21). The first inequality is obvious. ∎

REMARK 7.3. It is important to remark that, from the inequalities (7.20) and (7.21) we have the following Hermite-Hadamard's type results in the operator order of $B(H)$ and for the convex function $f : [m, M] \to \mathbb{R}$

$$\begin{aligned}f\left(\frac{A + \langle By,y\rangle 1_H}{2}\right) &\leq \int_0^1 f((1-t)A + t\langle By,y\rangle 1_H)\, dt\\&\leq \frac{1}{2}[f(A) + f(\langle By,y\rangle)1_H]\end{aligned} \tag{7.27}$$

for any $y \in H$ with $\|y\| = 1$ and any selfadjoint operators A, B with spectra in $[m, M]$.

In particular, we have from (7.27)

$$f\left(\frac{A+\langle Ay,y\rangle 1_H}{2}\right)\le\int_0^1 f\left((1-t)A+t\langle Ay,y\rangle 1_H\right)dt \le\frac{1}{2}\left[f(A)+f(\langle Ay,y\rangle)1_H\right] \tag{7.28}$$

for any $y\in H$ with $\|y\|=1$ and

$$f\left(\frac{A+s1_H}{2}\right)\le\int_0^1 f\left((1-t)A+ts1_H\right)dt \le\frac{1}{2}\left[f(A)+f(s)1_H\right] \tag{7.29}$$

for any $s\in[m,M]$.

As a particular case of the above theorem we have the following refinement of the Mond-Pečarić inequality:

COROLLARY 7.6 (Dragomir, 2010, [**14**]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a convex function on $[m,M]$, then*

$$\begin{aligned} f(\langle Ax,x\rangle) &\le\left\langle f\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)x,x\right\rangle \\ &\le\left\langle\left[\int_0^1 f\left((1-t)A+t\langle Ax,x\rangle 1_H\right)dt\right]x,x\right\rangle \\ &\le\frac{1}{2}\left[\langle f(A)x,x\rangle+f(\langle Ax,x\rangle)\right]\le\langle f(A)x,x\rangle. \end{aligned} \tag{7.30}$$

Finally, the case of log-convex functions is as follows:

COROLLARY 7.7 (Dragomir, 2010, [**14**]). *If g is a log-convex function on $[m,M]$, then*

$$\begin{aligned} &g\left(\frac{\langle Ax,x\rangle+\langle By,y\rangle}{2}\right) \\ &\le\exp\left[\int_0^1\ln g\left((1-t)\langle Ax,x\rangle+t\langle By,y\rangle\right)dt\right] \\ &\le\exp\left\langle\left[\int_0^1\ln g\left((1-t)A+t\langle By,y\rangle 1_H\right)dt\right]x,x\right\rangle \\ &\le\exp\left[\frac{1}{2}\left[\langle\ln g(A)x,x\rangle+\ln g(\langle By,y\rangle)\right]\right] \\ &\le\exp\left[\frac{1}{2}\left[\langle\ln g(A)x,x\rangle+\langle\ln g(B)y,y\rangle\right]\right] \end{aligned} \tag{7.31}$$

and

$$\begin{aligned} &g\left(\frac{\langle Ax,x\rangle+\langle By,y\rangle}{2}\right) \\ &\le\exp\left\langle\ln g\left(\frac{A+\langle By,y\rangle 1_H}{2}\right)x,x\right\rangle \\ &\le\exp\left\langle\left[\int_0^1\ln g\left((1-t)A+t\langle By,y\rangle 1_H\right)dt\right]x,x\right\rangle \end{aligned} \tag{7.32}$$

and

$$g(\langle Ax,x\rangle) \leq \exp\left\langle \ln g\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)x,x\right\rangle \leq \exp\left\langle \left[\int_0^1 \ln g((1-t)A+t\langle Ax,x\rangle 1_H)\,dt\right]x,x\right\rangle \leq \exp\left[\frac{1}{2}\left[\langle \ln g(A)x,x\rangle + \ln g(\langle Ax,x\rangle)\right]\right] \leq \exp\langle \ln g(A)x,x\rangle \tag{7.33}$$

respectively, for each $x\in H$ with $\|x\|=1$ and A,B selfadjoint operators with spectra in $[m,M]$.

It is obvious that all the above inequalities can be applied for particular convex or log-convex functions of interest. However, we will restrict ourselves to only a few examples that are connected with famous results such as the Hölder-McCarthy inequality or the Ky Fan inequality.

7.3. Applications for Hölder-McCarthy's Inequality. We can improve the Hölder-McCarthy's inequality above as follows:

PROPOSITION 7.8. *Let A be a selfadjoint positive operator on a Hilbert space H.*
If $r>1$, then

$$\langle Ax,x\rangle^r \leq \left\langle\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)^r x,x\right\rangle \leq \left\langle\left[\int_0^1 ((1-t)A+t\langle Ax,x\rangle 1_H)^r\,dt\right]x,x\right\rangle \leq \frac{1}{2}\left[\langle A^r x,x\rangle + \langle Ax,x\rangle^r\right] \leq \langle A^r x,x\rangle \tag{7.34}$$

for any $x\in H$ with $\|x\|=1$.
If $0<r<1$, then the inequalities reverse in (7.34).
If A is invertible and $r>0$, then

$$\langle Ax,x\rangle^{-r} \leq \left\langle\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)^{-r} x,x\right\rangle \leq \left\langle\left[\int_0^1 ((1-t)A+t\langle Ax,x\rangle 1_H)^{-r}\,dt\right]x,x\right\rangle \leq \frac{1}{2}\left[\left\langle A^{-r}x,x\right\rangle + \langle Ax,x\rangle^{-r}\right] \leq \left\langle A^{-r}x,x\right\rangle \tag{7.35}$$

for any $x\in H$ with $\|x\|=1$.

Follows from the inequality (7.31) applied for the power function.

Since the function $g(t)=t^{-r}$ for $r>0$ is log-convex, then by utilising the inequality (7.33) we can improve the Hölder-McCarthy inequality as follows:

PROPOSITION 7.9. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax,x\rangle^{-r} \le \exp\left\langle \ln\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)^{-r} x,x\right\rangle \le \exp\left\langle \left[\int_0^1 \ln\left((1-t)A+t\langle Ax,x\rangle 1_H\right)^{-r}dt\right]x,x\right\rangle \le \exp\left[\frac{1}{2}\left[\left\langle \ln A^{-r}x,x\right\rangle+\ln\langle Ax,x\rangle^{-r}\right]\right] \le \exp\left\langle \ln A^{-r}x,x\right\rangle \tag{7.36}$$

for all $r>0$ and $x\in H$ with $\|x\|=1$.

Now, from a different perspective, we can state the following operator power inequalities:

PROPOSITION 7.10. *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]\subset[0,\infty)$, then*

$$\frac{m^r+M^r}{2} \ge \left\langle \frac{A^r+((m+M)1_H-A)^r}{2}x,x\right\rangle \ge \frac{\langle Ax,x\rangle^r+(m+M-\langle Ax,x\rangle)^r}{2} \ge \left(\frac{m+M}{2}\right)^r \tag{7.37}$$

for each $x\in H$ with $\|x\|=1$ and $r>1$.

If $0<r<1$ then the inequalities reverse in (7.37).

If A is positive definite and $r>0$, then

$$\frac{m^{-r}+M^{-r}}{2} \ge \left\langle \frac{A^{-r}+((m+M)1_H-A)^{-r}}{2}x,x\right\rangle \ge \frac{\langle Ax,x\rangle^{-r}+(m+M-\langle Ax,x\rangle)^{-r}}{2} \ge \left(\frac{m+M}{2}\right)^{-r} \tag{7.38}$$

for each $x\in H$ with $\|x\|=1$.

The proof follows by the inequality (7.1).

Finally we have:

PROPOSITION 7.11. *Assume that A and B are selfadjoint operators with spectra in $[m,M]\subset[0,\infty)$ and $x\in H$ with $\|x\|=1$ and such that $\langle Ax,x\rangle\neq\langle Bx,x\rangle$.*

If $r>1$ or $r\in(\infty,-1)\cup(-1,0)$ then we have

$$\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle^r \le \frac{1}{r+1}\cdot\frac{\langle Ax,x\rangle^{r+1}-\langle Bx,x\rangle^{r+1}}{\langle Ax,x\rangle-\langle Bx,x\rangle} \le \left\langle\left[\int_0^1((1-t)A+tB)^r dt\right]x,x\right\rangle \le \frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}m^r+\frac{\left\langle\frac{A+B}{2}x,x\right\rangle-m}{M-m}M^r. \tag{7.39}$$

If $0<r<1$, then the inequalities reverse in (7.39).

If A and B are positive definite, then

$$\left\langle \left(\frac{A+B}{2}\right) x,x\right\rangle^{-1} \leq \frac{\ln\langle Bx,x\rangle - \ln\langle Ax,x\rangle}{\langle Bx,x\rangle - \langle Ax,x\rangle} \leq \left\langle \left[\int_0^1 ((1-t)A+tB)^{-1}\,dt\right] x,x\right\rangle \leq \frac{M-\left\langle \frac{A+B}{2}x,x\right\rangle}{(M-m)\,m} + \frac{\left\langle \frac{A+B}{2}x,x\right\rangle - m}{(M-m)\,M}. \tag{7.40}$$

7.4. Applications for Ky Fan's Inequality. The following results related to the Ky Fan inequality may be stated as well:

PROPOSITION 7.12. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}
&\left(\langle (1_H - A)x,x\rangle \langle Ax,x\rangle^{-1}\right)^r \\
&\leq \exp\left\langle \ln\left(\left[1_H - A + \langle (1_H-A)x,x\rangle 1_H\right](A+\langle Ax,x\rangle 1_H)^{-1}\right)^r x,x\right\rangle \\
&\leq \left\langle \exp\left[\int_0^1 \left[\ln\left((1-t)(1_H-A)+t\langle (1_H-A)x,x\rangle 1_H\right)\right.\right.\right. \\
&\quad \left.\left.\left. \times\left((1-t)A+t\langle Ax,x\rangle 1_H\right)^{-1}\right]^r dt\right] x,x\right\rangle \\
&\leq \exp\left[\frac{1}{2}\left[\left\langle \ln\left[(1_H-A)A^{-1}\right]^r x,x\right\rangle + \ln\left(\langle (1_H-A)x,x\rangle\langle Ax,x\rangle^{-1}\right)^r\right]\right] \\
&\leq \exp\left\langle \ln\left[(1_H-A)A^{-1}\right]^r x,x\right\rangle
\end{aligned} \tag{7.41}$$

for any $x \in H$ with $\|x\| = 1$.

It follows from the inequality (7.33) applied for the log-convex function $g:(0,1)\to\mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0$.

PROPOSITION 7.13. *Assume that A is a selfadjoint operator with $Sp(A) \subset \left(0,\frac{1}{2}\right)$ and $s \in \left(0,\frac{1}{2}\right)$. Then we have the following inequality in the operator order of $B(H)$:*

$$\begin{aligned}
&\ln\left[\left[(2-s)1_H - A\right](A+s1_H)^{-1}\right] \\
&\leq \int_0^1 \ln\left(\left[(1-ts)1_H - (1-t)A\right]\left((1-t)A+ts1_H\right)^{-1}\right) dt \\
&\leq \frac{1}{2}\left(\ln\left[(1_H-A)A^{-1}\right]^r + \ln\left(\frac{1-s}{s}\right)^r 1_H\right).
\end{aligned} \tag{7.42}$$

If follows from the inequality (7.29) applied for the log-convex function $g:(0,1)\to\mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0$.

8. HERMITE-HADAMARD'S TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS

8.1. Introduction. The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}. \tag{8.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [**29**]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [**36**].

E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [**3**]. In 1974, D.S. Mitrinović found Hermite's note in *Mathesis* [**29**]. Since (8.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [**36**].

Let X be a vector space, $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x.y] \subset X$, we have the Hermite-Hadamard integral inequality (see [**4**, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x + ty]dt \le \frac{f(x) + f(y)}{2}, \tag{8.2}$$

which can be derived from the classical Hermite-Hadamard inequality (8.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \le p < \infty$) is a convex function, we have the following norm inequality from (8.2) (see [**35**, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x + ty\|^p dt \le \frac{\|x\|^p + \|y\|^p}{2}, \tag{8.3}$$

for any $x, y \in X$.

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator convex functions. The operator quasilinearity of some associated functionals are also provided.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A + \lambda B) \le (\ge) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \le B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \le f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [**20**] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \le r \le 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \le r \le 2$ or $-1 \le r \le 0$ and is operator concave on $(0,\infty)$ if $0 \le r \le 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0,\infty)$. The exponential function$f(t) = e^t$ is neither operator convex nor operator monotone.

8.2. Some Hermite-Hadamard's Type Inequalities. We start with the following result:

THEOREM 8.1 (Dragomir, 2010, [**13**]). *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\left(f\left(\frac{A+B}{2}\right) \leq\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \leq \int_0^1 f\left((1-t)A+tB\right)dt \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\left(\leq \frac{f(A)+f(B)}{2}\right). \tag{8.4}$$

PROOF. First of all, since the function f is continuos, the operator valued integral

$$\int_0^1 f\left((1-t)A+tB\right)dt$$

exists for any selfadjoint operators A and B with spectra in I.

We give here two proofs, the first using only the definition of operator convex functions and the second using the classical Hermite-Hadamard inequality for real valued functions.

1. By the definition of operator convex functions we have the double inequality:

$$f\left(\frac{C+D}{2}\right) \leq \frac{1}{2}\left[f\left((1-t)C+tD\right)+f\left((1-t)D+tC\right)\right] \leq \frac{1}{2}\left[f(C)+f(D)\right] \tag{8.5}$$

for any $t \in [0,1]$ and any selfadjoint operators C and D with the spectra in I.

Integrating the inequality (8.5) over $t \in [0,1]$ and taking into account that

$$\int_0^1 f\left((1-t)C+tD\right)dt = \int_0^1 f\left((1-t)D+tC\right)dt$$

then we deduce the Hermite-Hadamard inequality for operator convex functions

$$f\left(\frac{C+D}{2}\right) \leq \int_0^1 f\left((1-t)C+tD\right)dt \leq \frac{1}{2}\left[f(C)+f(D)\right] \tag{HHO}$$

that holds for any selfadjoint operators C and D with the spectra in I.

Now, on making use of the change of variable $u = 2t$ we have

$$\int_0^{1/2} f\left((1-t)A+tB\right)dt = \frac{1}{2}\int_0^1 f\left((1-u)A+u\frac{A+B}{2}\right)du$$

and by the change of variable $u = 2t-1$ we have

$$\int_{1/2}^1 f\left((1-t)A+tB\right)dt = \frac{1}{2}\int_0^1 f\left((1-u)\frac{A+B}{2}+uB\right)du.$$

Utilising the Hermite-Hadamard inequality (HHO) we can write

$$f\left(\frac{3A+B}{4}\right) \le \int_0^1 f\left((1-u)A + u\frac{A+B}{2}\right) du \le \frac{1}{2}\left[f(A) + f\left(\frac{A+B}{2}\right)\right]$$

and

$$f\left(\frac{A+3B}{4}\right) \le \int_0^1 f\left((1-u)\frac{A+B}{2} + uB\right) du \le \frac{1}{2}\left[f(A) + f\left(\frac{A+B}{2}\right)\right],$$

which by summation and division by two produces the desired result (8.4).

2. Consider now $x \in H, \|x\| = 1$ and two selfadjoint operators A and B with spectra in I. Define the real-valued function $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ given by $\varphi_{x,A,B}(t) = \langle f((1-t)A + tB)x, x\rangle$.

Since f is operator convex, then for any $t_1, t_2 \in [0,1]$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ we have

$$\begin{aligned}
&\varphi_{x,A,B}(\alpha t_1 + \beta t_2) \\
&= \langle f((1-(\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x\rangle \\
&= \langle f(\alpha[(1-t_1)A + t_1 B] + \beta[(1-t_2)A + t_2 B])x, x\rangle \\
&\le \alpha\langle f([(1-t_1)A + t_1 B])x, x\rangle + \beta\langle f([(1-t_2)A + t_2 B])x, x\rangle \\
&= \alpha\varphi_{x,A,B}(t_1) + \beta\varphi_{x,A,B}(t_2)
\end{aligned}$$

showing that $\varphi_{x,A,B}$ is a convex function on $[0,1]$.

Now we use the Hermite-Hadamard inequality for real-valued convex functions

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(s)\,ds \le \frac{g(a)+g(b)}{2}$$

to get that

$$\varphi_{x,A,B}\left(\frac{1}{4}\right) \le 2\int_0^{1/2} \varphi_{x,A,B}(t)\,dt \le \frac{\varphi_{x,A,B}(0) + \varphi_{x,A,B}\left(\frac{1}{2}\right)}{2}$$

and

$$\varphi_{x,A,B}\left(\frac{3}{4}\right) \le 2\int_{1/2}^1 \varphi_{x,A,B}(t)\,dt \le \frac{\varphi_{x,A,B}\left(\frac{1}{2}\right) + \varphi_{x,A,B}(1)}{2}$$

which by summation and division by two produces

$$\begin{aligned}
(8.6)\qquad &\frac{1}{2}\left\langle\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right]x, x\right\rangle \\
&\le \int_0^1 \langle f((1-t)A + tB)x, x\rangle\,dt \\
&\le \frac{1}{2}\left\langle\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]x, x\right\rangle.
\end{aligned}$$

Finally, since by the continuity of the function f we have

$$\int_0^1 \langle f((1-t)A + tB)x, x\rangle\,dt = \left\langle\int_0^1 f((1-t)A + tB)\,dt x, x\right\rangle$$

for any $x \in H, \|x\| = 1$ and any two selfadjoint operators A and B with spectra in I, we deduce from (8.6) the desired result (8.4). ∎

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

COROLLARY 8.2 (Dragomir, 2010, [**13**]). *With the assumptions in Theorem 8.1 we have the inequality*

$$(0\leq)\int_0^1 f\left((1-t)A+tB\right)dt-f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}-\int_0^1 f\left((1-t)A+tB\right)dt. \tag{8.7}$$

REMARK 8.1. Utilising different examples of operator convex or concave functions, we can provide inequalities of interest.

If $r\in[-1,0]\cup[1,2]$ then we have the inequalities for powers of operators

$$\begin{aligned}\left(\left(\frac{A+B}{2}\right)^r\leq\right)&\frac{1}{2}\left[\left(\frac{3A+B}{4}\right)^r+\left(\frac{A+3B}{4}\right)^r\right]\\ &\leq\int_0^1\left((1-t)A+tB\right)^r dt\\ &\leq\frac{1}{2}\left[\left(\frac{A+B}{2}\right)^r+\frac{A^r+B^r}{2}\right]\left(\leq\frac{A^r+B^r}{2}\right)\end{aligned} \tag{8.8}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$.

If $r\in(0,1)$ the inequalities in (8.8) hold with " $\geq$ " instead of " $\leq$ ".

We also have the following inequalities for logarithm

$$\begin{aligned}\left(\ln\left(\frac{A+B}{2}\right)\geq\right)&\frac{1}{2}\left[\ln\left(\frac{3A+B}{4}\right)+\ln\left(\frac{A+3B}{4}\right)\right]\\ &\geq\int_0^1\ln\left((1-t)A+tB\right)dt\\ &\geq\frac{1}{2}\left[\ln\left(\frac{A+B}{2}\right)+\frac{\ln(A)+\ln(B)}{2}\right]\left(\geq\frac{\ln(A)+\ln(B)}{2}\right)\end{aligned} \tag{8.9}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$.

8.3. Some Operator Quasilinearity Properties. Consider an operator convex function $f:I\subset\mathbb{R}\to\mathbb{R}$ defined on the interval I and two distinct selfadjoint operators A,B with the spectra in I. We denote by $[A,B]$ the closed operator segment defined by the family of operators $\{(1-t)A+tB,\, t\in[0,1]\}$. We also define the operator-valued functional

$$\Delta_f(A,B;t):=(1-t)f(A)+tf(B)-f\left((1-t)A+tB\right)\geq 0 \tag{8.10}$$

in the operator order, for any $t\in[0,1]$.

The following result concerning an operator quasilinearity property for the functional $\Delta_f(\cdot,\cdot;t)$ may be stated:

THEOREM 8.3 (Dragomir, 2010, [**13**]). *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an operator convex function on the interval I. Then for each A,B two distinct selfadjoint operators A,B with the spectra in I and $C\in[A,B]$ we have*

$$(0\leq)\,\Delta_f(A,C;t)+\Delta_f(C,B;t)\leq\Delta_f(A,B;t) \tag{8.11}$$

for each $t\in[0,1]$, i.e., the functional $\Delta_f(\cdot,\cdot;t)$ is operator superadditive as a function of interval.

If $[C, D] \subset [A, B]$ *, then*

$$(0 \leq) \Delta_f (C, D; t) \leq \Delta_f (A, B; t) \tag{8.12}$$

for each $t \in [0, 1]$ *, i.e., the functional* $\Delta_f (\cdot, \cdot; t)$ *is operator nondecreasing as a function of interval.*

PROOF. Let $C = (1 - s) A + sB$ with $s \in (0, 1)$. For $t \in (0, 1)$ we have

$$\begin{aligned}\Delta_f (C, B; t) &= (1 - t) f ((1 - s) A + sB) + tf (B) \\ &\quad - f ((1 - t) [(1 - s) A + sB] + tB)\end{aligned}$$

and

$$\begin{aligned}\Delta_f (A, C; t) &= (1 - t) f (A) + tf ((1 - s) A + sB) \\ &\quad - f ((1 - t) A + t [(1 - s) A + sB])\end{aligned}$$

giving that

$$\begin{aligned}&\Delta_f (A, C; t) + \Delta_f (C, B; t) - \Delta_f (A, B; t) \\ &= f ((1 - s) A + sB) + f ((1 - t) A + tB) \\ &- f ((1 - t) (1 - s) A + [(1 - t) s + t] B) - f ((1 - ts) A + tsB) .\end{aligned} \tag{8.13}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$\frac{\varphi (t_1) - \varphi (t_2)}{t_1 - t_2} \leq \frac{\varphi (s_1) - \varphi (s_2)}{s_1 - s_2}. \tag{8.14}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi (t) := \frac{\varphi (t) - \varphi (a)}{t - a}$$

is monotonic nondecreasing where is defined. Utilising this property repeatedly we have

$$\begin{aligned}\frac{\varphi (t_1) - \varphi (t_2)}{t_1 - t_2} &\leq \frac{\varphi (s_1) - \varphi (t_2)}{s_1 - t_2} = \frac{\varphi (t_2) - \varphi (s_1)}{t_2 - s_1} \\ &\leq \frac{\varphi (s_2) - \varphi (s_1)}{s_2 - s_1} = \frac{\varphi (s_1) - \varphi (s_2)}{s_1 - s_2}\end{aligned}$$

which proves the inequality (8.14).

For a vector $x \in H$, with $\|x\| = 1$, consider the function $\varphi_x : [0, 1] \to \mathbb{R}$ given by $\varphi_x (t) := \langle f ((1 - t) A + tB) x, x\rangle$. Since f is operator convex on I it follows that φ_x is convex on $[0, 1]$. Now, if we consider, for given $t, s \in (0, 1)$,

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1 - t) s =: s_2,$$

then we have

$$\varphi_x (t_1) = \langle f ((1 - ts) A + tsB) x, x\rangle$$

and

$$\varphi_x (t_2) = \langle f ((1 - t) A + tB) x, x\rangle$$

giving that

$$\frac{\varphi_x (t_1) - \varphi_x (t_2)}{t_1 - t_2} = \left\langle \left[\frac{f ((1 - ts) A + tsB) - f ((1 - t) A + tB)}{t (s - 1)}\right] x, x \right\rangle .$$

Also

$$\varphi_x (s_1) = \langle f ((1 - s) A + sB) x, x\rangle$$

and

$$\varphi_x(s_2) = \langle f((1-t)(1-s)A + [(1-t)s+t]B)x, x\rangle$$

giving that

$$\begin{aligned}&\frac{\varphi_x(s_1)-\varphi_x(s_2)}{s_1-s_2}\\ &= \left\langle \frac{f((1-s)A+sB) - f((1-t)(1-s)A+[(1-t)s+t]B)}{t(s-1)}x, x\right\rangle.\end{aligned}$$

Utilising the inequality (8.14) and multiplying with $t(s-1) < 0$ we deduce the following inequality in the operator order

$$\begin{aligned}f((1-ts)A+tsB) - f((1-t)A+tB)&\\ \geq f((1-s)A+sB) - f((1-t)(1-s)A+[(1-t)s+t]B)&.\end{aligned} \tag{8.15}$$

Finally, by (8.13) and (8.15) we get the desired result (8.11).

Applying repeatedly the superadditivity property we have for $[C,D]\subset[A,B]$ that

$$\Delta_f(A,C;t) + \Delta_f(C,D;t) + \Delta_f(D,B;t) \leq \Delta_f(A,B;t)$$

giving that

$$0 \leq \Delta_f(A,C;t) + \Delta_f(D,B;t) \leq \Delta_f(A,B;t) - \Delta_f(C,D;t)$$

which proves (8.12). ∎

For $t=\frac{1}{2}$ we consider the functional

$$\Delta_f(A,B) := \Delta_f\left(A,B;\frac{1}{2}\right) = \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Delta_f(\cdot,\cdot;t)$. We are able then to state the following

COROLLARY 8.4 (Dragomir, 2010, [**13**]). *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be an operator convex function on the interval I. Then for each A,B two distinct selfadjoint operators A,B with the spectra in I we have the following bounds in the operator order*

$$\inf_{C\in[A,B]}\left[f\left(\frac{A+C}{2}\right) + f\left(\frac{C+B}{2}\right) - f(C)\right] = f\left(\frac{A+B}{2}\right) \tag{8.16}$$

and

$$\begin{aligned}&\sup_{C,D\in[A,B]}\left[\frac{f(C)+f(D)}{2} - f\left(\frac{C+D}{2}\right)\right]\\ &= \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right).\end{aligned} \tag{8.17}$$

PROOF. By the superadditivity of the functional $\Delta_f(\cdot,\cdot)$ we have for each $C\in[A,B]$ that

$$\begin{aligned}&\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\\ &\geq \frac{f(A)+f(C)}{2} - f\left(\frac{A+C}{2}\right) + \frac{f(C)+f(B)}{2} - f\left(\frac{C+B}{2}\right)\end{aligned}$$

which is equivalent with

$$f\left(\frac{A+C}{2}\right) + f\left(\frac{C+B}{2}\right) - f(C) \geq f\left(\frac{A+B}{2}\right). \tag{8.18}$$

Since the equality case in (8.18) is realized for either $C = A$ or $C = B$ we get the desired bound (8.16).

The bound (8.17) is obvious by the monotonicity of the functional $\Delta_f(\cdot,\cdot)$ as a function of interval. ∎

Consider now the following functional

$$\Gamma_f(A,B;t) := f(A) + f(B) - f((1-t)A + tB) - f((1-t)B + tA),$$

where, as above, $f : C \subset X \to \mathbb{R}$ is a convex function on the convex set C and $A, B \in C$ while $t \in [0,1]$.

We notice that

$$\Gamma_f(A,B;t) = \Gamma_f(B,A;t) = \Gamma_f(A,B;1-t)$$

and

$$\Gamma_f(A,B;t) = \Delta_f(A,B;t) + \Delta_f(A,B;1-t) \geq 0$$

for any $A, B \in C$ and $t \in [0,1]$.

Therefore, we can state the following result as well

COROLLARY 8.5 (Dragomir, 2010, [**13**]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators A, B with the spectra in I, the functional $\Gamma_f(\cdot,\cdot;t)$ is operator superadditive and operator nondecreasing as a function of interval.*

In particular, if $C \in [A,B]$ then we have the inequality

$$\begin{aligned}(8.19)\qquad &\frac{1}{2}\left[f((1-t)A + tB) + f((1-t)B + tA)\right] \\ &\leq \frac{1}{2}\left[f((1-t)A + tC) + f((1-t)C + tA)\right] \\ &+ \frac{1}{2}\left[f((1-t)C + tB) + f((1-t)B + tC)\right] - f(C).\end{aligned}$$

Also, if $C, D \in [A,B]$ then we have the inequality

$$\begin{aligned}(8.20)\qquad &f(A) + f(B) - f((1-t)A + tB) - f((1-t)B + tA) \\ &\geq f(C) + f(D) - f((1-t)C + tD) - f((1-t)C + tD)\end{aligned}$$

for any $t \in [0,1]$.

Perhaps the most interesting functional we can consider is the following one:

$$(8.21)\qquad \Theta_f(A,B) = \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt.$$

Notice that, by the second Hermite-Hadamard inequality for operator convex functions we have that $\Theta_f(A,B) \geq 0$ in the operator order.

We also observe that

$$(8.22)\qquad \Theta_f(A,B) = \int_0^1 \Delta_f(A,B;t)\,dt = \int_0^1 \Delta_f(A,B;1-t)\,dt.$$

Utilising this representation, we can state the following result as well:

COROLLARY 8.6 (Dragomir, 2010, [**13**]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators A, B with the* *spectra in I, the functional $\Theta_f(\cdot,\cdot)$ is operator superadditive and operator nondecreasing as a function of interval. Moreover, we have the bounds in the operator order*

$$\inf_{C\in[A,B]}\left[\int_0^1 [f((1-t)A+tC)+f((1-t)C+tB)]\,dt - f(C)\right] = \int_0^1 f((1-t)A+tB)\,dt \tag{8.23}$$

and

$$\sup_{C,D\in[A,B]}\left[\frac{f(C)+f(D)}{2}-\int_0^1 f((1-t)C+tD)\,dt\right] = \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt. \tag{8.24}$$

REMARK 8.2. The above inequalities can be applied to various concrete operator convex function of interest.

If we choose for instance the inequality (8.24), then we get the following bounds in the operator order

$$\sup_{C,D\in[A,B]}\left[\frac{C^r+D^r}{2}-\int_0^1 ((1-t)C+tD)^r\,dt\right] = \frac{A^r+B^r}{2}-\int_0^1 ((1-t)A+tB)^r\,dt, \tag{8.25}$$

where $r\in[-1,0]\cup[1,2]$ and A,B are selfadjoint operators with spectra in $(0,\infty)$.

If $r\in(0,1)$ then

$$\sup_{C,D\in[A,B]}\left[\int_0^1 ((1-t)C+tD)^r\,dt-\frac{C^r+D^r}{2}\right] = \int_0^1 ((1-t)A+tB)^r\,dt-\frac{A^r+B^r}{2}, \tag{8.26}$$

and A,B are selfadjoint operators with spectra in $(0,\infty)$.

We also have the operator bound for the logarithm

$$\sup_{C,D\in[A,B]}\left[\int_0^1 \ln((1-t)C+tD)\,dt-\frac{\ln(C)+\ln(D)}{2}\right] = \int_0^1 \ln((1-t)A+tB)\,dt-\frac{\ln(A)+\ln(B)}{2}, \tag{8.27}$$

where A,B are selfadjoint operators with spectra in $(0,\infty)$.

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CHAPTER 3

Inequalities for the Čebyšev Functional

1. Introduction

The *Čebyšev*, or in a different spelling, *Chebyshev inequality* which compares the integral/discrete mean of the product with the product of the integral/discrete means is famous in the literature devoted to Mathematical Inequalities. It has been extended, generalised, refined etc...by many authors during the last century. A simple search utilising either spellings and the key word "inequality" in the title in the comprehensive *MathSciNet* database of the *American Mathematical Society* produces more than 200 research articles devoted to this result.

The sister result due to Grüss which provides error bounds for the magnitude of the difference between the integral mean of the product and the product of the integral means has also attracted much interest since it has been discovered in 1935 with more than 180 papers published, as a simple search in the same database reveals. Far more publications have been devoted to the applications of these inequalities and an accurate picture of the impacted results in various fields of Modern Mathematics is difficult to provide.

In this chapter, however, we present only some recent results due to the author for the corresponding operator versions of these two famous inequalities. Applications for particular functions of selfadjoint operators such as the power, logarithmic and exponential functions are provided as well.

2. Čebyšev's Inequality

2.1. Čebyšev's Inequality for Real Numbers. First of all, let us recall a number of classical results for sequences of real numbers concerning the celebrated *Čebyšev inequality.*

Consider the real sequences $(n-\text{tuples})$ $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)$ and the nonnegative sequence $\mathbf{p}=(p_1,\dots,p_n)$ with $P_n:=\sum_{i=1}^n p_i>0$. Define the *weighted Čebyšev's functional*

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}):=\frac{1}{P_n}\sum_{i=1}^n p_i a_i b_i-\frac{1}{P_n}\sum_{i=1}^n p_i a_i\cdot\frac{1}{P_n}\sum_{i=1}^n p_i b_i. \tag{2.1}$$

In 1882 – 1883, Čebyšev [**7**] and [**8**] proved that if $\mathbf{a}$ and $\mathbf{b}$ are *monotonic* in the same (opposite) sense, then

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b})\ge(\le)\,0. \tag{2.2}$$

In the special case $\mathbf{p}=\mathbf{a}\ge\mathbf{0}$, it appears that the inequality (2.2) has been obtained by Laplace long before Čebyšev (see for example [**51**, p. 240]).

The inequality (2.2) was mentioned by Hardy, Littlewood and Pólya in their survey [**46**] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are *synchronous* (*asynchronous*), this means that

$$(a_i-a_j)(b_i-b_j)\ge(\le)\,0 \text{ for any } i,j\in\{1,\dots,n\}, \tag{2.3}$$

then (2.2) holds true as well.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [**5**], which showed that, if $\mathbf{a}$, $\mathbf{b}$ are *monotonic in mean* in the same sense, i.e., for $P_k := \sum_{i=1}^{k} p_i$, $k = 1, \dots, n-1$;

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i a_i \leq (\geq) \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i a_i, \quad k \in \{1, \dots, n-1\} \tag{2.4}$$

and

$$\frac{1}{P_k}\sum_{i=1}^{k} p_i b_i \leq (\geq) \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i b_i, \quad k \in \{1, \dots, n-1\}, \tag{2.5}$$

then (2.2) holds with " $\geq$ ". If if $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the opposite sense then (2.2) holds with " $\leq$ ".

If one would like to drop the assumption of nonnegativity for the components of $\mathbf{p}$, then one may state the following inequality obtained by Mitrinović and Pečarić in 1991, [**50**]: If $0 \leq P_i \leq P_n$ for each $i \in \{1, \dots, n-1\}$, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq 0 \tag{2.6}$$

provided $\mathbf{a}$ and $\mathbf{b}$ are sequences with the same monotonicity.

If $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the opposite sense, the sign of the inequality (2.6) reverses.

Similar integral inequalities may be stated, however we do not present them here.

For other recent results on the Čebyšev inequality in either discrete or integral form see [**6**], [**19**], [**20**], [**26**], [**39**], [**40**], [**51**], [**49**], [**52**], [**57**], [**58**], [**59**], and the references therein.

The main aim of the present section is to provide operator versions for the Čebyšev inequality in different settings. Related results and some particular cases of interest are also given.

2.2. A Version of the Čebyšev Inequality for One Operator. We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0 \quad \text{for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [**42**] and [**41**].

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators.

THEOREM 2.1 (Dragomir, 2008, [**30**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A) g(A) x, x\rangle \geq (\leq) \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. We consider only the case of synchronous functions. In this case we have then

$$f(t) g(t) + f(s) g(s) \geq f(t) g(s) + f(s) g(t) \tag{2.8}$$

for each $t, s \in [a, b]$.

If we fix $s \in [a, b]$ and apply the property (P) for the inequality (2.8) then we have for each $x \in H$ with $\|x\| = 1$ that

$$\langle (f (A) g (A) + f (s) g (s) 1_H) x, x\rangle \geq \langle (g (s) f (A) + f (s) g (A)) x, x\rangle ,$$

which is clearly equivalent with

$$\langle f (A) g (A) x, x\rangle + f (s) g (s) \geq g (s) \langle f (A) x, x\rangle + f (s) \langle g (A) x, x\rangle \tag{2.9}$$

for each $s \in [a, b]$.

Now, if we apply again the property (P) for the inequality (2.9), then we have for any $y \in H$ with $\|y\| = 1$ that

$$\begin{aligned} &\langle (\langle f (A) g (A) x, x\rangle 1_H + f (A) g (A)) y, y\rangle \\ &\geq \langle (\langle f (A) x, x\rangle g (A) + \langle g (A) x, x\rangle f (A)) y, y\rangle , \end{aligned}$$

which is clearly equivalent with

$$\begin{aligned} &\langle f (A) g (A) x, x\rangle + \langle f (A) g (A) y, y\rangle \\ &\geq \langle f (A) x, x\rangle \langle g (A) y, y\rangle + \langle f (A) y, y\rangle \langle g (A) x, x\rangle \end{aligned} \tag{2.10}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. This is an inequality of interest in itself.

Finally, on making $y = x$ in (2.10) we deduce the desired result (2.7). ∎

Some particular cases are of interest for applications. In the first instance we consider the case of power functions.

EXAMPLE 2.1. *Assume that A is a positive operator on the Hilbert space H and $p, q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\left\langle A^{p+q}x, x\right\rangle \geq \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle . \tag{2.11}$$

If A is positive definite then the inequality (2.11) also holds for $p, q < 0$.

If A is positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then the reverse inequality holds in (2.11).

Another case of interest for applications is the exponential function.

EXAMPLE 2.2. *Assume that A is a selfadjoint operator on H. If $\alpha, \beta > 0$ or $\alpha, \beta < 0$, then*

$$\langle \exp [(\alpha + \beta) A] x, x\rangle \geq \langle \exp (\alpha A) x, x\rangle \cdot \langle \exp (\beta A) x, x\rangle \tag{2.12}$$

for each $x \in H$ with $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then the reverse inequality holds in (2.12).

The following particular cases may be of interest as well:

EXAMPLE 2.3. ***a.*** *Assume that A is positive definite and $p > 0$. Then*

$$\langle A^p \log Ax, x\rangle \geq \langle A^p x, x\rangle \cdot \langle \log Ax, x\rangle \tag{2.13}$$

for each $x \in H$ with $\|x\| = 1$. If $p < 0$ then the reverse inequality holds in (2.13).

b. *Assume that A is positive definite and $Sp (A) \subset (0, 1)$. If $r, s > 0$ or $r, s < 0$ then*

$$\begin{aligned} &\left\langle (1_H - A^r)^{-1} (1_H - A^s)^{-1} x, x\right\rangle \\ &\geq \left\langle (1_H - A^r)^{-1} x, x\right\rangle \cdot \left\langle (1_H - A^s)^{-1} x, x\right\rangle \end{aligned} \tag{2.14}$$

for each $x \in H$ with $\|x\| = 1$.

If either $r > 0, s < 0$ or $r < 0, s > 0$, then the reverse inequality holds in (2.14).

REMARK 2.1. We observe, from the proof of the above theorem that, if A and B are selfadjoint operators and $Sp(A), Sp(B) \subseteq [m, M]$, then for any continuous synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the more general result

$$\begin{aligned} &\langle f(A) g(A) x, x\rangle + \langle f(B) g(B) y, y\rangle \\ &\geq (\leq) \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(B) y, y\rangle \langle g(A) x, x\rangle \end{aligned} \tag{2.15}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If $f : [m, M] \longrightarrow (0, \infty)$ is continuous then the functions f^p, f^q are synchronous in the case when $p, q > 0$ or $p, q < 0$ and asynchronous when either $p > 0, q < 0$ or $p < 0, q > 0$. In this situation if A and B are positive definite operators then we have the inequality

$$\begin{aligned} &\left\langle f^{p+q}(A) x, x\right\rangle + \left\langle f^{p+q}(B) y, y\right\rangle \\ &\geq \langle f^p(A) x, x\rangle \langle f^q(B) y, y\rangle + \langle f^p(B) y, y\rangle \langle f^q(A) x, x\rangle \end{aligned} \tag{2.16}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$ where either $p, q > 0$ or $p, q < 0$. If $p > 0, q < 0$ or $p < 0, q > 0$ then the reverse inequality also holds in (2.16).

As particular cases, we should observe that for $p = q = 1$ and $f(t) = t$, we get from (2.16) the inequality

$$\left\langle A^2 x, x\right\rangle + \left\langle B^2 y, y\right\rangle \geq 2 \cdot \langle Ax, x\rangle \langle By, y\rangle \tag{2.17}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

For $p = 1$ and $q = -1$ we have from (2.16)

$$\langle Ax, x\rangle \left\langle B^{-1} y, y\right\rangle + \langle By, y\rangle \left\langle A^{-1} x, x\right\rangle \leq 2 \tag{2.18}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

2.3. A Version of the Čebyšev Inequality for n Operators. The following multiple operator version of Theorem 2.1 holds:

THEOREM 2.2 (Dragomir, 2008, [**30**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\begin{aligned} &\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle \\ &\geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle, \end{aligned} \tag{2.19}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

PROOF. As in [**44**, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \text{ and } \left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle .$$

Applying Theorem 2.1 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (2.19). ∎

The following particular cases may be of interest for applications.

EXAMPLE 2.4. *Assume that $A_j, j \in \{1,\ldots,n\}$ are positive operators on the Hilbert space H and $p,q > 0$. Then for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$ we have the inequality*

$$\left\langle \sum_{j=1}^{n} A_j^{p+q}x_j,x_j\right\rangle \geq \sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle . \tag{2.20}$$

If A_j are positive definite then the inequality (2.20) also holds for $p,q<0$.

If A_j are positive definite and either $p>0,q<0$ or $p<0,q>0$, then the reverse inequality holds in (2.20).

Another case of interest for applications is the exponential function.

EXAMPLE 2.5. *Assume that $A_j, j \in \{1,\ldots,n\}$ are selfadjoint operators on H. If $\alpha,\beta>0$ or $\alpha,\beta<0$, then*

$$\begin{aligned}\left\langle \sum_{j=1}^{n}\exp\left[\left(\alpha+\beta\right)A_j\right]x_j,x_j\right\rangle \\ \geq \sum_{j=1}^{n}\left\langle \exp\left(\alpha A_j\right)x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle \exp\left(\beta A_j\right)x_j,x_j\right\rangle\end{aligned} \tag{2.21}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If either $\alpha>0,\beta<0$ or $\alpha<0,\beta>0$, then the reverse inequality holds in (2.21).

The following particular cases may be of interest as well:

EXAMPLE 2.6. ***a.*** *Assume that $A_j, j \in \{1,\ldots,n\}$ are positive definite operators and $p>0$. Then*

$$\left\langle \sum_{j=1}^{n} A_j^{p}\log A_j x_j,x_j\right\rangle \geq \sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle \log A_j x_j,x_j\right\rangle \tag{2.22}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$. If $p<0$ then the reverse inequality holds in (2.22).

b. *If A_j are positive definite and $Sp\left(A_j\right)\subset(0,1)$ for $j\in\{1,\ldots,n\}$ then for $r,s>0$ or $r,s<0$ we have the inequality*

$$\begin{aligned}\left\langle \sum_{j=1}^{n}\left(1_H - A_j^{r}\right)^{-1}\left(1_H - A_j^{s}\right)^{-1}x_j,x_j\right\rangle \\ \geq \sum_{j=1}^{n}\left\langle \left(1_H - A_j^{r}\right)^{-1}x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle \left(1_H - A_j^{s}\right)^{-1}x_j,x_j\right\rangle\end{aligned} \tag{2.23}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If either $r>0,s<0$ or $r<0,s>0$, then the reverse inequality holds in (2.23).

2.4. Another Version of the Čebyšev Inequality for n Operators. The following different version of the Čebyšev inequality for a sequence of operators also holds:

THEOREM 2.3 (Dragomir, 2008, [**30**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \tag{2.24}$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$.

In particular

$$\left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \frac{1}{n} \sum_{j=1}^{n} f(A_j) x, x \right\rangle \cdot \left\langle \frac{1}{n} \sum_{j=1}^{n} g(A_j) x, x \right\rangle, \tag{2.25}$$

for each $x \in H$ with $\|x\| = 1$.

PROOF. We provide here two proofs. The first is based on the inequality (2.15) and generates as a by-product a more general result. The second is derived from Theorem 2.2.

1. If we make use of the inequality (2.15), then we can write

$$\langle f(A_j) g(A_j) x, x \rangle + \langle f(B_k) g(B_k) y, y \rangle \geq (\leq) \langle f(A_j) x, x \rangle \langle g(B_k) y, y \rangle + \langle f(B_k) y, y \rangle \langle g(A_j) x, x \rangle, \tag{2.26}$$

which holds for any A_j and B_k selfadjoint operators with $Sp(A_j), Sp(B_k) \subseteq [m, M]$, $j, k \in \{1, \ldots, n\}$ and for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, if $p_j \geq 0, q_k \geq 0, j, k \in \{1, \ldots, n\}$ and $\sum_{j=1}^{n} p_j = \sum_{k=1}^{n} q_k = 1$ then, by multiplying (2.26) with $p_j \geq 0, q_k \geq 0$ and summing over j and k from 1 to n we deduce the following inequality that is of interest in its own right:

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle + \left\langle \sum_{k=1}^{n} q_k f(B_k) g(B_k) y, y \right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \left\langle \sum_{k=1}^{n} q_k g(B_k) y, y \right\rangle + \left\langle \sum_{k=1}^{n} q_k f(B_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \tag{2.27}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, the choice $B_k = A_k, q_k = p_k$ and $y = x$ in (2.27) produces the desired result (2.24).

2. In we choose in Theorem 2.2 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (2.19) becomes (2.24). The details are omitted. ∎

REMARK 2.2. We remark that the case $n = 1$ in (2.24) produces the inequality (2.7).

The following particular cases are of interest:

EXAMPLE 2.7. *Assume that* $A_j, j \in \{1, \dots, n\}$ *are positive operators on the Hilbert space* H, $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $p, q > 0$. *Then for each* $x \in H$ *with* $\|x\| = 1$ *we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \geq \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle . \tag{2.28}$$

If $A_j, j \in \{1, \dots, n\}$ *are positive definite then the inequality (2.28) also holds for* $p, q < 0$.

If $A_j, j \in \{1, \dots, n\}$ *are positive definite and either* $p > 0, q < 0$ *or* $p < 0, q > 0$, *then the reverse inequality holds in (2.28).*

Another case of interest for applications is the exponential function.

EXAMPLE 2.8. *Assume that* $A_j, j \in \{1, \dots, n\}$ *are selfadjoint operators on* H *and* $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$. *If* $\alpha, \beta > 0$ *or* $\alpha, \beta < 0$, *then*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j \exp\left[(\alpha+\beta) A_j\right] x, x \right\rangle \\ &\geq \left\langle \sum_{j=1}^{n} p_j \exp\left(\alpha A_j\right) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \exp\left(\beta A_j\right) x, x \right\rangle \end{aligned} \tag{2.29}$$

for each $x \in H$ *with* $\|x\| = 1$.

If either $\alpha > 0, \beta < 0$ *or* $\alpha < 0, \beta > 0$, *then the reverse inequality holds in (2.29).*

The following particular cases may be of interest as well:

EXAMPLE 2.9. ***a.*** *Assume that* $A_j, j \in \{1, \dots, n\}$ *are positive definite operators on the Hilbert space* H, $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $p > 0$. *Then*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j A_j^{p} \log A_j x, x \right\rangle \\ &\geq \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle . \end{aligned} \tag{2.30}$$

If $p < 0$ *then the reverse inequality holds in (2.30).*

b. *Assume that* $A_j, j \in \{1, \dots, n\}$ *are positive definite operators on the Hilbert space* H, $Sp(A_j) \subset (0, 1)$ *and* $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$. *If* $r, s > 0$ *or* $r, s < 0$ *then*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^{r}\right)^{-1} \left(1_H - A_j^{s}\right)^{-1} x, x \right\rangle \\ &\geq \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^{r}\right)^{-1} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \left(1_H - A_j^{s}\right)^{-1} x, x \right\rangle \end{aligned} \tag{2.31}$$

for each $x \in H$ *with* $\|x\| = 1$.

If either $r > 0, s < 0$ *or* $r < 0, s > 0$, *then the reverse inequality holds in (2.31).*

We remark that the following operator norm inequality can be stated as well:

COROLLARY 2.4. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, asynchronous on $[m, M]$ and for $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n p_j = 1$ the operator $\sum_{j=1}^n p_j f(A_j) g(A_j)$ is positive, then*

$$\left\| \sum_{j=1}^n p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^n p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^n p_j g(A_j) \right\|. \tag{2.32}$$

PROOF. We have from (2.24) that

$$0 \leq \left\langle \sum_{j=1}^n p_j f(A_j) g(A_j) x, x \right\rangle \leq \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^n p_j g(A_j) x, x \right\rangle$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum in this inequality over $x \in H$ with $\|x\| = 1$ we deduce the desired result (2.32). ∎

The above Corollary 2.4 provides some interesting norm inequalities for sums of positive operators as follows:

EXAMPLE 2.10. ***a.*** *If $A_j, j \in \{1, \ldots, n\}$ are positive definite and either $p > 0, q < 0$ or $p < 0, q > 0$, then for $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^n p_j A_j^{p+q} \right\| \leq \left\| \sum_{j=1}^n p_j A_j^p \right\| \cdot \left\| \sum_{j=1}^n p_j A_j^q \right\|. \tag{2.33}$$

In particular

$$1 \leq \left\| \sum_{j=1}^n p_j A_j^r \right\| \cdot \left\| \sum_{j=1}^n p_j A_j^{-r} \right\| \tag{2.34}$$

for any $r > 0$.

b. *Assume that $A_j, j \in \{1, \ldots, n\}$ are selfadjoint operators on H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n p_j = 1$. If either $\alpha > 0, \beta < 0$ or $\alpha < 0, \beta > 0$, then*

$$\begin{aligned} &\left\| \sum_{j=1}^n p_j \exp[(\alpha + \beta) A_j] \right\| \\ &\leq \left\| \sum_{j=1}^n p_j \exp(\alpha A_j) \right\| \cdot \left\| \sum_{j=1}^n p_j \exp(\beta A_j) \right\|. \end{aligned} \tag{2.35}$$

In particular

$$1 \leq \left\| \sum_{j=1}^n p_j \exp(\gamma A_j) \right\| \cdot \left\| \sum_{j=1}^n p_j \exp(-\gamma A_j) \right\|.$$

for any $\gamma > 0$.

2.5. Related Results for One Operator. The following result that is related to the Čebyšev inequality may be stated:

THEOREM 2.5 (Dragomir, 2008, [**30**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\begin{aligned} &\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \\ &\geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] \cdot [g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle] \end{aligned} \tag{2.36}$$

for any $x \in H$ *with* $\|x\| = 1.$

If f, g *are asynchronous, then*

$$\begin{aligned} \text{(2.37)} \quad & \langle f(A)x, x\rangle \cdot \langle g(A)x, x\rangle - \langle f(A)g(A)x, x\rangle \\ & \geq [\langle f(A)x, x\rangle - f(\langle Ax, x\rangle)] \cdot [\langle g(A)x, x\rangle - g(\langle Ax, x\rangle)] \end{aligned}$$

for any $x \in H$ *with* $\|x\| = 1.$

PROOF. Since f, g are synchronous and $m \leq \langle Ax, x\rangle \leq M$ for any $x \in H$ with $\|x\| = 1,$ then we have

$$\text{(2.38)} \quad [f(t) - f(\langle Ax, x\rangle)][g(t) - g(\langle Ax, x\rangle)] \geq 0$$

for any $t \in [a, b]$ and $x \in H$ with $\|x\| = 1.$

On utilising the property (P) for the inequality (2.38) we have that

$$\text{(2.39)} \quad \langle [f(B) - f(\langle Ax, x\rangle)][g(B) - g(\langle Ax, x\rangle)]y, y\rangle \geq 0$$

for any B a bounded linear operator with $Sp(B) \subseteq [m, M]$ and $y \in H$ with $\|y\| = 1.$

Since

$$\begin{aligned} \text{(2.40)} \quad & \langle [f(B) - f(\langle Ax, x\rangle)][g(B) - g(\langle Ax, x\rangle)]y, y\rangle \\ & = \langle f(B)g(B)y, y\rangle + f(\langle Ax, x\rangle)g(\langle Ax, x\rangle) \\ & - \langle f(B)y, y\rangle g(\langle Ax, x\rangle) - f(\langle Ax, x\rangle)\langle g(B)y, y\rangle, \end{aligned}$$

then from (2.39) we get

$$\begin{aligned} & \langle f(B)g(B)y, y\rangle + f(\langle Ax, x\rangle)g(\langle Ax, x\rangle) \\ & \geq \langle f(B)y, y\rangle g(\langle Ax, x\rangle) + f(\langle Ax, x\rangle)\langle g(B)y, y\rangle \end{aligned}$$

which is clearly equivalent with

$$\begin{aligned} \text{(2.41)} \quad & \langle f(B)g(B)y, y\rangle - \langle f(A)y, y\rangle \cdot \langle g(A)y, y\rangle \\ & \geq [\langle f(B)y, y\rangle - f(\langle Ax, x\rangle)] \cdot [g(\langle Ax, x\rangle) - \langle g(B)y, y\rangle] \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1.$ This inequality is of interest in its own right.

Now, if we choose $B = A$ and $y = x$ in (2.41), then we deduce the desired result (2.36). ∎

The following result which improves the Čebyšev inequality may be stated:

COROLLARY 2.6 (Dragomir, 2008, [**30**]). *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M.$ *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous, synchronous and one is convex while the other is concave on* $[m, M]$, *then*

$$\begin{aligned} \text{(2.42)} \quad & \langle f(A)g(A)x, x\rangle - \langle f(A)x, x\rangle \cdot \langle g(A)x, x\rangle \\ & \geq [\langle f(A)x, x\rangle - f(\langle Ax, x\rangle)] \cdot [g(\langle Ax, x\rangle) - \langle g(A)x, x\rangle] \geq 0 \end{aligned}$$

for any $x \in H$ *with* $\|x\| = 1.$

If f, g *are asynchronous and either both of them are convex or both of them concave on* $[m, M]$*, then*

$$\begin{aligned} \text{(2.43)} \quad & \langle f(A)x, x\rangle \cdot \langle g(A)x, x\rangle - \langle f(A)g(A)x, x\rangle \\ & \geq [\langle f(A)x, x\rangle - f(\langle Ax, x\rangle)] \cdot [\langle g(A)x, x\rangle - g(\langle Ax, x\rangle)] \geq 0 \end{aligned}$$

for any $x \in H$ *with* $\|x\| = 1.$

PROOF. The second inequality follows by making use of the result due to Mond & Pečarić, see [**55**], [**54**] or [**44**, p. 5]:

$$\langle h(A)x,x\rangle \geq (\leq)\, h(\langle Ax,x\rangle) \tag{MP}$$

for any $x \in H$ with $\|x\| = 1$ provided that A is a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and h is convex (concave) on the given interval $[m,M]$. ∎

The above Corollary 2.6 offers the possibility to improve some of the results established before for power function as follows:

EXAMPLE 2.11. ***a.*** *Assume that A is a positive operator on the Hilbert space H. If $p \in (0,1)$ and $q \in (1,\infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\begin{aligned} &\left\langle A^{p+q}x,x\right\rangle - \langle A^p x,x\rangle \cdot \langle A^q x,x\rangle \\ &\geq [\langle A^q x,x\rangle - \langle Ax,x\rangle^q]\,[\langle Ax,x\rangle^p - \langle A^p x,x\rangle] \geq 0. \end{aligned} \tag{2.44}$$

If A is positive definite and $p > 1, q < 0$, then

$$\begin{aligned} &\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle - \left\langle A^{p+q}x,x\right\rangle \\ &\geq [\langle A^q x,x\rangle - \langle Ax,x\rangle^q]\,[\langle A^p x,x\rangle - \langle Ax,x\rangle^p] \geq 0 \end{aligned} \tag{2.45}$$

for each $x \in H$ with $\|x\| = 1$.

b. *Assume that A is positive definite and $p > 1$. Then*

$$\begin{aligned} &\langle A^p \log Ax,x\rangle - \langle A^p x,x\rangle \cdot \langle \log Ax,x\rangle \\ &\geq [\langle A^p x,x\rangle - \langle Ax,x\rangle^p]\,[\log\langle Ax,x\rangle - \langle \log Ax,x\rangle] \geq 0 \end{aligned} \tag{2.46}$$

for each $x \in H$ with $\|x\| = 1$.

2.6. Related Results for n Operators. We can state now the following generalisation of Theorem 2.5 for n operators:

THEOREM 2.7 (Dragomir, 2008, [**30**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for $j \in \{1,\dots,n\}$ and for some scalars $m < M$.*

(i) If $f,g : [m,M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m,M]$, then

$$\begin{aligned} &\sum_{j=1}^n \langle f(A_j)\,g(A_j)\,x_j,x_j\rangle - \sum_{j=1}^n \langle f(A_j)\,x_j,x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j)\,x_j,x_j\rangle \\ &\geq \left[\sum_{j=1}^n \langle f(A_j)\,x_j,x_j\rangle - f\left(\sum_{j=1}^n \langle A_j x_j,x_j\rangle\right)\right] \\ &\times \left[g\left(\sum_{j=1}^n \langle A_j x_j,x_j\rangle\right) - \sum_{j=1}^n \langle g(A_j)\,x_j,x_j\rangle\right] \end{aligned} \tag{2.47}$$

for each $x_j \in H, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$. Moreover, if one function is convex while the other is concave on $[m,M]$, then the right hand side of (2.47) is nonnegative.

(ii) If f, g are asynchronous on $[m, M]$, then

$$\begin{aligned}(2.48)\qquad &\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle\cdot\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle-\sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j,x_j\rangle\\ &\geq\left[\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle-f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)\right]\\ &\times\left[\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle-g\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)\right]\end{aligned}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$. Moreover, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (2.48) is nonnegative as well.

PROOF. The argument is similar to the one from the proof of Theorem 2.2 on utilising the results from one operator obtained in Theorem 2.5.

The nonnegativity of the right hand sides of the inequalities (2.47) and (2.48) follows by the use of the Jensen's type result from [**44**, p. 5]

$$(2.49)\qquad \sum_{j=1}^{n}\langle h(A_j)x_j,x_j\rangle\geq(\leq)\,h\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$, which holds provided that A_j are selfadjoint operators with $Sp(A_j)\subseteq[m, M]$ for $j\in\{1,\dots,n\}$ and for some scalars $m<M$ and h is convex (concave) on $[m, M]$.

The details are omitted. ∎

EXAMPLE 2.12. ***a.*** *Assume that $A_j, j\in\{1,\dots,n\}$ are positive operators on the Hilbert space H. If $p\in(0,1)$ and $q\in(1,\infty)$, then for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$ we have the inequality*

$$\begin{aligned}(2.50)\qquad &\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\\ &\geq\left[\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)^{q}\right]\\ &\times\left[\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)^{p}-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\right]\\ &\geq 0.\end{aligned}$$

If A_j are positive definite and $p > 1, q < 0$, then

$$\begin{aligned}(2.51)\qquad &\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle A_j^q x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^{p+q} x_j, x_j\right\rangle \\ &\geq \left[\sum_{j=1}^{n}\left\langle A_j^q x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^q\right] \\ &\times \left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p\right] \\ &\geq 0\end{aligned}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

b. *Assume that A_j are positive definite and $p > 1$. Then*

$$\begin{aligned}(2.52)\qquad &\sum_{j=1}^{n}\left\langle A_j^p \log A x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle \\ &\geq \left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p\right] \\ &\times \left[\sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle - \log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)\right] \\ &\geq 0\end{aligned}$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

The following result may be stated as well:

THEOREM 2.8 (Dragomir, 2008, [**30**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$.*

(i) If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, then

$$\begin{aligned}(2.53)\qquad &\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle \\ &- \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle \\ &\geq \left[f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle\right] \\ &\times \left[\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle - g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\right]\end{aligned}$$

for any $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$. Moreover, if one is convex while the other is concave on $[m, M]$, then the right hand side of (2.53) is nonnegative.

(ii) If f, g are asynchronous on $[m, M]$, then

$$\begin{aligned}(2.54)\quad & \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \\ & - \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \\ & \geq \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle - f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right] \\ & \times \left[\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle - g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \right]\end{aligned}$$

for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$. Moreover, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (2.54) is nonnegative as well.

PROOF. Follows from Theorem 2.7 on choosing $x_j = \sqrt{p_j} \cdot x,\ j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}, \sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$.

Also, the positivity of the right hand term in (2.53) follows by the Jensen's type inequality from the inequality (2.49) for the same choices, namely $x_j = \sqrt{p_j} \cdot x,\ j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}, \sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. ∎

Finally, we can list some particular inequalities that may be of interest for applications. They improve some result obtained above:

EXAMPLE 2.13. ***a.*** *Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space H and $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$. If $p \in (0, 1)$ and $q \in (1, \infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality*

$$\begin{aligned}(2.55)\quad & \left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle \\ & \geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^{q} x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{q} \right] \\ & \times \left[\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{p} - \left\langle \sum_{j=1}^{n} p_j A_j^{p} x, x \right\rangle \right] \\ & \geq 0.\end{aligned}$$

If $A_j, j \in \{1, \dots, n\}$ are positive definite and $p > 1, q < 0$, then

$$\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^{p+q} x, x \right\rangle \tag{2.56}$$
$$\geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^q x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^q \right]$$
$$\times \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right]$$
$$\geq 0$$

for each $x \in H$ with $\|x\| = 1$.

***b.** Assume that A_j, $j \in \{1, \dots, n\}$ are positive definite and $p > 1$. Then*

$$\left\langle \sum_{j=1}^{n} p_j A_j^p \log A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \tag{2.57}$$
$$\geq \left[\left\langle \sum_{j=1}^{n} p_j A_j^p x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^p \right]$$
$$\times \left[\log \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j \log A_j x, x \right\rangle \right]$$
$$\geq 0$$

for each $x \in H$ with $\|x\| = 1$.

3. GRÜSS INEQUALITY

3.1. Some Elementary Inequalities of Grüss Type. In 1935, G. Grüss [**45**] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \tag{3.1}$$
$$\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma),$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi, \qquad \gamma \leq g(x) \leq \Gamma \tag{3.2}$$

for each $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [**51**, Chapter X] established the following discrete version of Grüss' inequality:

Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be two $n-$tuples of real numbers such that $r \le a_i \le R$ and $s \le b_i \le S$ for $i = 1, \dots, n$. Then one has

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \le \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s), \tag{3.3}$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$.

For a simple proof of (3.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [**51**]. For other related results see the papers [**1**]-[**4**], [**11**]-[**9**], [**12**]-[**13**], [**15**]-[**37**], [**43**], [**56**], [**62**] and the references therein.

3.2. An Inequality of Grüss' Type for One Operator.

The following result may be stated:

THEOREM 3.1 (Dragomir, 2008, [**31**]). *Let A be a selfadjoint operator on the Hilbert space $(H; \langle .,. \rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$ and $\Gamma := \max_{t \in [m,M]} f(t)$ then*

$$\begin{aligned} &|\langle f(A) g(A) y, y \rangle - \langle f(A) y, y \rangle \cdot \langle g(A) x, x \rangle \\ &- \frac{\gamma + \Gamma}{2} [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle] \Big| \\ &\le \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2} \end{aligned} \tag{3.4}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. First of all, observe that, for each $\lambda \in \mathbb{R}$ and $x, y \in H$, $\|x\| = \|y\| = 1$ we have the identity

$$\begin{aligned} &\langle (f(A) - \lambda \cdot 1_H)(g(A) - \langle g(A) x, x \rangle \cdot 1_H) y, y \rangle \\ &= \langle f(A) g(A) y, y \rangle - \lambda \cdot [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle] \\ &- \langle g(A) x, x \rangle \langle f(A) y, y \rangle . \end{aligned} \tag{3.5}$$

Taking the modulus in (3.5) we have

$$\begin{aligned} &|\langle f(A) g(A) y, y \rangle - \lambda \cdot [\langle g(A) y, y \rangle - \langle g(A) x, x \rangle] \\ &\qquad - \langle g(A) x, x \rangle \langle f(A) y, y \rangle| \\ &= |\langle (g(A) - \langle g(A) x, x \rangle \cdot 1_H) y, (f(A) - \lambda \cdot 1_H) y \rangle| \\ &\le \|g(A) y - \langle g(A) x, x \rangle y\| \|f(A) y - \lambda y\| \\ &= \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2} \\ &\qquad \times \|f(A) y - \lambda y\| \\ &\le \left[\|g(A) y\|^2 + \langle g(A) x, x \rangle^2 - 2 \langle g(A) x, x \rangle \langle g(A) y, y \rangle \right]^{1/2} \\ &\qquad \times \|f(A) - \lambda \cdot 1_H\| \end{aligned} \tag{3.6}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, since $\gamma = \min_{t \in [m,M]} f(t)$ and $\Gamma = \max_{t \in [m,M]} f(t)$, then by the property (P) we have that $\gamma \le \langle f(A) y, y \rangle \le \Gamma$ for each $y \in H$ with $\|y\| = 1$ which is clearly equivalent with

$$\left| \langle f(A) y, y \rangle - \frac{\gamma + \Gamma}{2} \|y\|^2 \right| \le \frac{1}{2} (\Gamma - \gamma)$$

or with

$$\left|\left\langle\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)y,y\right\rangle\right|\le\frac{1}{2}(\Gamma-\gamma)$$

for each $y\in H$ with $\|y\|=1.$

Taking the supremum in this inequality we get

$$\left\|f(A)-\frac{\gamma+\Gamma}{2}\cdot 1_H\right\|\le\frac{1}{2}(\Gamma-\gamma),$$

which together with the inequality (3.6) applied for $\lambda=\frac{\gamma+\Gamma}{2}$ produces the desired result (3.4). ∎

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

COROLLARY 3.2 (Dragomir, 2008, [**31**]). *With the assumptions in Theorem 3.1 we have*

$$\begin{aligned}(3.7)\qquad &|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\\ &\le\frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)x\|^2-\langle g(A)x,x\rangle^2\right]^{1/2}\\ &\left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1$, *where* $\delta:=\min_{t\in[m,M]}g(t)$ *and* $\Delta:=\max_{t\in[m,M]}g(t)$.

PROOF. The first inequality follows from (3.4) by putting $y=x.$

Now, if we write the first inequality in (3.7) for $f=g$ we get

$$\begin{aligned}0&\le\|g(A)x\|^2-\langle g(A)x,x\rangle^2=\left\langle g^2(A)x,x\right\rangle-\langle g(A)x,x\rangle^2\\ &\le\frac{1}{2}(\Delta-\delta)\left[\|g(A)x\|^2-\langle g(A)x,x\rangle^2\right]^{1/2}\end{aligned}$$

which implies that

$$\left[\|g(A)x\|^2-\langle g(A)x,x\rangle^2\right]^{1/2}\le\frac{1}{2}(\Delta-\delta)$$

for each $x\in H$ with $\|x\|=1.$

This together with the first part of (3.7) proves the desired bound. ∎

The following particular cases that hold for power function are of interest:

EXAMPLE 3.1. *Let* A *be a selfadjoint operator with* $Sp(A)\subseteq[m,M]$ *for some scalars* $m<M.$

If A *is positive* $(m\ge0)$ *and* $p,q>0$, *then*

$$\begin{aligned}(3.8)\qquad &(0\le)\left\langle A^{p+q}x,x\right\rangle-\langle A^px,x\rangle\cdot\langle A^qx,x\rangle\\ &\le\frac{1}{2}\cdot(M^p-m^p)\left[\|A^qx\|^2-\langle A^qx,x\rangle^2\right]^{1/2}\\ &\left[\le\frac{1}{4}\cdot(M^p-m^p)(M^q-m^q)\right]\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1.$

If A is positive definite $(m>0)$ and $p,q<0$, then

$$(0\leq)\left\langle A^{p+q}x,x\right\rangle-\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle \tag{3.9}$$
$$\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\|A^{q}x\|^{2}-\langle A^{q}x,x\rangle^{2}\right]^{1/2}$$
$$\left[\leq\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]$$

for each $x\in H$ with $\|x\|=1$.

If A is positive definite $(m>0)$ and $p<0$, $q>0$ then

$$(0\leq)\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle-\left\langle A^{p+q}x,x\right\rangle \tag{3.10}$$
$$\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\|A^{q}x\|^{2}-\langle A^{q}x,x\rangle^{2}\right]^{1/2}$$
$$\left[\leq\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left(M^{q}-m^{q}\right)\right]$$

for each $x\in H$ with $\|x\|=1$.

If A is positive definite $(m>0)$ and $p>0$, $q<0$ then

$$(0\leq)\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle-\left\langle A^{p+q}x,x\right\rangle \tag{3.11}$$
$$\leq\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\left[\|A^{q}x\|^{2}-\langle A^{q}x,x\rangle^{2}\right]^{1/2}$$
$$\left[\leq\frac{1}{4}\cdot\left(M^{p}-m^{p}\right)\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]$$

for each $x\in H$ with $\|x\|=1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.8)-(3.11) follows from the Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

EXAMPLE 3.2. *Let A be a positive definite operator with $Sp(A)\subseteq[m,M]$ for some scalars $0<m<M$.*

If $p>0$ then

$$(0\leq)\langle A^{p}\ln Ax,x\rangle-\langle A^{p}x,x\rangle\cdot\langle\ln Ax,x\rangle \tag{3.12}$$
$$\leq\begin{cases}\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\left[\|\ln Ax\|^{2}-\langle\ln Ax,x\rangle^{2}\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\|A^{p}x\|^{2}-\langle A^{p}x,x\rangle^{2}\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\ln\sqrt{\frac{M}{m}}\right]$$

for each $x\in H$ with $\|x\|=1$.

If $p<0$ *then*

$$(0\leq)\left\langle A^{p}x,x\right\rangle\cdot\left\langle \ln Ax,x\right\rangle-\left\langle A^{p}\ln Ax,x\right\rangle \leq\begin{cases}\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\left\Vert \ln Ax\right\Vert^{2}-\left\langle \ln Ax,x\right\rangle^{2}\right]^{1/2}\\ \ln\sqrt{\frac{M}{m}}\cdot\left[\left\Vert A^{p}x\right\Vert^{2}-\left\langle A^{p}x,x\right\rangle^{2}\right]^{1/2}\end{cases} \left[\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right] \tag{3.13}$$

for each $x\in H$ *with* $\|x\|=1.$

3.3. An Inequality of Grüss' Type for n Operators. The following multiple operator version of Theorem 3.1 holds:

THEOREM 3.3 (Dragomir, 2008, [**31**]). *Let* A_j *be selfadjoint operators with* $Sp\left(A_j\right)\subseteq[m,M]$ *for* $j\in\{1,\dots,n\}$ *and for some scalars* $m<M$. *If* $f,g:[m,M]\longrightarrow\mathbb{R}$ *are continuous and* $\gamma:=\min_{t\in[m,M]}f(t)$ *and* $\Gamma:=\max_{t\in[m,M]}f(t)$ *then*

$$\left|\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)y_{j},y_{j}\right\rangle-\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)y_{j},y_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle -\frac{\gamma+\Gamma}{2}\left[\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)y_{j},y_{j}\right\rangle-\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right]\right| \leq\frac{1}{2}\left(\Gamma-\gamma\right)\left[\sum_{j=1}^{n}\left\Vert g\left(A_{j}\right)y_{j}\right\Vert^{2}+\left(\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right)^{2} -2\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)y_{j},y_{j}\right\rangle\right]^{\frac{1}{2}} \tag{3.14}$$

for each $x_j,y_j\in H,j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=\sum_{j=1}^{n}\|y_j\|^2=1.$

PROOF. Follows from Theorem 3.1. ∎

The following particular case provides a refinement of the Mond-Pečarić result.

COROLLARY 3.4 (Dragomir, 2008, [**31**]). *With the assumptions of Theorem 3.3 we have*

$$\left|\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle-\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right| \leq\frac{1}{2}\cdot\left(\Gamma-\gamma\right)\left[\sum_{j=1}^{n}\left\Vert g\left(A_{j}\right)x_{j}\right\Vert^{2}-\left(\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right)^{2}\right]^{1/2} \left(\leq\frac{1}{4}\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)\right) \tag{3.15}$$

for each $x_j\in H,j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$ *where* $\delta:=\min_{t\in[m,M]}g(t)$ *and* $\Delta:=\max_{t\in[m,M]}g(t)$.

EXAMPLE 3.3. *Let $A_j, j \in \{1, \dots, n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \dots, n\}$ for some scalars $m < M$.*

If A_j are positive $(m \geq 0)$ and $p, q > 0$, then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \tag{3.16}$$
$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p, q < 0$, then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \tag{3.17}$$
$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p < 0, q > 0$ then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \tag{3.18}$$
$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right]$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p > 0, q < 0$ then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle \tag{3.19}$$
$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle \right)^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.16)-(3.19) follows from the Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

EXAMPLE 3.4. *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$ for some scalars $0 < m < M$.*

If $p > 0$ then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \tag{3.20}$$

$$\leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \|\ln A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right)^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \left\| A_j^p x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \right)^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If $p < 0$ then

$$(0 \leq) \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle \tag{3.21}$$

$$\leq \begin{cases} \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \|\ln A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right)^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{j=1}^{n} \left\| A_j^p x_j \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \right)^2 \right]^{1/2} \end{cases}$$

$$\left[\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}} \right]$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

3.4. Another Inequality of Grüss' Type for n Operators.

The following different result for n operators can be stated as well:

THEOREM 3.5 (Dragomir, 2008, [**31**]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$ and $\Gamma := \max_{t \in [m,M]} f(t)$ then for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with*

$\sum_{j=1}^{n} p_j = 1$ *we have*

$$\begin{aligned}(3.22)\quad & \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) y, y \right\rangle \right. \\ & - \frac{\gamma+\Gamma}{2} \cdot \left[\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right] \\ & \left. - \left\langle \sum_{k=1}^{n} p_k f(A_k) y, y \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right| \\ & \leq \frac{\Gamma-\gamma}{2} \left[\sum_{k=1}^{n} p_k \|g(A_k) y\|^2 - 2 \left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right. \\ & \left. + \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle^2 \right]^{1/2},\end{aligned}$$

for each $x, y \in H$ *with* $\|x\| = \|y\| = 1.$

PROOF. Follows from Theorem 3.3 on choosing $x_j = \sqrt{p_j} \cdot x, y_j = \sqrt{p_j} \cdot y, j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

REMARK 3.1. The case $n = 1$ (therefore $p = 1$) in (3.22) provides the result from Theorem 3.1.

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

COROLLARY 3.6 (Dragomir, 2008, [**31**]). *With the assumptions of Theorem 3.5 we have*

$$\begin{aligned} & \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right| \\ (3.23)\quad & \leq \frac{\Gamma-\gamma}{2} \left(\sum_{k=1}^{n} p_k \|g(A_k) x\|^2 - \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle^2 \right)^{1/2} \\ & \left[\leq \frac{1}{4} \cdot (\Gamma-\gamma)(\Delta-\delta) \right]\end{aligned}$$

for each $x \in H$ *with* $\|x\| = 1$, *where* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

PROOF. It is similar with the proof from Corollary 3.2 and the details are omitted. ∎

The following particular cases that hold for power function are of interest:

EXAMPLE 3.5. *Let* $A_j, j \in \{1, \ldots, n\}$ *be a selfadjoint operators with* $Sp(A_j) \subseteq [m, M], j \in \{1, \ldots, n\}$ *for some scalars* $m < M$ *and* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1.$

If A_j, $j \in \{1, \dots, n\}$ *are positive* $(m \geq 0)$ *and* $p, q > 0$, *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \tag{3.24}$$
$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j \in \{1, \dots, n\}$ *are positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \tag{3.25}$$
$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j \in \{1, \dots, n\}$ *are positive definite* $(m > 0)$ *and* $p < 0$, $q > 0$ *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \tag{3.26}$$
$$\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j \in \{1, \dots, n\}$ *are positive definite* $(m > 0)$ *and* $p > 0$, $q < 0$ *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \tag{3.27}$$
$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.24)-(3.27) follows from the Theorem 2.1.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

EXAMPLE 3.6. *Let* $A_j,\ j \in \{1, \dots, n\}$ *be positive definite operators with* $Sp(A_j) \subseteq [m, M],\ j \in \{1, \dots, n\}$ *for some scalars* $0 < m < M$ *and* $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1.$

If $p > 0$ *then*

$$\left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle \tag{3.28}$$
$$\leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \cdot \left[\sum_{k=1}^{n} p_k \|\ln A_k x\|^2 - \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle^2\right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \|A_k^p x\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^2\right]^{1/2} \end{cases}$$
$$\left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}}\right]$$

for each $x \in H$ *with* $\|x\| = 1.$

If $p < 0$ *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle \tag{3.29}$$
$$- \left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle$$
$$\leq \begin{cases} \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \|\ln A_k x\|^2 - \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle^2\right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \|A_k^p x\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^2\right]^{1/2} \end{cases}$$
$$\left[\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}}\right]$$

for each $x \in H$ *with* $\|x\| = 1.$

The following norm inequalities may be stated as well:

COROLLARY 3.7 (Dragomir, 2008, [**31**]). *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \dots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous, then for each* $p_j \geq 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\| \tag{3.30}$$
$$+ \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta),$$

where $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$.

PROOF. Utilising the inequality (3.23) we deduce the inequality

$$\left|\left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle\right| \le \left|\left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle\right| \cdot \left|\left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle\right| + \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ we deduce the desired inequality (3.30). ∎

EXAMPLE 3.7. ***a.*** *Let* $A_j,\ j \in \{1, \dots, n\}$ *be a selfadjoint operators with* $Sp(A_j) \subseteq [m, M], j \in \{1, \dots, n\}$ *for some scalars* $m < M$ *and* $p_j \ge 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$.

If $A_j, j \in \{1, \dots, n\}$ *are positive* $(m \ge 0)$ *and* $p, q > 0$, *then*

$$\left\|\sum_{k=1}^{n} p_k A_k^{p+q}\right\| \le \left\|\sum_{k=1}^{n} p_k A_k^{p}\right\| \cdot \left\|\sum_{k=1}^{n} p_k A_k^{q}\right\| + \frac{1}{4}\cdot (M^p - m^p)(M^q - m^q). \tag{3.31}$$

If $A_j, j \in \{1, \dots, n\}$ *are positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$\left\|\sum_{k=1}^{n} p_k A_k^{p+q}\right\| \le \left\|\sum_{k=1}^{n} p_k A_k^{p}\right\| \cdot \left\|\sum_{k=1}^{n} p_k A_k^{q}\right\| + \frac{1}{4}\cdot \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q}m^{-q}}. \tag{3.32}$$

b. *Let* $A_j,\ j \in \{1, \dots, n\}$ *be positive definite operators with* $Sp(A_j) \subseteq [m, M],\ j \in \{1, \dots, n\}$ *for some scalars* $0 < m < M$ *and* $p_j \ge 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$.

If $p > 0$ *then*

$$\left\|\sum_{k=1}^{n} p_k A_k^{p} \ln A_k\right\| \le \left\|\sum_{k=1}^{n} p_k A_k^{p}\right\| \cdot \left\|\sum_{k=1}^{n} p_k \ln A_k\right\| + \frac{1}{2}\cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}}. \tag{3.33}$$

4. MORE INEQUALITIES OF GRÜSS TYPE

4.1. Some Vectorial Grüss' Type Inequalities. The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author:

LEMMA 4.1 (Dragomir, 2003 & 2004, **[23]**, **[28]**). *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, *and* $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ *such that*

$$\operatorname{Re}\langle \beta e - u, u - \alpha e \rangle \ge 0, \qquad \operatorname{Re}\langle \delta e - v, v - \gamma e \rangle \ge 0 \tag{4.1}$$

or equivalently,

$$\left\|u - \frac{\alpha + \beta}{2} e\right\| \le \frac{1}{2}|\beta - \alpha|, \qquad \left\|v - \frac{\gamma + \delta}{2} e\right\| \le \frac{1}{2}|\delta - \gamma|. \tag{4.2}$$

Then

$$(4.3)\qquad \begin{aligned} &\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right| \\ &\leq \frac{1}{4}\cdot|\beta-\alpha|\,|\delta-\gamma| \\ &-\left\{\begin{array}{l} \left[\operatorname{Re}\langle\beta e-u,u-\alpha e\rangle\operatorname{Re}\langle\delta e-v,v-\gamma e\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle u,e\rangle-\frac{\alpha+\beta}{2}\right|\left|\langle v,e\rangle-\frac{\gamma+\delta}{2}\right|. \end{array}\right. \end{aligned}$$

The first inequality has been obtained in [**23**] (see also [**27**, p. 44]) while the second result was established in [**28**] (see also [**27**, p. 90]). They provide refinements of the earlier result from [**16**] where only the first part of the bound, i.e., $\frac{1}{4}|\beta-\alpha|\,|\delta-\gamma|$ has been given. Notice that, as pointed out in [**28**], the upper bounds for the Grüss functional incorporated in (4.3) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

LEMMA 4.2 (Dragomir, 2004 & 2006, [**24**], [**29**]). *With the assumptions in Lemma 4.1 and if* $\operatorname{Re}(\beta\overline{\alpha})>0, \operatorname{Re}(\delta\overline{\gamma})>0$ *then*

$$(4.4)\qquad \begin{aligned} &\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right| \\ &\leq\left\{\begin{array}{l} \frac{1}{4}\cdot\frac{|\beta-\alpha||\delta-\gamma|}{\left[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}}\left|\langle u,e\rangle\langle e,v\rangle\right|, \\ \\ \left[\left(|\alpha+\beta|-2\left[\operatorname{Re}(\beta\overline{\alpha})\right]^{\frac{1}{2}}\right)\left(|\delta+\gamma|-2\left[\operatorname{Re}(\delta\overline{\gamma})\right]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\ \times\left[\left|\langle u,e\rangle\langle e,v\rangle\right|\right]^{\frac{1}{2}}. \end{array}\right. \end{aligned}$$

The first inequality has been established in [**24**] (see [**27**, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [**29**]. The details are omitted.

Finally, another inequality of Grüss type that has been obtained in [**25**] (see also [**27**, p. 65]) can be stated as:

LEMMA 4.3 (Dragomir, 2004, [**25**]). *With the assumptions in Lemma 4.1 and if* $\beta\neq-\alpha$, $\delta\neq-\gamma$ *then*

$$(4.5)\qquad \begin{aligned} &\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right| \\ &\leq\frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{\left[|\beta+\alpha|\,|\delta+\gamma|\right]^{\frac{1}{2}}}\left[\left(\|u\|+|\langle u,e\rangle|\right)\left(\|v\|+|\langle v,e\rangle|\right)\right]^{\frac{1}{2}}. \end{aligned}$$

4.2. Some Inequalities of Grüss' Type for One Operator. The following results incorporates some new inequalities of Grüss' type for two functions of a selfadjoint operator.

THEOREM 4.4 (Dragomir, 2008, [**32**]). *Let* A *be a selfadjoint operator on the Hilbert space* $(H;\langle .,.\rangle)$ *and assume that* $Sp(A)\subseteq[m,M]$ *for some scalars* $m<M$. *If* f *and* g *are continuous on* $[m,M]$ *and* $\gamma:=\min_{t\in[m,M]}f(t)$, $\Gamma:=\max_{t\in[m,M]}f(t)$, $\delta:=\min_{t\in[m,M]}g(t)$ *and*

$\Delta := \max_{t\in[m,M]} g(t)$ *then*

$$\begin{aligned} &|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle| \\ &\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta) \\ &-\begin{cases} [\langle \Gamma x - f(A)x, f(A)x-\gamma x\rangle\langle \Delta x - g(A)x, g(A)x-\delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A)x,x\rangle - \frac{\Gamma+\gamma}{2}\right|\left|\langle g(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \end{aligned} \tag{4.6}$$

for each $x \in H$ *with* $\|x\| = 1$.

Moreover if γ *and* δ *are positive, then we also have*

$$\begin{aligned} &|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle| \\ &\leq \begin{cases} \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\langle f(A)x,x\rangle\langle g(A)x,x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)[\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{\frac{1}{2}}, \end{cases} \end{aligned} \tag{4.7}$$

while for $\Gamma+\gamma, \Delta+\delta \neq 0$ *we have*

$$\begin{aligned} &|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle| \\ &\leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{[|\Gamma+\gamma||\Delta+\delta|]^{\frac{1}{2}}} \\ &\times[(\|f(A)x\| + |\langle f(A)x,x\rangle|)(\|g(A)x\| + |\langle g(A)x,x\rangle|)]^{\frac{1}{2}} \end{aligned} \tag{4.8}$$

for each $x \in H$ *with* $\|x\| = 1$.

PROOF. Since $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, the by the property (P) we have that

$$\gamma\cdot 1_H \leq f(A) \leq \Gamma\cdot 1_H \quad \text{and} \quad \delta\cdot 1_H \leq g(A) \leq \Delta\cdot 1_H$$

in the operator order, which imply that

$$\begin{aligned} [f(A)-\gamma\cdot 1][\Gamma\cdot 1_H - f(A)] &\geq 0 \text{ and} \\ [\Delta\cdot 1_H - g(A)][g(A)-\delta\cdot 1_H] &\geq 0 \end{aligned} \tag{4.9}$$

in the operator order.

We then have from (4.9)

$$\langle [f(A)-\gamma\cdot 1][\Gamma\cdot 1_H - f(A)]x,x\rangle \geq 0$$

and

$$\langle [\Delta\cdot 1_H - g(A)][g(A)-\delta\cdot 1_H]x,x\rangle \geq 0,$$

for each $x \in H$ with $\|x\| = 1$, which, by the fact that the involved operators are selfadjoint, are equivalent with the inequalities

$$\begin{aligned} &\langle \Gamma x - f(A)x, f(A)x-\gamma x\rangle \geq 0 \text{ and} \\ &\langle \Delta x - g(A)x, g(A)x-\delta x\rangle \geq 0, \end{aligned} \tag{4.10}$$

for each $x \in H$ with $\|x\| = 1$.

Now, if we apply Lemma 4.1 for $u = f(A)x$, $v = g(A)x$, $e = x$, and the real scalars Γ, γ, Δ and δ defined in the statement of the theorem, then we can state the inequality

$$\begin{aligned} (4.11)\qquad & |\langle f(A)x, g(A)x\rangle - \langle f(A)x,x\rangle\langle x, g(A)x\rangle| \\ & \leq \frac{1}{4}\cdot(\Gamma-\gamma)(\Delta-\delta) \\ & - \begin{cases} [\operatorname{Re}\langle \Gamma x - f(A)x, f(A)x-\gamma x\rangle \operatorname{Re}\langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A)x,x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent with the inequality (4.6).

The inequalities (4.7) and (4.8) follow by Lemma 4.2 and Lemma 4.3 respectively and the details are omitted. ∎

REMARK 4.1. The first inequality in (4.7) can be written in a more convenient way as

$$(4.12)\qquad \left|\frac{\langle f(A)g(A)x,x\rangle}{\langle f(A)x,x\rangle\langle g(A)x,x\rangle} - 1\right| \leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}$$

for each $x \in H$ with $\|x\| = 1$, while the second inequality has the following equivalent form

$$\begin{aligned} (4.13)\qquad & \left|\frac{\langle f(A)g(A)x,x\rangle}{[\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{1/2}} - [\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{1/2}\right| \\ & \leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

We know, from **[30]** that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, i.e., we recall that

$$[f(t) - f(s)][g(t) - g(s)]\ (\geq) \leq 0 \quad \text{for each } t, s \in [m, M],$$

then we have the inequality

$$(4.14)\qquad \langle f(A)g(A)x,x\rangle \geq (\leq) \langle f(A)x,x\rangle\langle g(A)x,x\rangle$$

for each $x \in H$ with $\|x\| = 1$,provided f, g are continuous on $[m, M]$ and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$.

Therefore, if f, g are synchronous then we have from (4.12) and from (4.13) the following results:

$$(4.15)\qquad 0 \leq \frac{\langle f(A)g(A)x,x\rangle}{\langle f(A)x,x\rangle\langle g(A)x,x\rangle} - 1 \leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}$$

and

$$\begin{aligned} (4.16)\qquad 0 & \leq \frac{\langle f(A)g(A)x,x\rangle}{[\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{1/2}} - [\langle f(A)x,x\rangle\langle g(A)x,x\rangle]^{1/2} \\ & \leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, respectively.

If f, g are asynchronous then

$$(4.17)\qquad 0 \leq 1 - \frac{\langle f(A)g(A)x,x\rangle}{\langle f(A)x,x\rangle\langle g(A)x,x\rangle} \leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}$$

and

$$0 \leq [\langle f(A)x,x\rangle \langle g(A)x,x\rangle]^{1/2} - \frac{\langle f(A)g(A)x,x\rangle}{[\langle f(A)x,x\rangle \langle g(A)x,x\rangle]^{1/2}} \leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{4.18}$$

for each $x \in H$ with $\|x\| = 1$, respectively.

It is obvious that all the inequalities from Theorem 4.4 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [**31**]. However we give here only a few provided by the inequalities (4.15) and (4.16) above.

EXAMPLE 4.1. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$.*

If A is positive $(m \geq 0)$ and $p, q > 0$, then

$$0 \leq \frac{\langle A^{p+q}x,x\rangle}{\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle} - 1 \leq \frac{1}{4}\cdot\frac{(M^p-m^p)(M^q-m^q)}{M^{\frac{p+q}{2}}m^{\frac{p+q}{2}}} \tag{4.19}$$

and

$$0 \leq \frac{\langle A^{p+q}x,x\rangle}{[\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2}} - [\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2} \leq \left(M^{\frac{p}{2}}-m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}}-m^{\frac{q}{2}}\right) \tag{4.20}$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite $(m > 0)$ and $p, q < 0$, then

$$0 \leq \frac{\langle A^{p+q}x,x\rangle}{\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle} - 1 \leq \frac{1}{4}\cdot\frac{(M^{-p}-m^{-p})(M^{-q}-m^{-q})}{M^{-\frac{p+q}{2}}m^{-\frac{p+q}{2}}} \tag{4.21}$$

and

$$0 \leq \frac{\langle A^{p+q}x,x\rangle}{[\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2}} - [\langle A^p x,x\rangle \cdot \langle A^q x,x\rangle]^{1/2} \leq \frac{\left(M^{-\frac{p}{2}}-m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}}-m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}}m^{-\frac{p+q}{2}}} \tag{4.22}$$

for each $x \in H$ with $\|x\| = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

EXAMPLE 4.2. *Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then*

$$0 \leq \frac{\langle A^p \ln Ax,x\rangle}{\langle A^p x,x\rangle \cdot \langle \ln Ax,x\rangle} - 1 \leq \frac{1}{4}\cdot\frac{(M^p-m^p)\ln\frac{M}{m}}{M^{\frac{p}{2}}m^{\frac{p}{2}}\sqrt{\ln M \cdot \ln m}} \tag{4.23}$$

and

$$0 \leq \frac{\langle A^p \ln Ax, x\rangle}{[\langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle]^{1/2}} - [\langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle]^{1/2} \leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left[\sqrt{\ln M} - \sqrt{\ln m}\right], \tag{4.24}$$

for each $x \in H$ *with* $\|x\| = 1$.

4.3. Some Inequalities of Grüss' Type for n Operators. The following extension for sequences of operators can be stated:

THEOREM 4.5 (Dragomir, 2008, [**32**]). *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$ *and* $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$ *then*

$$\begin{aligned} &\left|\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle\right| \\ &\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \\ &- \begin{cases} \left[\sum_{j=1}^n \langle \Gamma x_j - f(A_j) x_j, f(A_j) x_j - \gamma x_j\rangle \right. \\ \qquad \left. \times \sum_{j=1}^n \langle \Delta x_j - g(A_j) x_j, g(A_j) x - \delta x_j\rangle\right]^{\frac{1}{2}}, \\ \left|\sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \end{aligned} \tag{4.25}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$.

Moreover if γ *and* δ *are positive, then we also have*

$$\begin{aligned} &\left|\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle\right| \\ &\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \\ \times \left[\sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle\right]^{\frac{1}{2}}, \end{cases} \end{aligned} \tag{4.26}$$

while for $\Gamma+\gamma,\Delta+\delta\neq 0$ *we have*

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle-\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right|\\
&\leq\frac{1}{4}\cdot\frac{\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)}{\left[\left|\Gamma+\gamma\right|\left|\Delta+\delta\right|\right]^{\frac{1}{2}}}\\
&\times\left[\left(\left(\sum_{j=1}^{n}\left\|f\left(A_{j}\right)x_{j}\right\|^{2}\right)^{1/2}+\left|\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\right|\right)\right.\\
&\times\left.\left(\left(\sum_{j=1}^{n}\left\|g\left(A_{j}\right)x_{j}\right\|^{2}\right)^{1/2}+\left|\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right|\right)\right]^{1/2},
\end{aligned}\tag{4.27}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\left\|x_j\right\|^2=1$.

PROOF. Follows from Theorem 4.4. The details are omitted. ∎

REMARK 4.2. The first inequality in (4.26) can be written in a more convenient way as

$$\begin{aligned}
&\left|\frac{\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle}{\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle}-1\right|\\
&\leq\frac{1}{4}\cdot\frac{\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)}{\sqrt{\Gamma\gamma\Delta\delta}}
\end{aligned}\tag{4.28}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2=1$, while the second inequality has the following equivalent form

$$\begin{aligned}
&\left|\frac{\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle}{\left[\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right]^{1/2}}\right.\\
&\left.-\left[\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle\right]^{1/2}\right|\\
&\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)
\end{aligned}\tag{4.29}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2=1$.

We know, from [**30**] that if f,g are synchronous (asynchronous) functions on the interval $[m,M]$, then we have the inequality

$$\begin{aligned}
&\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle\\
&\geq(\leq)\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle
\end{aligned}\tag{4.30}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2=1$,provided f,g are continuous on $[m,M]$ and A_j are selfadjoint operators with $Sp\left(A_j\right)\subseteq[m,M]$, $j\in\{1,\dots,n\}$.

Therefore, if f, g are synchronous then we have from (4.28) and from (4.29) the following results:

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} - 1 \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \tag{4.31}$$

and

$$\begin{aligned} 0 &\leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} \\ &\quad - \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} \\ &\leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \end{aligned} \tag{4.32}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \tag{4.33}$$

and

$$\begin{aligned} 0 &\leq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} \\ &\quad - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} \\ &\leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \end{aligned} \tag{4.34}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

It is obvious that all the inequalities from Theorem 4.5 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [**31**]. However we give here only a few provided by the inequalities (4.31) and (4.32) above.

EXAMPLE 4.3. *Let A_j $j \in \{1, \ldots, n\}$ be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$.*

If A_j are positive $(m \geq 0)$ and $p, q > 0$, then

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}} \tag{4.35}$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} \leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}} - m^{\frac{q}{2}}\right) \tag{4.36}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

If A *is positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle} - 1 \leq \frac{1}{4} \cdot \frac{\left(M^{-p} - m^{-p}\right)\left(M^{-q} - m^{-q}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.37}$$

and

$$0 \leq \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} - \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x, x \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} \leq \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \tag{4.38}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

EXAMPLE 4.4. *Let* A *be a positive definite operator with* $Sp(A) \subseteq [m, M]$ *for some scalars* $1 < m < M$. *If* $p > 0$ *then*

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p} \ln A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle} - 1 \leq \frac{1}{4} \cdot \frac{\left(M^{p} - m^{p}\right) \ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}} \tag{4.39}$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p} \ln A_j x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2} \leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left[\sqrt{\ln M} - \sqrt{\ln m}\right], \tag{4.40}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1.$

Similar inequalities may be stated for $p < 0$. The details are omitted.
The following result for n operators can be stated as well:

COROLLARY 4.6. *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$ *and* $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$ *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right| \tag{4.41}$$
$$\leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)$$
$$-\begin{cases} \left[\sum_{j=1}^{n} p_j \langle \Gamma x - f(A_j) x, f(A_j) x - \gamma x\rangle \right. \\ \qquad \left. \times \sum_{j=1}^{n} p_j \langle \Delta x - g(A_j) x, g(A_j) x - \delta x\rangle\right]^{\frac{1}{2}}, \\ \left|\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases}$$

for each $x \in H$, *with* $\|x\|^2 = 1.$

Moreover if γ *and* δ *are positive, then we also have*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right| \tag{4.42}$$
$$\leq \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \\ \times \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{\frac{1}{2}}. \end{cases}$$

while for $\Gamma+\gamma,\Delta+\delta\neq 0$ *we have*

$$\left|\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right|$$
$$\leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{[|\Gamma+\gamma||\Delta+\delta|]^{\frac{1}{2}}} \tag{4.43}$$
$$\times\left[\left(\left(\sum_{j=1}^{n} p_j \|f(A_j)x\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle\right|\right)\right.$$
$$\left.\times\left(\left(\sum_{j=1}^{n} p_j \|g(A_j)x\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right|\right)\right]^{1/2}$$

for each $x\in H$, *with* $\|x\|^2=1$.

PROOF. Follows from Theorem 4.5 on choosing $x_j=\sqrt{p_j}\cdot x,\ j\in\{1,\dots,n\}$, where $p_j\geq 0, j\in\{1,\dots,n\}, \sum_{j=1}^{n}p_j=1$ and $x\in H$, with $\|x\|=1$. The details are omitted. ∎

REMARK 4.3. The first inequality in (4.42) can be written in a more convenient way as

$$\left|\frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle} - 1\right| \leq \frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.44}$$

for each $x\in H$, with $\|x\|^2=1$, while the second inequality has the following equivalent form

$$\left|\frac{\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle}{\left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2}} - \left[\left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle\right]^{1/2}\right| \leq \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{4.45}$$

for each $x\in H$, with $\|x\|^2=1$.

We know, from [**30**] that if f,g are synchronous (asynchronous) functions on the interval $[m,M]$, then we have the inequality

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x\right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle \tag{4.46}$$

for each $x\in H$, with $\|x\|^2=1$, provided f,g are continuous on $[m,M]$ and A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]$, $j\in\{1,\dots,n\}$.

Therefore, if f,g are synchronous then we have from (4.44) and from (4.45) the following results:

$$0\leq\frac{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)g\left(A_{j}\right)x,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle}-1\leq\frac{1}{4}\cdot\frac{\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)}{\sqrt{\Gamma\gamma\Delta\delta}}\tag{4.47}$$

and

$$\begin{aligned}0&\leq\frac{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)g\left(A_{j}\right)x,x\right\rangle}{\left[\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle\right]^{1/2}}\\&\quad-\left[\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle\right]^{1/2}\\&\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\end{aligned}\tag{4.48}$$

for each $x\in H$, with $\|x\|=1$, respectively.

If f,g are asynchronous then

$$0\leq1-\frac{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)g\left(A_{j}\right)x,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle}\leq\frac{1}{4}\cdot\frac{\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)}{\sqrt{\Gamma\gamma\Delta\delta}}\tag{4.49}$$

and

$$\begin{aligned}0&\leq\left[\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle\right]^{1/2}\\&\quad-\frac{\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)g\left(A_{j}\right)x,x\right\rangle}{\left[\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle\right]^{1/2}}\\&\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\end{aligned}\tag{4.50}$$

for each $x\in H$, with $\|x\|=1$, respectively.

The above inequalities (4.47) - (4.50) can be used to state various particular inequalities as in the previous examples, however the details are left to the interested reader.

5. More Inequalities for the Čebyšev Functional

5.1. A Refinement and Some Related Results. The following result can be stated:

Theorem 5.1 (Dragomir, 2008, [**33**]). *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f,g:[m,M]\longrightarrow\mathbb{R}$ are continuous with $\delta:=$*

$\min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then*

$$\begin{aligned}|C(f,g;A;x)| &\\ \le \frac{1}{2}(\Delta-\delta)\langle|f(A)-\langle f(A)x,x\rangle\cdot 1_H|x,x\rangle &\\ \le \frac{1}{2}(\Delta-\delta)C^{1/2}(f,f;A;x), &\end{aligned} \tag{5.1}$$

for any $x\in H$ *with* $\|x\|=1$.

PROOF. Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we have

$$\left|g(t)-\frac{\Delta+\delta}{2}\right| \le \frac{1}{2}(\Delta-\delta), \tag{5.2}$$

for any $t\in[m,M]$ and for any $x\in H$ with $\|x\|=1$.

If we multiply the inequality (5.2) with $|f(t)-\langle f(A)x,x\rangle|$ we get

$$\begin{aligned}&\left|f(t)g(t)-\langle f(A)x,x\rangle g(t)-\frac{\Delta+\delta}{2}f(t)+\frac{\Delta+\delta}{2}\langle f(A)x,x\rangle\right| \\ &\le \frac{1}{2}(\Delta-\delta)|f(t)-\langle f(A)x,x\rangle|,\end{aligned} \tag{5.3}$$

for any $t\in[m,M]$ and for any $x\in H$ with $\|x\|=1$.

Now, if we apply the property (P) for the inequality (5.3) and a selfadjoint operator B with $Sp(B)\subset[m,M]$, then we get the following inequality of interest in itself:

$$\begin{aligned}&|\langle f(B)g(B)y,y\rangle-\langle f(A)x,x\rangle\langle g(B)y,y\rangle \\ &-\frac{\Delta+\delta}{2}\langle f(B)y,y\rangle+\frac{\Delta+\delta}{2}\langle f(A)x,x\rangle\Big| \\ &\le \frac{1}{2}(\Delta-\delta)\langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|y,y\rangle,\end{aligned} \tag{5.4}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

If we choose in (5.4) $y=x$ and $B=A$, then we deduce the first inequality in (5.1).

Now, by the Schwarz inequality in H we have

$$\begin{aligned}&\langle|f(A)-\langle f(A)x,x\rangle\cdot 1_H|x,x\rangle \\ &\le \||f(A)-\langle f(A)x,x\rangle\cdot 1_H|x\| \\ &= \|f(A)x-\langle f(A)x,x\rangle\cdot x\| \\ &= \left[\|f(A)x\|^2-\langle f(A)x,x\rangle^2\right]^{1/2} \\ &= C^{1/2}(f,f;A;x),\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, and the second part of (5.1) is also proved. ∎

Let U be a selfadjoint operator on the Hilbert space $(H,\langle.,.\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be its *spectral family*. Then for any continuous function $f:[m,M]\to\mathbb{R}$, it is well known that we have the following representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x,x\rangle = \int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x,x\rangle), \tag{5.5}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m-0) = 0 \quad \text{and} \quad g_x(M) = 1 \tag{5.6}$$

for any $x \in H$ with $\|x\| = 1$.

The following result is of interest:

THEOREM 5.2 (Dragomir, 2008, [**33**]). *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r-L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \le L|s-t|^r \quad \textit{for any } s, t \in [m, M],$$

then we have the Ostrowski type inequality for selfadjoint operators:

$$|f(s) - \langle f(A)x, x\rangle| \le L\left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^r, \tag{5.7}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Moreover, we have

$$\begin{aligned} &|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle| \\ &\le \langle |f(B) - \langle f(A)x, x\rangle \cdot 1_H| y, y\rangle \\ &\le L\left[\frac{1}{2}(M-m) + \left\langle \left|B - \frac{m+M}{2}\cdot 1_H\right| y, y\right\rangle\right]^r, \end{aligned} \tag{5.8}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. We use the following Ostrowski type inequality for the Riemann-Stieltjes integral obtained by the author in [**22**]:

$$\begin{aligned} &\left|f(s)[u(b) - u(a)] - \int_a^b f(t)\,du(t)\right| \\ &\le L\left[\frac{1}{2}(b-a) + \left|s - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) \end{aligned} \tag{5.9}$$

for any $s \in [a, b]$, provided that f is of $r-L-$Hölder type on $[a, b]$, u is of bounded variation on $[a, b]$ and $\bigvee_a^b (u)$ denotes the total variation of u on $[a, b]$.

Now, applying this inequality for $u(\lambda) = g_x(\lambda) := \langle E_\lambda x, x\rangle$ where $x \in H$ with $\|x\| = 1$ we get

$$\begin{aligned} &\left|f(s) - \int_{m-0}^M f(\lambda)\,d(\langle E_\lambda x, x\rangle)\right| \\ &\le L\left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^r \bigvee_{m-0}^M (g_x) \end{aligned} \tag{5.10}$$

which, by (5.5) and (5.6) is equivalent with (5.7).

By applying the property (P) for the inequality (5.7) and the operator B we have

$$\begin{aligned}
&\left\langle \left| f(B) - \langle f(A)x,x\rangle \cdot 1_H \right| y,y \right\rangle \\
&\le L\left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2}\cdot 1_H \right| \right]^r y,y \right\rangle \\
&\le L\left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2} \right| \cdot 1_H \right] y,y \right\rangle^r \\
&= L\left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2}\cdot 1_H \right| y,y \right\rangle \right]^r
\end{aligned}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$, which proves the second inequality in (5.8).

Further, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [**44**, p. 5]) applied for the modulus, we can state that

$$|\langle h(A)x,x\rangle| \le \langle |h(A)|x,x\rangle \tag{M}$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m,M]$.

Now, if we apply the inequality (M), then we have

$$|\langle [f(B) - \langle f(A)x,x\rangle \cdot 1_H]y,y\rangle| \le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H| y,y\rangle$$

which shows the first part of (5.8), and the proof is complete. ∎

REMARK 5.1. With the above assumptions for f, A and B we have the following particular inequalities of interest:

$$\left| f\left(\frac{m+M}{2}\right) - \langle f(A)x,x\rangle \right| \le \frac{1}{2^r} L(M-m)^r \tag{5.11}$$

and

$$\begin{aligned}
&|f(\langle Ax,x\rangle) - \langle f(A)x,x\rangle| \\
&\le L\left[\frac{1}{2}(M-m) + \left| \langle Ax,x\rangle - \frac{m+M}{2} \right| \right]^r,
\end{aligned} \tag{5.12}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequalities:

$$\begin{aligned}
&|\langle f(A)y,y\rangle - \langle f(A)x,x\rangle| \\
&\le \langle |f(A) - \langle f(A)x,x\rangle \cdot 1_H| y,y\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2}\cdot 1_H \right| y,y \right\rangle \right]^r,
\end{aligned} \tag{5.13}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$,

$$\begin{aligned}
&|\langle [f(B) - f(A)]x,x\rangle| \\
&\le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H| x,x\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2}\cdot 1_H \right| x,x \right\rangle \right]^r
\end{aligned} \tag{5.14}$$

and, more particularly,

$$\begin{aligned}
&\langle |f(A) - \langle f(A)x,x\rangle \cdot 1_H| x,x\rangle \\
&\le L\left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2}\cdot 1_H \right| x,x \right\rangle \right]^r,
\end{aligned} \tag{5.15}$$

for any $x \in H$ with $\|x\| = 1.$

We also have the norm inequality

$$\|f(B) - f(A)\| \le L\left[\frac{1}{2}(M-m) + \left\|B - \frac{m+M}{2} \cdot 1_H\right\|\right]^r. \tag{5.16}$$

The following corollary of the above Theorem 5.2 can be useful for applications:

COROLLARY 5.3 (Dragomir, 2008, [**33**]). *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous then we have the Ostrowski type inequality for selfadjoint operators:*

$$\begin{aligned} &|f(s) - \langle f(A)x, x\rangle| \\ &\le \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m, M]; \\ \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \end{aligned} \tag{5.17}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$, where $\|\cdot\|_{p,[m,M]}$ are the Lebesgue norms, i.e.,

$$\|h\|_{\infty,[m,M]} := ess \sup_{t \in [m,M]} \|h(t)\|$$

and

$$\|h\|_{p,[m,M]} := \left(\int_m^M |h(t)|^p\right)^{1/p}, \quad p \ge 1.$$

Moreover, we have

$$\begin{aligned} &|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle| \\ &\le \langle |f(B) - \langle f(A)x, x\rangle \cdot 1_H| y, y\rangle \\ &\le \begin{cases} \left[\frac{M-m}{2} + \left\langle \left|B - \frac{m+M}{2} \cdot 1_H\right| y, y\right\rangle\right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m, M]; \\ \\ \left[\frac{M-m}{2} + \left\langle \left|B - \frac{m+M}{2} \cdot 1_H\right| y, y\right\rangle\right]^{\frac{1}{q}} \|f'\|_{p,[m,M]} & \text{if } f' \in L_p[m, M], \\ & p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \end{aligned} \tag{5.18}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

Now, on utilising Theorem 5.1 we can provide the following upper bound for the Čebyšev functional that may be more useful in applications:

COROLLARY 5.4 (Dragomir, 2008, [**33**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then for any $f : [m, M] \longrightarrow \mathbb{R}$ of $r - L-$Hölder type we have the inequality:*

$$\begin{aligned} &|C(f, g; A; x)| \\ &\le \frac{1}{2}(\Delta - \delta) L\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x, x\right\rangle\right]^r, \end{aligned} \tag{5.19}$$

for any $x \in H$ with $\|x\| = 1.$

REMARK 5.2. With the assumptions from Corollary 5.4 for g and A and if f is absolutely continuos on $[m,M]$, then we have the inequalities:

$$
\begin{aligned}
(5.20)\quad & |C(f,g;A;x)| \\
& \le \frac{1}{2}(\Delta-\delta) \\
& \times \begin{cases} \left[\frac{1}{2}(M-m)+\left\langle \left|A-\frac{m+M}{2}\cdot 1_H\right| x,x\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f'\in L_\infty[m,M]; \\ \left[\frac{1}{2}(M-m)+\left\langle \left|A-\frac{m+M}{2}\cdot 1_H\right| x,x\right\rangle\right]^{1/q}\|f'\|_{p,[m,M]} & \begin{array}{l}\text{if } f'\in L_\infty[m,M], \\ p,q>1, \frac{1}{p}+\frac{1}{q}=1\end{array} \end{cases}
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$.

5.2. Some Inequalities for Sequences of Operators. Consider the sequence of selfadjoint operators $\mathbf{A}=(A_1,\dots,A_n)$ with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,\dots,n\}$ and for some scalars $m<M$. If $\mathbf{x}=(x_1,\dots,x_n)\in H^n$ are such that $\sum_{j=1}^n\|x_j\|^2=1$, then we can consider the following Čebyšev type functional

$$
\begin{aligned}
C(f,g;\mathbf{A},\mathbf{x}) := & \sum_{j=1}^n \langle f(A_j)g(A_j)x_j,x_j\rangle \\
& -\sum_{j=1}^n \langle f(A_j)x_j,x_j\rangle\cdot\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle .
\end{aligned}
$$

As a particular case of the above functional and for a probability sequence $\mathbf{p}=(p_1,\dots,p_n)$, i.e., $p_j\ge 0$ for $j\in\{1,\dots,n\}$ and $\sum_{j=1}^n p_j=1$, we can also consider the functional

$$
\begin{aligned}
C(f,g;\mathbf{A},\mathbf{p},x) := & \left\langle \sum_{j=1}^n p_j f(A_j)g(A_j)x,x\right\rangle \\
& -\left\langle \sum_{j=1}^n p_j f(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^n p_j g(A_j)x,x\right\rangle
\end{aligned}
$$

where $x\in H$, $\|x\|=1$.

We know, from [**30**] that for the sequence of selfadjoint operators $\mathbf{A}=(A_1,\dots,A_n)$ with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,\dots,n\}$ and for the synchronous (asynchronous) functions $f,g:[m,M]\longrightarrow\mathbb{R}$ we have the inequality

$$
C(f,g;\mathbf{A},\mathbf{x})\ge(\le)\,0 \tag{5.21}
$$

for any $\mathbf{x}=(x_1,\dots,x_n)\in H^n$ with $\sum_{j=1}^n\|x_j\|^2=1$. Also, for any probability distribution $\mathbf{p}=(p_1,\dots,p_n)$ and any $x\in H$, $\|x\|=1$ we have

$$
C(f,g;\mathbf{A},\mathbf{p},x)\ge(\le)\,0. \tag{5.22}
$$

On the other hand, the following Grüss' type inequality is valid as well [**31**]:

$$
\begin{aligned}
|C(f,g;\mathbf{A},\mathbf{x})| & \le \frac{1}{2}\cdot(\Gamma-\gamma)\left[C(g,g;\mathbf{A},\mathbf{x})\right]^{1/2} \\
& \left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)
\end{aligned} \tag{5.23}
$$

for any $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where f and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ and any $x \in H$, $\|x\| = 1$ we also have the inequality:

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \le \frac{1}{2} \cdot (\Gamma - \gamma) [C(g, g; \mathbf{A}, \mathbf{p}, x)]^{1/2} \left(\le \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right). \tag{5.24}$$

We can state now the following new result:

THEOREM 5.5 (Dragomir, 2008, [**33**]). *Consider the sequence of selfadjoint operators* $\mathbf{A} = (A_1, \dots, A_n)$ *with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \dots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous with* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then*

$$\begin{aligned} |C(f, g; \mathbf{A}; \mathbf{x})| &\le \frac{1}{2}(\Delta - \delta) \sum_{j=1}^{n} \left\langle \left| f(A_j) - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \cdot 1_H \right| x_j, x_j \right\rangle \\ &\le \frac{1}{2}(\Delta - \delta) C^{1/2}(f, f; \mathbf{A}; \mathbf{x}), \end{aligned} \tag{5.25}$$

for any $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

PROOF. Follows from Theorem 5.1 and the details are omitted. ∎

The following particular results is of interest for applications:

COROLLARY 5.6 (Dragomir, 2008, [**33**]). *Consider the sequence of selfadjoint operators* $\mathbf{A} = (A_1, \dots, A_n)$ *with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \dots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous with* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then for any* $p_j \ge 0, j \in \{1, \dots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have*

$$\begin{aligned} |C(f, g; \mathbf{A}, \mathbf{p}, x)| &\le \frac{1}{2}(\Delta - \delta) \left\langle \sum_{j=1}^{n} p_j \left| f(A_j) - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot 1_H \right| x, x \right\rangle \\ &\le \frac{1}{2}(\Delta - \delta) C^{1/2}(f, f; \mathbf{A}, \mathbf{p}, x). \end{aligned} \tag{5.26}$$

PROOF. In we choose in Theorem 5.5 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \dots, n\}$, where $p_j \ge 0, j \in \{1, \dots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (5.25) becomes (5.26). The details are omitted. ∎

In a similar manner we can prove the following result as well:

THEOREM 5.7 (Dragomir, 2008, [**33**]). *Consider the sequences of selfadjoint operators* $\mathbf{A} = (A_1, \dots, A_n)$, $\mathbf{B} = (B_1, \dots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \dots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L-$*Hölder type, then we have the*

Ostrowski type inequality for sequences of selfadjoint operators:

$$\begin{aligned} \left| f(s) - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \right| \\ \leq L \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r, \end{aligned} \tag{5.27}$$

for any $s \in [m, M]$ *and any* $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Moreover, we have

$$\begin{aligned} &\left| \sum_{j=1}^{n} \langle f(B_j) y_j, y_j \rangle - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \right| \\ &\leq \sum_{j=1}^{n} \left\langle \left| f(B_j) - \sum_{k=1}^{n} \langle f(A_k) x_k, x_k \rangle \cdot 1_H \right| y_j, y_j \right\rangle \\ &\leq L \left[\frac{1}{2}(M-m) + \sum_{j=1}^{n} \left\langle \left| B_j - \frac{m+M}{2} \cdot 1_H \right| y_j, y_j \right\rangle \right]^r, \end{aligned} \tag{5.28}$$

for any $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in H^n$ *such that* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

COROLLARY 5.8 (Dragomir, 2008, [**33**]). *Consider the sequences of selfadjoint operators* $\mathbf{A} = (A_1, \ldots, A_n), \mathbf{B} = (B_1, \ldots, B_n)$ *with* $Sp(A_j), Sp(B_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is of* $r - L-$*Hölder type, then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have the weighted Ostrowski type inequality for sequences of selfadjoint operators:*

$$\begin{aligned} \left| f(s) - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \right| \\ \leq L \left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right| \right]^r, \end{aligned} \tag{5.29}$$

for any $s \in [m, M]$.

Moreover, we have

$$\begin{aligned} &\left| \left\langle \sum_{j=1}^{n} q_j f(B_j) y, y \right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \right| \\ &\leq \left\langle \sum_{j=1}^{n} q_j \left| f(B_j) - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot 1_H \right| y, y \right\rangle \\ &\leq L \left[\frac{1}{2}(M-m) + \left\langle \sum_{j=1}^{n} q_j \left| B_j - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle \right]^r, \end{aligned} \tag{5.30}$$

for any $q_k \geq 0, k \in \{1, \ldots, n\}$ *with* $\sum_{k=1}^{n} q_k = 1$ *and* $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

5.3. Some Reverses of Jensen's Inequality. It is clear that all the above inequalities can be applied for various particular instances of functions f and g. However, in the following we only consider the inequalities

$$\text{(5.31)}\qquad \begin{aligned} &\left|f\left(\left\langle Ax,x\right\rangle\right)-\left\langle f\left(A\right)x,x\right\rangle\right| \\ \leq\ & L\left[\frac{1}{2}\left(M-m\right)+\left|\left\langle Ax,x\right\rangle-\frac{m+M}{2}\right|\right]^{r}\end{aligned}$$

for any $x\in H$ with$\left\|x\right\|=1$, where the function $f:[m,M]\to\mathbb{R}$ is of $r-L-$Hölder type, and

$$\text{(5.32)}\qquad \begin{aligned} &\left|f\left(\left\langle Ax,x\right\rangle\right)-\left\langle f\left(A\right)x,x\right\rangle\right| \\ &\leq\begin{cases}\left[\frac{1}{2}\left(M-m\right)+\left|\left\langle Ax,x\right\rangle-\frac{m+M}{2}\right|\right]\left\|f'\right\|_{\infty,[m,M]}, & \text{if } f'\in L_{\infty}\left[m,M\right]\\ \\ \left[\frac{1}{2}\left(M-m\right)+\left|\left\langle Ax,x\right\rangle-\frac{m+M}{2}\right|\right]^{q}\left\|f'\right\|_{p,[m,M]}, & \begin{array}{l}\text{if } f'\in L_{p}\left[m,M\right];\\ p>1,\frac{1}{p}+\frac{1}{q}=1\end{array}\end{cases}\end{aligned}$$

for any $x\in H$ with $\left\|x\right\|=1$, where the function $f:[m,M]\to\mathbb{R}$ is absolutely continuous on $[m,M]$, which are related to the *Jensen's inequality* for convex functions.

1. Now, if we consider the concave function $f:[m,M]\subset[0,\infty)\to\mathbb{R}$, $f(t)=t^{r}$ with $r\in(0,1)$ and take into account that it is of $r-L-$Hölder type with the constant $L=1$, then from (5.31) we derive the following reverse for the *Hölder-McCarthy inequality* [**48**]

$$\text{(5.33)}\qquad 0\leq\left\langle A^{r}x,x\right\rangle-\left\langle Ax,x\right\rangle^{r}\leq\left[\frac{1}{2}\left(M-m\right)+\left|\left\langle Ax,x\right\rangle-\frac{m+M}{2}\right|\right]^{r}$$

for any $x\in H$ with $\left\|x\right\|=1$.

2. Now, if we consider the functions $f:[m,M]\subset(0,\infty)\to\mathbb{R}$ with $f(t)=t^{s}$ and $s\in(-\infty,0)\cup(0,\infty)$, then they are absolutely continuous and

$$\left\|f'\right\|_{\infty,[m,M]}=\begin{cases}sM^{s-1} & \text{for } s\in[1,\infty),\\ \\ \left|s\right|m^{s-1} & \text{for } s\in(-\infty,0)\cup(0,1).\end{cases}$$

If $p\geq1$, then

$$\begin{aligned}\left\|f'\right\|_{p,[m,M]}&=\left|s\right|\left(\int_{m}^{M}t^{p(s-1)}dt\right)^{1/p}\\ &=\left|s\right|\times\begin{cases}\left(\frac{M^{p(s-1)+1}-m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s\neq1-\frac{1}{p}\\ \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s=1-\frac{1}{p}.\end{cases}\end{aligned}$$

On making use of the first inequality from (5.32) we deduce for a given $s\in(-\infty,0)\cup(0,\infty)$ that

$$\text{(5.34)}\qquad \begin{aligned}&\left|\left\langle Ax,x\right\rangle^{s}-\left\langle A^{s}x,x\right\rangle\right|\\ &\leq\left[\frac{1}{2}\left(M-m\right)+\left|\left\langle Ax,x\right\rangle-\frac{m+M}{2}\right|\right]\\ &\times\begin{cases}sM^{s-1} & \text{for } s\in[1,\infty),\\ \\ \left|s\right|m^{s-1} & \text{for } s\in(-\infty,0)\cup(0,1).\end{cases}\end{aligned}$$

for any $x\in H$ with $\left\|x\right\|=1$.

The second part of (5.32) will produce the following reverse of the *Hölder-McCarthy inequality* as well:

(5.35)
$$\begin{aligned}
&\left|\langle Ax,x\rangle^{s}-\langle A^{s}x,x\rangle\right| \\
&\leq |s|\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^{q} \\
&\times\begin{cases}\left(\frac{M^{p(s-1)+1}-m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s\neq 1-\frac{1}{p} \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s=1-\frac{1}{p}\end{cases}
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, where $s\in(-\infty,0)\cup(0,\infty)$, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

3. Now, if we consider the function $f(t)=\ln t$ defined on the interval $[m,M]\subset(0,\infty)$, then f is also absolutely continuous and

$$\|f'\|_{p,[m,M]}=\begin{cases} m^{-1} & \text{for } p=\infty, \\ \left(\frac{M^{p-1}-m^{p-1}}{(p-1)M^{p-1}m^{p-1}}\right)^{1/p} & \text{for } p>1, \\ \ln\left(\frac{M}{m}\right) & \text{for } p=1.\end{cases}$$

Making use of the first inequality in (5.32) we deduce

(5.36)
$$\begin{aligned}
0&\leq \ln(\langle Ax,x\rangle)-\langle \ln(A)x,x\rangle \\
&\leq\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]m^{-1}
\end{aligned}$$

and

(5.37)
$$\begin{aligned}
0&\leq \ln(\langle Ax,x\rangle)-\langle \ln(A)x,x\rangle \\
&\leq\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^{q}\left(\frac{M^{p-1}-m^{p-1}}{(p-1)M^{p-1}m^{p-1}}\right)^{1/p}
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Similar results can be stated for sequences of operators, however the details are left to the interested reader.

5.4. Some Particular Grüss' Type Inequalities.

In this last section we provide some particular cases that can be obtained via the Grüss' type inequalities established before. For this purpose we select only two examples as follows.

Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $g:[m,M]\longrightarrow\mathbb{R}$ is continuous with $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$, then for any $f:[m,M]\longrightarrow\mathbb{R}$ of $r-L-$Hölder type we have the inequality:

(5.38)
$$\begin{aligned}
&\left|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle\right| \\
&\leq\frac{1}{2}(\Delta-\delta)L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{r},
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

Moreover, if f is absolutely continuos on $[m,M]$, then we have the inequalities:

$$\begin{aligned}(5.39)\quad &|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\\&\le\frac{1}{2}(\Delta-\delta)\\&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f'\in L_\infty[m,M];\\ \\ \left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{1/q}\|f'\|_{p,[m,M]} & \text{if } f'\in L_p[m,M],\\ & p,q>1,\frac{1}{p}+\frac{1}{q}=1\end{cases}\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

1. If we consider the concave function $f:[m,M]\subset[0,\infty)\to\mathbb{R}$, $f(t)=t^r$ with $r\in(0,1)$ and take into account that it is of $r-L-$Hölder type with the constant $L=1$, then from (5.38) we derive the following result:

$$\begin{aligned}(5.40)\quad &|\langle A^r g(A)x,x\rangle-\langle A^r x,x\rangle\cdot\langle g(A)x,x\rangle|\\&\le\frac{1}{2}(\Delta-\delta)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, where $g:[m,M]\longrightarrow\mathbb{R}$ is continuous with $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$.

Now, consider the function $g:[m,M]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^p$ with $p\in(-\infty,0)\cup(0,\infty)$. Obviously,

$$\Delta-\delta=\begin{cases}M^p-m^p & \text{if } p>0,\\ \\ \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} & \text{if } p<0,\end{cases}$$

and by (5.40) we get for any $x\in H$ with $\|x\|=1$ that

$$\begin{aligned}(5.41)\quad &0\le\left\langle A^{r+p}x,x\right\rangle-\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle\\&\le\frac{1}{2}(M^p-m^p)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}$$

when $p>0$ and

$$\begin{aligned}(5.42)\quad &0\le\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle-\left\langle A^{r+p}x,x\right\rangle\\&\le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}$$

when $p<0$.

If $g:[m,M]\subset(0,\infty)\to\mathbb{R}$, $g(t)=\ln t$, then by (5.40) we also get the inequality for logarithm:

$$\begin{aligned}(5.43)\quad &0\le\langle A^r\ln Ax,x\rangle-\langle A^r x,x\rangle\cdot\langle\ln Ax,x\rangle\\&\le\ln\sqrt{\frac{M}{m}}\cdot\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

2. Now consider the functions $f,g:[m,M]\subset(0,\infty)\to\mathbb{R}$, with $f(t)=t^s$ and $g(t)=t^w$ with $s,w\in(-\infty,0)\cup(0,\infty)$. We have

$$\|f'\|_{\infty,[m,M]}=\begin{cases}sM^{s-1} & \text{for } s\in[1,\infty),\\ \\ |s|\,m^{s-1} & \text{for } s\in(-\infty,0)\cup(0,1).\end{cases}$$

and, for $p \geq 1$,

$$\|f'\|_{p,[m,M]} = |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}$$

If $w > 0$, then by the first inequality in (5.39) we have

$$\begin{aligned} &\left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\leq \frac{1}{2}\left(M^w - m^w\right)\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x, x\right\rangle\right] \\ &\times \begin{cases} sM^{s-1} & \text{for } s \in [1,\infty), \\ \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0,1). \end{cases} \end{aligned} \tag{5.44}$$

for any $x \in H$ with $\|x\| = 1$.

If $w < 0$, then by the same inequality we also have

$$\begin{aligned} &\left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\leq \frac{1}{2}\cdot\frac{M^{-w} - m^{-w}}{M^{-w}m^{-w}}\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x, x\right\rangle\right] \\ &\times \begin{cases} sM^{s-1} & \text{for } s \in [1,\infty), \\ \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0,1), \end{cases} \end{aligned} \tag{5.45}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we assume that $p > 1$ and $w > 0$, then by the second inequality in (5.39) we have

$$\begin{aligned} &\left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\leq \frac{1}{2}|s|\left(M^w - m^w\right)\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x, x\right\rangle\right]^{1/q} \\ &\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases} \end{aligned} \tag{5.46}$$

while for $w < 0$, we also have

$$\begin{aligned} &\left|\left\langle A^{s+w}x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\leq \frac{1}{2}|s|\cdot\frac{M^{-w} - m^{-w}}{M^{-w}m^{-w}}\left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x, x\right\rangle\right]^{1/q} \\ &\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}, \end{cases} \end{aligned} \tag{5.47}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in H$ with $\|x\| = 1$.

6. BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF LIPSCHITZIAN FUNCTIONS

6.1. The Case of Lipschitzian Functions. The following result can be stated:

THEOREM 6.1 (Dragomir, 2008, [**34**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$|C(f, g; A; x)| \leq \frac{1}{2}(\Delta - \delta) L \langle \ell_{A,x}(A) x, x\rangle \leq (\Delta - \delta) LC(e, e; A; x) \tag{6.1}$$

for any $x \in H$ with $\|x\| = 1$, where

$$\ell_{A,x}(t) := \langle |t \cdot 1_H - A| x, x\rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ and

$$C(e, e; A; x) = \|Ax\|^2 - \langle Ax, x\rangle^2 (\geq 0). \tag{6.2}$$

PROOF. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [**44**, p. 5]) applied for the modulus, we can state that

$$|\langle h(A) x, x\rangle| \leq \langle |h(A)| x, x\rangle \tag{M}$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Since f is Lipschitzian with the constant $L > 0$, then for any $t, s \in [m, M]$ we have

$$|f(t) - f(s)| \leq L|t - s|. \tag{6.3}$$

Now, if we fix $t \in [m, M]$ and apply the property (P) for the inequality (6.3) and the operator A we get

$$\langle |f(t) \cdot 1_H - f(A)| x, x\rangle \leq L\langle |t \cdot 1_H - A| x, x\rangle, \tag{6.4}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the property (M) we get

$$|f(t) - \langle f(A) x, x\rangle| = |\langle f(t) \cdot 1_H - f(A) x, x\rangle| \leq \langle |f(t) \cdot 1_H - f(A)| x, x\rangle$$

which together with (6.4) gives

$$|f(t) - \langle f(A) x, x\rangle| \leq L\ell_{A,x}(t) \tag{6.5}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we also have

$$\left|g(t) - \frac{\Delta + \delta}{2}\right| \leq \frac{1}{2}(\Delta - \delta) \tag{6.6}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (6.5) with (6.6) we get

$$\begin{aligned}
(6.7)\quad & \left| f(t)g(t) - \langle f(A)x,x\rangle g(t) - \frac{\Delta+\delta}{2}f(t) + \frac{\Delta+\delta}{2}\langle f(A)x,x\rangle \right| \\
& \le \frac{1}{2}(\Delta-\delta)L\ell_{A,x}(t) = \frac{1}{2}(\Delta-\delta)L\langle |t\cdot 1_H - A|x,x\rangle \\
& \le \frac{1}{2}(\Delta-\delta)L\left\langle |t\cdot 1_H - A|^2 x,x\right\rangle^{1/2} \\
& = \frac{1}{2}(\Delta-\delta)L\left(\left\langle A^2x,x\right\rangle - 2\langle Ax,x\rangle t + t^2\right)^{1/2},
\end{aligned}$$

for any $t \in [m,M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (6.7) and a selfadjoint operator B with $Sp(B) \subset [m,M]$, then we get the following inequality of interest in itself:

$$\begin{aligned}
(6.8)\quad & \langle f(B)g(B)y,y\rangle - \langle f(A)x,x\rangle\langle g(B)y,y\rangle \\
& \left. -\frac{\Delta+\delta}{2}\langle f(B)y,y\rangle + \frac{\Delta+\delta}{2}\langle f(A)x,x\rangle \right| \\
& \le \frac{1}{2}(\Delta-\delta)L\langle \ell_{A,x}(B)y,y\rangle \\
& \le \frac{1}{2}(\Delta-\delta)L\left\langle \left(\left\langle A^2x,x\right\rangle 1_H - 2\langle Ax,x\rangle B + B^2\right)^{1/2} y,y\right\rangle \\
& \le \frac{1}{2}(\Delta-\delta)L\left(\left\langle A^2x,x\right\rangle - 2\langle Ax,x\rangle\langle By,y\rangle + \left\langle B^2y,y\right\rangle\right),
\end{aligned}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we choose in (6.8) $y = x$ and $B = A$, then we deduce the desired result (6.1). ∎

In the case of two Lipschitzian functions, the following result may be stated as well:

THEOREM 6.2 (Dragomir, 2008, [**34**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f,g : [m,M] \longrightarrow \mathbb{R}$ are Lipschitzian with the constants $L,K > 0$, then*

$$(6.9)\qquad |C(f,g;A;x)| \le LKC(e,e;A;x),$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Since $f,g : [m,M] \longrightarrow \mathbb{R}$ are Lipschitzian, then

$$|f(t) - f(s)| \le L|t-s| \text{ and } |g(t) - g(s)| \le K|t-s|$$

for any $t,s \in [m,M]$, which gives the inequality

$$|f(t)g(t) - f(t)g(s) - f(s)g(t) + f(s)g(s)| \le KL\left(t^2 - 2ts + s^2\right)$$

for any $t,s \in [m,M]$.

Now, fix $t \in [m, M]$ and if we apply the properties (P) and (M) for the operator A we get successively

$$\begin{aligned} (6.10) \qquad & |f(t) g(t) - \langle g(A) x, x\rangle f(t) \\ & - \langle f(A) x, x\rangle g(t) + \langle f(A) g(A) x, x\rangle| \\ & = |\langle [f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)] x, x\rangle| \\ & \le \langle |f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)| x, x\rangle \\ & \le KL \left\langle \left(t^2 \cdot 1_H - 2tA + A^2\right) x, x\right\rangle \\ & = KL \left(t^2 - 2t \langle Ax, x\rangle + \left\langle A^2 x, x\right\rangle\right) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Further, fix $x \in H$ with $\|x\| = 1$. On applying the same properties for the inequality (6.10) and another selfadjoint operator B with $Sp(B) \subset [m, M]$, we have

$$\begin{aligned} (6.11) \qquad & |\langle f(B) g(B) y, y\rangle - \langle g(A) x, x\rangle \langle f(B) y, y\rangle \\ & - \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(A) g(A) x, x\rangle| \\ & = |\langle [f(B) g(B) - \langle g(A) x, x\rangle f(B) \\ & - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H] y, y\rangle| \\ & \le \langle |f(B) g(B) - \langle g(A) x, x\rangle f(B) \\ & - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H| y, y\rangle \\ & \le KL \left\langle \left(B^2 - 2\langle Ax, x\rangle B + \left\langle A^2 x, x\right\rangle 1_H\right) y, y\right\rangle \\ & = KL \left(\left\langle B^2 y, y\right\rangle - 2\langle Ax, x\rangle \langle By, y\rangle + \left\langle A^2 x, x\right\rangle\right) \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is an inequality of interest in its own right.

Finally, on making $B = A$ and $y = x$ in (6.11) we deduce the desired result (6.9). ∎

6.2. Some Inequalities for Sequences of Operators. Consider the sequence of selfadjoint operators $\mathbf{A} = (A_1, \dots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$. If $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then we can consider the following Čebyšev type functional

$$\begin{aligned} C(f, g; \mathbf{A}, \mathbf{x}) := & \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle \\ & - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle . \end{aligned}$$

As a particular case of the above functional and for a probability sequence $\mathbf{p} = (p_1, \dots, p_n)$, i.e., $p_j \ge 0$ for $j \in \{1, \dots, n\}$ and $\sum_{j=1}^{n} p_j = 1$, we can also consider the functional

$$\begin{aligned} C(f, g; \mathbf{A}, \mathbf{p}, x) := & \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \\ & - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \end{aligned}$$

where $x \in H$, $\|x\| = 1$.

We know, from [**30**] that for the sequence of selfadjoint operators $\mathbf{A} = (A_1, \ldots, A_n)$ with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for the synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the inequality

$$C(f, g; \mathbf{A}, \mathbf{x}) \geq (\leq) \, 0 \tag{6.12}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Also, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we have

$$C(f, g; \mathbf{A}, \mathbf{p}, x) \geq (\leq) \, 0. \tag{6.13}$$

On the other hand, the following Grüss' type inequality is valid as well [**30**]:

$$\begin{aligned} |C(f, g; \mathbf{A}, \mathbf{x})| &\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[C(g, g; \mathbf{A}, \mathbf{x})\right]^{1/2} \\ &\qquad \left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)\right) \end{aligned} \tag{6.14}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Similarly, for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$ and any $x \in H$, $\|x\| = 1$ we also have the inequality:

$$\begin{aligned} |C(f, g; \mathbf{A}, \mathbf{p}, x)| &\leq \frac{1}{2} \cdot (\Gamma - \gamma) \left[C(g, g; \mathbf{A}, \mathbf{p}, x)\right]^{1/2} \\ &\qquad \left(\leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)\right). \end{aligned} \tag{6.15}$$

We can state now the following new result:

THEOREM 6.3 (Dragomir, 2008, [**34**]). *Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a sequence of selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \ldots, n\}$ and for some scalars $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$, then*

$$\begin{aligned} |C(f, g; \mathbf{A}, \mathbf{x})| &\leq \frac{1}{2} (\Delta - \delta) L \sum_{k=1}^{n} \langle \ell_{\mathbf{A},\mathbf{x}}(A_k) x_k, x_k \rangle \\ &\leq (\Delta - \delta) L C(e, e; \mathbf{A}; \mathbf{x}) \end{aligned} \tag{6.16}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where

$$\ell_{\mathbf{A},\mathbf{x}}(t) := \sum_{j=1}^{n} \langle |t \cdot 1_H - A_j| \, x_j, x_j \rangle$$

is a continuous function on $[m, M]$, $e(t) = t$ and

$$C(e, e; \mathbf{A}; \mathbf{x}) = \sum_{j=1}^{n} \|Ax_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^2 (\geq 0).$$

PROOF. Follows from Theorem 6.1. The details are omitted. ∎

As a particular case we have:

COROLLARY 6.4 (Dragomir, 2008, [**34**]). *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [m, M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq \frac{1}{2}(\Delta - \delta) L \left\langle \sum_{k=1}^{n} p_k \ell_{\mathbf{A},\mathbf{p},x}(A_k) x, x \right\rangle \leq (\Delta - \delta) L C(e, e; \mathbf{A}, \mathbf{p}, x) \tag{6.17}$$

where

$$\ell_{\mathbf{A},\mathbf{p},x}(t) := \left\langle \sum_{j=1}^{n} p_j |t \cdot 1_H - A_j| x, x \right\rangle$$

is a continuous function on $[m, M]$ *and*

$$C(e, e; \mathbf{A}, \mathbf{p}, x) = \sum_{j=1}^{n} p_j \|Ax_j\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 (\geq 0).$$

PROOF. In we choose in Theorem 6.3 $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (6.16) becomes (6.17). The details are omitted. ∎

In a similar way we obtain the following results as well:

THEOREM 6.5 (Dragomir, 2008, [**34**]). *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then*

$$|C(f, g; \mathbf{A}, \mathbf{x})| \leq LKC(e, e; \mathbf{A}, \mathbf{x}), \tag{6.18}$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

COROLLARY 6.6. *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, \ldots, n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are Lipschitzian with the constants* $L, K > 0$, *then for any* $p_j \geq 0, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *we have*

$$|C(f, g; \mathbf{A}, \mathbf{p}, x)| \leq LKC(e, e; \mathbf{A}, \mathbf{p}, x), \tag{6.19}$$

for any $x \in H$ *with* $\|x\| = 1$.

6.3. The Case of (φ, Φ) **–Lipschitzian Functions.** The following lemma may be stated.

LEMMA 6.7. *Let* $u : [a, b] \to \mathbb{R}$ *and* $\varphi, \Phi \in \mathbb{R}$ *with* $\Phi > \varphi$. *The following statements are equivalent:*

(i) *The function* $u - \frac{\varphi+\Phi}{2} \cdot e$, *where* $e(t) = t, t \in [a, b]$, *is* $\frac{1}{2}(\Phi - \varphi)$ *–Lipschitzian;*
(ii) *We have the inequality:*

$$\varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t \neq s; \tag{6.20}$$

(iii) *We have the inequality:*

$$\varphi(t - s) \leq u(t) - u(s) \leq \Phi(t - s) \quad \text{for each } t, s \in [a, b] \text{ with } t > s. \tag{6.21}$$

Following [**47**], we can introduce the concept:

DEFINITION 6.1. The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ, Φ) −Lipschitzian on $[a, b]$.

Notice that in [**47**], the definition was introduced on utilising the statement (iii) and only the equivalence (i) ⇔ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ, Φ) −Lipschitzian functions.

PROPOSITION 6.8. *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty \tag{6.22}$$

then u is (γ, Γ) −Lipschitzian on $[a, b]$.

The following result can be stated:

THEOREM 6.9 (Dragomir, 2008, [**34**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is (φ, Φ) −Lipschitzian on $[a, b]$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$\begin{aligned} \left| C(f, g; A; x) - \frac{\varphi + \Phi}{2} C(e, g; A; x) \right| &\le \frac{1}{4} (\Delta - \delta)(\Phi - \varphi) \langle \ell_{A,x}(A) x, x\rangle \\ &\le \frac{1}{2} (\Delta - \delta)(\Phi - \varphi) C(e, e; A; x) \end{aligned} \tag{6.23}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 6.1 applied for the $\frac{1}{2}(\Phi - \varphi)$ −Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ (see Lemma 6.7) and the details are omitted.

THEOREM 6.10 (Dragomir, 2008, [**34**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \longrightarrow \mathbb{R}$. If f is (φ, Φ) −Lipschitzian and g is (ψ, Ψ) −Lipschitzian on $[a, b]$, then*

$$\begin{aligned} &\left| C(f, g; A; x) - \frac{\Phi + \varphi}{2} C(e, g; A; x) \right. \\ &\left. - \frac{\Psi + \psi}{2} C(f, e; A; x) + \frac{\Phi + \varphi}{2} \cdot \frac{\Psi + \psi}{2} C(e, e; A; x) \right| \\ &\le \frac{1}{4} (\Phi - \varphi)(\Psi - \psi) C(e, e; A; x), \end{aligned} \tag{6.24}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 6.2 applied for the $\frac{1}{2}(\Phi - \varphi)$ −Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ and the $\frac{1}{2}(\Psi - \psi)$ −Lipschitzian function $g - \frac{\Psi+\psi}{2} \cdot e$. The details are omitted.

Similar results can be derived for sequences of operators, however they will not be presented here.

6.4. Some Applications. It is clear that all the inequalities obtained in the previous sections can be applied to obtain particular inequalities of interest for different selections of the functions f and g involved. However we will present here only some particular results that can be derived from the inequality

$$|C(f,g;A;x)| \leq LKC(e,e;A;x), \tag{6.25}$$

that holds for the Lipschitzian functions f and g, the first with the constant $L > 0$ and the second with the constant $K > 0$.

1. Now, if we consider the functions $f, g : [m, M] \subset (0,\infty) \to \mathbb{R}$ with $f(t) = t^p$, $g(t) = t^q$ and $p, q \in (-\infty, 0) \cup (0,\infty)$ then they are Lipschitzian with the constants $L = \|f'\|_\infty$ and $K = \|g'\|_\infty$. Since $f'(t) = pt^{p-1}$, $g(t) = qt^{q-1}$, hence

$$\|f'\|_\infty = \begin{cases} pM^{p-1} & \text{for } p \in [1,\infty), \\ \\ |p|\, m^{p-1} & \text{for } p \in (-\infty,0) \cup (0,1) \end{cases}$$

and

$$\|g'\|_\infty = \begin{cases} qM^{q-1} & \text{for } q \in [1,\infty), \\ \\ |q|\, m^{q-1} & \text{for } q \in (-\infty,0) \cup (0,1) \end{cases}.$$

Therefore we can state the following inequalities for the powers of a positive definite operator A with $Sp(A) \subset [m, M] \subset (0,\infty)$.

If $p, q \geq 1$, then

$$\begin{aligned} (0 \leq\ ) \left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \\ \leq pqM^{p+q-2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \end{aligned} \tag{6.26}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \geq 1$ and $q \in (-\infty, 0) \cup (0, 1)$, then

$$\begin{aligned} &\left|\left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle\right| \\ &\leq p\,|q|\, M^{p-1}m^{q-1}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \end{aligned} \tag{6.27}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \in (-\infty, 0) \cup (0, 1)$ and $q \geq 1$, then

$$\begin{aligned} &\left|\left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle\right| \\ &\leq |p|\, qM^{q-1}m^{p-1}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \end{aligned} \tag{6.28}$$

for each $x \in H$ with $\|x\| = 1$.

If $p, q \in (-\infty, 0) \cup (0, 1)$, then

$$\begin{aligned} &\left|\left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle\right| \\ &\leq |pq|\, m^{p+q-2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \end{aligned} \tag{6.29}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover, if we take $p = 1$ and $q = -1$ in (6.27), then we get the following result

$$(0 \leq) \langle Ax, x\rangle \cdot \left\langle A^{-1}x, x\right\rangle - 1 \leq m^{-2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \tag{6.30}$$

for each $x \in H$ with $\|x\| = 1$.

2. Consider now the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^p, p \in (-\infty, 0) \cup (0, \infty)$ and $g(t) = \ln t$. Then g is also Lipschitzian with the constant $K = \|g'\|_\infty = m^{-1}$. Applying the inequality (6.25) we then have for any $x \in H$ with $\|x\| = 1$ that

$$(0 \le) \langle A^p \ln Ax, x\rangle - \langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle \le pM^{p-1}m^{-1}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \tag{6.31}$$

if $p \ge 1$,

$$(0 \le) \langle A^p \ln Ax, x\rangle - \langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle \le pm^{p-2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \tag{6.32}$$

if $p \in (0, 1)$ and

$$(0 \le) \langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle - \langle A^p \ln Ax, x\rangle \le (-p)\, m^{p-2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \tag{6.33}$$

if $p \in (-\infty, 0)$.

3. Now consider the functions $f, g : [m, M] \subset \mathbb{R} \to \mathbb{R}$ given by $f(t) = \exp(\alpha t)$ and $g(t) = \exp(\beta t)$ with α, β nonzero real numbers. It is obvious that

$$\|f'\|_\infty = |\alpha| \times \begin{cases} \exp(\alpha M) & \text{for } \alpha > 0, \\ \exp(\alpha m) & \text{for } \alpha < 0 \end{cases}$$

and

$$\|g'\|_\infty = |\beta| \times \begin{cases} \exp(\beta M) & \text{for } \beta > 0, \\ \exp(\beta m) & \text{for } \beta < 0 \end{cases}.$$

Finally, on applying the inequality (6.25) we get

$$\begin{aligned}(0 \le)\ &\langle \exp[(\alpha+\beta)A]x, x\rangle - \langle \exp(\alpha A)x, x\rangle \cdot \langle \exp(\beta A)x, x\rangle \\ &\le |\alpha\beta|\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp[(\alpha+\beta)M] & \text{for } \alpha, \beta > 0, \\ \exp[(\alpha+\beta)m] & \text{for } \alpha, \beta < 0 \end{cases}\end{aligned}$$

and

$$\begin{aligned}(0 \le)\ &\langle \exp(\alpha A)x, x\rangle \cdot \langle \exp(\beta A)x, x\rangle - \langle \exp[(\alpha+\beta)A]x, x\rangle \\ &\le |\alpha\beta|\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp(\alpha M + \beta m) & \text{for } \alpha > 0, \beta < 0 \\ \exp(\alpha m + \beta M) & \text{for } \alpha < 0, \beta > 0 \end{cases}\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

7. QUASI GRÜSS' TYPE INEQUALITIES

7.1. Introduction. In [**16**], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

THEOREM 7.1 (Dragomir, 1999, [**16**]). *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H, \|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \ge 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \ge 0 \tag{7.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \le \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{7.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [**27**] and the references therein.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation theorem in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{7.3}$$

which in terms of vectors can be written as

$$\langle f(U) x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle , \tag{7.4}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m - 0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

7.2. Vector Inequalities. In this section we provide various bounds for the magnitude of the difference

$$\langle f(A) x, y\rangle - \langle x, y\rangle \langle f(A) x, x\rangle$$

under different assumptions on the continuous function, the selfadjoint operator $A : H \to H$ and the vectors $x, y \in H$ with $\|x\| = 1$.

THEOREM 7.2 (Dragomir, 2010, [**35**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $x, y \in H$, $\|x\| = 1$ are such that there exists $\gamma, \Gamma \in \mathbb{C}$ with either*

$$\operatorname{Re}\langle \Gamma x - y, y - \gamma x\rangle \ge 0 \tag{7.5}$$

or, equivalently

$$\left\| y - \frac{\gamma + \Gamma}{2} x \right\| \le \frac{1}{2} |\Gamma - \gamma| .$$

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned} & |\langle f(A) x, y\rangle - \langle x, y\rangle \langle f(A) x, x\rangle| \\ & \le \max_{\lambda \in [m,M]} |\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle| \bigvee_m^M (f) \\ & \le \max_{\lambda \in [m,M]} (\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle)^{1/2} \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \\ & \le \frac{1}{2} \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \le \frac{1}{4} |\Gamma - \gamma| \bigvee_m^M (f) . \end{aligned} \tag{7.6}$$

2. If $f:[m,M]\to\mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,y\right\rangle-\left\langle x,y\right\rangle\left\langle f\left(A\right)x,x\right\rangle\right| \\
&\leq L\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,y\right\rangle-\left\langle E_{\lambda}x,x\right\rangle\left\langle x,y\right\rangle\right|d\lambda \\
&\leq L\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\int_{m-0}^{M}\left(\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right)^{1/2}d\lambda \\
&\leq L\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\left\langle\left(M1_{H}-A\right)x,x\right\rangle^{1/2}\left\langle\left(A-m1_{H}\right)x,x\right\rangle^{1/2} \\
&\leq\frac{1}{2}\left(M-m\right)L\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|\left(M-m\right)L.
\end{aligned}\tag{7.7}$$

3. If $f:[m,M]\to\mathbb{R}$ is a continuous monotonic nondecreasing function on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,y\right\rangle-\left\langle x,y\right\rangle\left\langle f\left(A\right)x,x\right\rangle\right| \\
&\leq\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,y\right\rangle-\left\langle E_{\lambda}x,x\right\rangle\left\langle x,y\right\rangle\right|df\left(\lambda\right) \\
&\leq\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\int_{m-0}^{M}\left(\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right)^{1/2}df\left(\lambda\right) \\
&\leq\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2} \\
&\times\left\langle\left(f\left(M\right)1_{H}-f\left(A\right)\right)x,x\right\rangle^{1/2}\left\langle\left(f\left(A\right)-f\left(m\right)1_{H}\right)x,x\right\rangle^{1/2} \\
&\leq\frac{1}{2}\left[f\left(M\right)-f\left(m\right)\right]\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|\left[f\left(M\right)-f\left(m\right)\right].
\end{aligned}\tag{7.8}$$

PROOF. First of all, we notice that by the Schwarz inequality in H we have for any $u,v,e\in H$ with $\|e\|=1$ that

$$\left|\left\langle u,v\right\rangle-\left\langle u,e\right\rangle\left\langle e,v\right\rangle\right|\leq\left(\left\|u\right\|^{2}-\left|\left\langle u,e\right\rangle\right|^{2}\right)^{1/2}\left(\left\|v\right\|^{2}-\left|\left\langle v,e\right\rangle\right|^{2}\right)^{1/2}.\tag{7.9}$$

Now on utilizing (7.9), we can state that

$$\begin{aligned}
&\left|\left\langle E_{\lambda}x,y\right\rangle-\left\langle E_{\lambda}x,x\right\rangle\left\langle x,y\right\rangle\right| \\
&\leq\left(\left\|E_{\lambda}x\right\|^{2}-\left|\left\langle E_{\lambda}x,x\right\rangle\right|^{2}\right)^{1/2}\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}
\end{aligned}\tag{7.10}$$

for any $\lambda\in[m,M]$.

Since E_λ are projections and $E_\lambda\geq 0$ then

$$\begin{aligned}
\left\|E_{\lambda}x\right\|^{2}-\left|\left\langle E_{\lambda}x,x\right\rangle\right|^{2}&=\left\langle E_{\lambda}x,x\right\rangle-\left\langle E_{\lambda}x,x\right\rangle^{2} \\
&=\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\leq\frac{1}{4}
\end{aligned}\tag{7.11}$$

for any $\lambda\in[m,M]$ and $x\in H$ with $\|x\|=1$.

Also, by making use of the Grüss' type inequality in inner product spaces obtained by the author in [**16**] we have

$$\left(\left\|y\right\|^{2}-\left|\left\langle y,x\right\rangle\right|^{2}\right)^{1/2}\leq\frac{1}{2}\left|\Gamma-\gamma\right|.\tag{7.12}$$

Combining the relations (7.10)-(7.12) we deduce the following inequality that is of interest in itself

$$\begin{aligned}(7.13)\qquad & \left|\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right| \\ & \leq \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\ & \leq \frac{1}{2}\left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \leq \frac{1}{4}|\Gamma - \gamma|\end{aligned}$$

for any $\lambda \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$(7.14)\qquad \left|\int_a^b p(t)\, dv(t)\right| \leq \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property of the Riemann-Stieltjes integral and the inequality (7.13) we have

$$\begin{aligned}(7.15)\qquad & \left|\int_{m-0}^{M} [\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle]\, df(\lambda)\right| \\ & \leq \max_{\lambda\in[m,M]} |\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle| \bigvee_m^M (f) \\ & \leq \max_{\lambda\in[m,M]} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \\ & \leq \frac{1}{2}\left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \leq \frac{1}{4}|\Gamma - \gamma| \bigvee_m^M (f)\end{aligned}$$

for x and y as in the assumptions of the theorem.

Now, integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation theorem we have

$$\begin{aligned}(7.16)\qquad & \int_{m-0}^{M} [\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle]\, df(\lambda) \\ & = [\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle] f(\lambda)\Big|_{m-0}^{M} \\ & - \int_{m-0}^{M} f(\lambda)\, d[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle] \\ & = \langle x, y\rangle \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle - \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle \\ & = \langle x, y\rangle \langle f(A) x, x\rangle - \langle f(A) x, y\rangle\end{aligned}$$

which together with (7.15) produces the desired result (7.6).

Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have from (7.13) that

$$\begin{aligned}(7.17)\qquad &\left|\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right]df(\lambda)\right|\\ &\le L\int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|d\lambda\\ &\le L\left(\|y\|^2-|\langle y,x\rangle|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda.\end{aligned}$$

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation theorem we have successively

$$\begin{aligned}(7.18)\qquad &\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda\\ &\le\left[\int_{m-0}^{M}\langle E_\lambda x,x\rangle\,d\lambda\right]^{1/2}\left[\int_{m-0}^{M}\langle(1_H-E_\lambda)x,x\rangle\,d\lambda\right]^{1/2}\\ &=\left[\langle E_\lambda x,x\rangle\lambda\Big|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle E_\lambda x,x\rangle\right]^{1/2}\\ &\times\left[\langle(1_H-E_\lambda)x,x\rangle\lambda\Big|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle(1_H-E_\lambda)x,x\rangle\right]\\ &=\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2}.\end{aligned}$$

On utilizing (7.18), (7.17) and (7.16) we deduce the first three inequalities in (7.7).

The fourth inequality follows from the fact that

$$\begin{aligned}&\langle(M1_H-A)x,x\rangle\langle(A-m1_H)x,x\rangle\\ &\le\frac{1}{4}\left[\langle(M1_H-A)x,x\rangle+\langle(A-m1_H)x,x\rangle\right]^2=\frac{1}{4}(M-m)^2.\end{aligned}$$

The last part follows from (7.12).

Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$(7.19)\qquad \left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t).$$

Utilising this property and the inequality (7.13) we have successively

$$\begin{aligned}(7.20)\qquad &\left|\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right]df(\lambda)\right|\\ &\le\int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|df(\lambda)\\ &\le\left(\|y\|^2-|\langle y,x\rangle|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}df(\lambda).\end{aligned}$$

Applying the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic integrators and the spectral representation theorem we have

$$\begin{aligned}
\text{(7.21)}\quad & \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} df(\lambda) \\
& \leq \left[\int_{m-0}^{M} \langle E_\lambda x, x\rangle \, df(\lambda)\right]^{1/2} \left[\int_{m-0}^{M} \langle (1_H - E_\lambda) x, x\rangle \, df(\lambda)\right]^{1/2} \\
& = \left[\langle E_\lambda x, x\rangle f(\lambda)\big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle\right]^{1/2} \\
& \times \left[\langle (1_H - E_\lambda) x, x\rangle f(\lambda)\big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda)\, d\langle (1_H - E_\lambda) x, x\rangle\right]^{1/2} \\
& = \langle (f(M) 1_H - f(A)) x, x\rangle^{1/2} \langle (f(A) - f(m) 1_H) x, x\rangle^{1/2} \\
& \leq \frac{1}{2}\left[f(M) - f(m)\right]
\end{aligned}$$

and the proof is complete. ∎

REMARK 7.1. If we drop the conditions on x, y, we can obtain from the inequalities (7.6)-(7.7) the following results that can be easily applied for particular functions:

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned}
\text{(7.22)}\quad & \left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| \\
& \leq \frac{1}{2} \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \bigvee_{m}^{M}(f)
\end{aligned}$$

for any $x, y \in H, x \neq 0$.

2. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality

$$\begin{aligned}
\text{(7.23)}\quad & \left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| \\
& \leq L \left(\|y\|^2 \|x\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\
& \times \left[\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle\right]^{1/2} \\
& \leq \frac{1}{2}(M - m) L \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x\rangle|^2\right)^{1/2}
\end{aligned}$$

for any $x, y \in H, x \neq 0$.

3. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality

$$\begin{aligned}
\text{(7.24)}\quad & \left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| \\
& \leq \left(\|y\|^2 \|x\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\
& \times \left[\langle (f(M) 1_H - f(A)) x, x\rangle \langle (f(A) - f(m) 1_H) x, x\rangle\right]^{1/2} \\
& \leq \frac{1}{2}\left[f(M) - f(m)\right] \|x\|^2 \left(\|y\|^2 \|x\|^2 - |\langle y, x\rangle|^2\right)^{1/2}
\end{aligned}$$

for any $x, y \in H, x \neq 0$.

We are able now to provide the following corollary:

COROLLARY 7.3 (Dragomir, 2010, [**35**]). *With the assumptions of Theorem 7.2 and if* $f : [m, M] \to \mathbb{R}$ *is a* (φ, Φ)*-Lipschitzian function then we have*

$$\begin{aligned}
(7.25) \qquad & |\langle f(A)x, y\rangle - \langle x, y\rangle \langle f(A)x, x\rangle| \\
&\le \frac{1}{2}(\Phi - \varphi) \int_{m-0}^{M} |\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle| \, d\lambda \\
&\le \frac{1}{2}(\Phi - \varphi) \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} d\lambda \\
&\le \frac{1}{2}(\Phi - \varphi) \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\
&\times \langle (M1_H - A)x, x\rangle^{1/2} \langle (A - m1_H)x, x\rangle^{1/2} \\
&\le \frac{1}{4}(M - m)(\Phi - \varphi) \left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\
&\le \frac{1}{8}|\Gamma - \gamma| (M - m)(\Phi - \varphi).
\end{aligned}$$

The proof follows from the second part of Theorem 7.2 applied for the $\frac{1}{2}(\Phi - \varphi)$-Lipschitzian function $f - \frac{\Phi + \varphi}{2} \cdot e$ by performing the required calculations in the first term of the inequality. The details are omitted.

7.3. Applications for Grüss' Type Inequalities.

The following result provides some Grüss' type inequalities for two function of two selfadjoint operators.

PROPOSITION 7.4 (Dragomir, 2010, [**35**]). *Let* A, B *be two selfadjoint operators in the Hilbert space* H *with the spectra* $Sp(A), Sp(B) \subseteq [m, M]$ *for some real numbers* $m < M$ *and let* $\{E_\lambda\}_\lambda$ *be the spectral family of* A*. Assume that* $g : [m, M] \to \mathbb{R}$ *is a continuous function and denote* $n := \min_{t \in [m,M]} g(t)$ *and* $N := \max_{t \in [m,M]} g(t)$.

1. If $f : [m, M] \to \mathbb{C}$ *is a continuous function of bounded variation on* $[m, M]$*, then we have the inequality*

$$\begin{aligned}
(7.26) \qquad & |\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle g(B)x, x\rangle| \\
&\le \max_{\lambda \in [m,M]} |\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle| \bigvee_m^M (f) \\
&\le \max_{\lambda \in [m,M]} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} \\
&\times \left(\|g(B)x\|^2 - |\langle g(B)x, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \\
&\le \frac{1}{2}\left(\|g(B)x\|^2 - |\langle g(B)x, x\rangle|^2\right)^{1/2} \bigvee_m^M (f) \le \frac{1}{4}(N - n) \bigvee_m^M (f)
\end{aligned}$$

for any $x \in H, \|x\| = 1$.

2. If $f:[m,M]\to\mathbb{C}$ *is a Lipschitzian function with the constant* $L>0$ *on* $[m,M]$*, then we have the inequality*

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,g\left(B\right)x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle\left\langle g\left(B\right)x,x\right\rangle\right| \\
&\leq L\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,g\left(B\right)x\right\rangle-\left\langle E_{\lambda}x,x\right\rangle\left\langle x,g\left(B\right)x\right\rangle\right|d\lambda \\
&\leq L\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\times\int_{m-0}^{M}\left(\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right)^{1/2}d\lambda \\
&\leq L\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\times\left\langle\left(M1_{H}-A\right)x,x\right\rangle^{1/2}\left\langle\left(A-m1_{H}\right)x,x\right\rangle^{1/2} \\
&\leq\frac{1}{2}\left(M-m\right)L\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\leq\frac{1}{4}\left(N-n\right)\left(M-m\right)L
\end{aligned} \tag{7.27}$$

for any $x\in H,\|x\|=1.$

3. If $f:[m,M]\to\mathbb{R}$ *is a continuous monotonic nondecreasing function on* $[m,M]$*, then we have the inequality*

$$\begin{aligned}
&\left|\left\langle f\left(A\right)x,g\left(B\right)x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle\left\langle g\left(B\right)x,x\right\rangle\right| \\
&\leq\int_{m-0}^{M}\left|\left\langle E_{\lambda}x,g\left(B\right)x\right\rangle-\left\langle E_{\lambda}x,x\right\rangle\left\langle x,g\left(B\right)x\right\rangle\right|df\left(\lambda\right) \\
&\leq\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\times\int_{m-0}^{M}\left(\left\langle E_{\lambda}x,x\right\rangle\left\langle\left(1_{H}-E_{\lambda}\right)x,x\right\rangle\right)^{1/2}df\left(\lambda\right) \\
&\leq\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\times\left\langle\left(f\left(M\right)1_{H}-f\left(A\right)\right)x,x\right\rangle^{1/2}\left\langle\left(f\left(A\right)-f\left(m\right)1_{H}\right)x,x\right\rangle^{1/2} \\
&\leq\frac{1}{2}\left[f\left(M\right)-f\left(m\right)\right]\left(\left\|g\left(B\right)x\right\|^{2}-\left|\left\langle g\left(B\right)x,x\right\rangle\right|^{2}\right)^{1/2} \\
&\leq\frac{1}{4}\left(N-n\right)\left[f\left(M\right)-f\left(m\right)\right]
\end{aligned} \tag{7.28}$$

for any $x\in H,\|x\|=1.$

PROOF. We notice that, since $n:=\min_{t\in[m,M]}g\left(t\right)$ and $N:=\max_{t\in[m,M]}g\left(t\right)$, then $n\leq\left\langle g\left(B\right)x,x\right\rangle\leq N$ which implies that $\left\langle g\left(B\right)x-nx,Mx-g\left(B\right)x\right\rangle\geq0$ for any $x\in H,\|x\|=1$. On applying Theorem 7.2 for $y=Bx,\Gamma=N$ and $\gamma=n$ we deduce the desired result. ∎

REMARK 7.2. We observe that if the function f takes real values and is a (φ, Φ)-Lipschitzian function on $[m, M]$, then the inequality (7.27) can be improved as follows

$$
\begin{aligned}
(7.29)\qquad & |\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle \langle g(B)x,x\rangle| \\
&\le \frac{1}{2}(\Phi-\varphi)\int_{m-0}^{M} |\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x,x\rangle \langle x, g(B)x\rangle|\, d\lambda \\
&\le \frac{1}{2}(\Phi-\varphi)\left(\|g(B)x\|^2 - |\langle g(B)x,x\rangle|^2\right)^{1/2} \\
&\times \int_{m-0}^{M} \left(\langle E_\lambda x,x\rangle \langle (1_H - E_\lambda)x,x\rangle\right)^{1/2} d\lambda \\
&\le \frac{1}{2}(\Phi-\varphi)\left(\|g(B)x\|^2 - |\langle g(B)x,x\rangle|^2\right)^{1/2} \\
&\times \langle (M1_H - A)x,x\rangle^{1/2} \langle (A - m1_H)x,x\rangle^{1/2} \\
&\le \frac{1}{4}(M-m)(\Phi-\varphi)\left(\|g(B)x\|^2 - |\langle g(B)x,x\rangle|^2\right)^{1/2} \\
&\le \frac{1}{8}(N-n)(M-m)(\Phi-\varphi)
\end{aligned}
$$

for any $x \in H$, $\|x\| = 1$.

7.4. Applications. By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f, g : (0,\infty) \to (0,\infty)$ with $f(t) = t^p, g(t) = t^q$ and $p, q > 0$, then for any selfadjoint operators A, B with $Sp(A), Sp(B) \subseteq [m, M] \subset (0,\infty)$ we get from (7.28) the inequalities

$$
\begin{aligned}
(7.30)\qquad & |\langle A^p x, B^q x\rangle - \langle A^p x,x\rangle \langle B^q x,x\rangle| \\
&\le p\left(\|B^q x\|^2 - |\langle B^q x,x\rangle|^2\right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x,x\rangle \langle (1_H - E_\lambda)x,x\rangle\right)^{1/2} \lambda^{p-1} d\lambda \\
&\le \left(\|B^q x\|^2 - |\langle B^q x,x\rangle|^2\right)^{1/2} \langle (M^p 1_H - A^p)x,x\rangle^{1/2} \langle (A^p - m^p 1_H)x,x\rangle^{1/2} \\
&\le \frac{1}{2}(M^p - m^p)\left(\|B^q x\|^2 - |\langle B^q x,x\rangle|^2\right)^{1/2} \le \frac{1}{4}(M^q - m^q)(M^p - m^p)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$, where $\{E_\lambda\}_\lambda$ is the spectral family *of* A.

The same choice of functions considered in the inequality (7.29) produce the result

$$
\begin{aligned}
&\left|\langle A^p x, B^q x\rangle - \langle A^p x, x\rangle \langle B^q x, x\rangle\right| \\
&\leq \frac{1}{2}\Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} d\lambda \\
&\leq \frac{1}{2}\Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \langle (M^p 1_H - A^p) x, x\rangle^{1/2} \langle (A^p - m^p 1_H) x, x\rangle^{1/2} \\
&\leq \frac{1}{4}(M-m)\Delta_p \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\leq \frac{1}{8}(M^q - m^q)(M-m)\Delta_p
\end{aligned} \tag{7.31}
$$

where

$$
\Delta_p := p \times \begin{cases} M^{p-1} - m^{p-1} \text{ if } p \geq 1 \\ \frac{M^{1-p} - m^{1-p}}{M^{1-p} m^{1-p}} \text{ if } 0 < p < 1. \end{cases} \tag{7.32}
$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we choose $f(t) = \ln t, t > 0$ and keep the same g then we have the inequalities

$$
\begin{aligned}
&\left|\langle \ln A x, B^q x\rangle - \langle \ln A x, x\rangle \langle B^q x, x\rangle\right| \\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} \lambda^{-1} d\lambda \\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \langle (\ln M 1_H - \ln A) x, x\rangle^{1/2} \langle (\ln A - \ln m 1_H) x, x\rangle^{1/2} \\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \ln \sqrt{\frac{M}{m}} \\
&\leq \frac{1}{2}(M^q - m^q) \ln \sqrt{\frac{M}{m}}
\end{aligned} \tag{7.33}
$$

and

(7.34)
$$\begin{aligned}
&\left|\langle \ln Ax, B^q x\rangle - \langle \ln Ax, x\rangle \langle B^q x, x\rangle\right| \\
&\le \frac{1}{2}\left(\frac{M-m}{mM}\right)\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} d\lambda \\
&\le \frac{1}{2}\left(\frac{M-m}{mM}\right)\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\times \langle (M 1_H - A) x, x\rangle^{1/2} \langle (A - m 1_H) x, x\rangle^{1/2} \\
&\le \frac{1}{4}\frac{(M-m)^2}{mM}\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \\
&\le \frac{1}{8}(M^q - m^q)\frac{(M-m)^2}{mM}
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

8. TWO OPERATORS GRÜSS' TYPE INEQUALITIES

8.1. Some Representation Results. We start with the following representation result that will play a key role in obtaining various bounds for different choices of functions including continuous functions of bounded variation, Lipschitzian functions or monotonic and continuous functions.

THEOREM 8.1 (Dragomir, 2010, [**36**]). *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. If $f, g : [m, M] \to \mathbb{C}$ are continuous, then we have the representation*

(8.1)
$$\begin{aligned}
&\langle f(A) x, g(B) x\rangle - \langle f(A) x, x\rangle \langle x, g(B) x\rangle \\
&= \int_{m-0}^{M}\left(\int_{m-0}^{M}\left[\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right] d(g(\mu))\right) d(f(\lambda))
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

PROOF. Integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation theorem we have

(8.2)
$$\begin{aligned}
&\int_{m-0}^{M}\left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right] df(\lambda) \\
&= \left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right] f(\lambda)\Big|_{m-0}^{M} \\
&- \int_{m-0}^{M} f(\lambda)\, d\left[\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right] \\
&= \langle x, y\rangle \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle - \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle \\
&= \langle x, y\rangle \langle f(A) x, x\rangle - \langle f(A) x, y\rangle
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = 1$.

Now, if we chose $y = g(B)x$ in (8.2) then we get that

$$\int_{m-0}^{M} [\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle]\, df(\lambda) = \langle x, g(B)x\rangle \langle f(A)x, x\rangle - \langle f(A)x, g(B)x\rangle \tag{8.3}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the spectral representation theorem for B we also have for each fixed $\lambda \in [m, M]$ that

$$\begin{aligned} &\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle \langle x, g(B)x\rangle \\ &= \left\langle E_\lambda x, \int_{m-0}^{M} g(\mu)\, dF_\mu x\right\rangle - \langle E_\lambda x, x\rangle \left\langle x, \int_{m-0}^{M} g(\mu)\, dF_\mu x\right\rangle \\ &= \int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) - \langle E_\lambda x, x\rangle \int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle) \end{aligned} \tag{8.4}$$

for any $x \in H$ with $\|x\| = 1$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} \int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) &= g(\mu)\langle E_\lambda x, F_\mu x\rangle]_{m-0}^{M} - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, dg(\mu) \\ &= g(M)\langle E_\lambda x, x\rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu)) \end{aligned}$$

and

$$\begin{aligned} \int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle) &= g(\mu)\langle x, F_\mu x\rangle]_{m-0}^{M} - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu)) \\ &= g(M) - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu)), \end{aligned}$$

therefore

$$\begin{aligned} &\int_{m-0}^{M} g(\mu)\, d(\langle E_\lambda x, F_\mu x\rangle) - \langle E_\lambda x, x\rangle \int_{m-0}^{M} g(\mu)\, d(\langle x, F_\mu x\rangle) \\ &= g(M)\langle E_\lambda x, x\rangle - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu)) \\ &- \langle E_\lambda x, x\rangle \left(g(M) - \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu))\right) \\ &= \langle E_\lambda x, x\rangle \int_{m-0}^{M} \langle x, F_\mu x\rangle\, d(g(\mu)) - \int_{m-0}^{M} \langle E_\lambda x, F_\mu x\rangle\, d(g(\mu)) \\ &= \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle]\, d(g(\mu)) \end{aligned} \tag{8.5}$$

for any $x \in H$ with $\|x\| = 1$ and $\lambda \in [m, M]$.

Utilising (8.3)-(8.5) we deduce the desired result (8.1). ∎

REMARK 8.1. In particular, if we take $B = A,$ then we get from (8.1) the equality

$$\begin{aligned} \langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle\langle x, g(A)x\rangle \\ = \int_{m-0}^{M}\left(\int_{m-0}^{M}\left[\langle E_\lambda x, x\rangle\langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d(g(\mu))\right) d(f(\lambda)) \end{aligned} \tag{8.6}$$

for any $x \in H$ with $\|x\| = 1,$ which for $g = f$ produces the representation result for the variance of the selfadjoint operator $f(A)$,

$$\begin{aligned} \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \\ = \int_{m-0}^{M}\left(\int_{m-0}^{M}\left[\langle E_\lambda x, x\rangle\langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right] d(f(\mu))\right) d(f(\lambda)) \end{aligned} \tag{8.7}$$

for any $x \in H$ with $\|x\| = 1.$

8.2. Bounds for f of Bounded Variation. The first vectorial Grüss' type inequality when one of the functions is of bounded variation is as follows:

THEOREM 8.2 (Dragomir, 2010, [**36**]). *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. Also, assume that $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$.*

1. If $g : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned} &|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle\langle x, g(B)x\rangle| \\ &\le \max_{(\lambda,\mu)\in[m,M]^2}|\langle E_\lambda x, x\rangle\langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \bigvee_m^M(g)\bigvee_m^M(f) \\ &\le \max_{\lambda\in[m,M]}\left[\langle E_\lambda x, x\rangle\langle(1_H - E_\lambda)x, x\rangle\right]^{1/2} \\ &\times \max_{\mu\in[m,M]}\left[\langle F_\mu x, x\rangle\langle(1_H - F_\mu)x, x\rangle\right]^{1/2}\bigvee_m^M(g)\bigvee_m^M(f) \le \frac{1}{4}\bigvee_m^M(g)\bigvee_m^M(f) \end{aligned} \tag{8.8}$$

for any $x \in H$ with $\|x\| = 1.$

2. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle x,g(B)x\rangle\right| \\
&\leq K\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|d\mu\right]\bigvee_m^M(f) \\
&\leq K\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}d\mu \\
&\leq\frac{1}{2}K\bigvee_m^M(f)\langle(M1_H-B)x,x\rangle^{1/2}\langle(B-m1_H)x,x\rangle^{1/2} \\
&\leq\frac{1}{4}K(M-m)\bigvee_m^M(f)
\end{aligned}\tag{8.9}$$

for any $x\in H$ with $\|x\|=1$.

3. If $g:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle x,g(B)x\rangle\right| \\
&\leq\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|dg(\mu)\right]\bigvee_m^M(f) \\
&\leq\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}dg(\mu) \\
&\leq\frac{1}{2}\bigvee_m^M(f)\langle(g(M)1_H-g(B))x,x\rangle^{1/2}\langle(g(B)-g(m)1_H)x,x\rangle^{1/2} \\
&\leq\frac{1}{4}\left[g(M)-g(m)\right]\bigvee_m^M(f)
\end{aligned}\tag{8.10}$$

for any $x\in H$ with $\|x\|=1$.

PROOF. 1. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),\tag{8.11}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Now, on utilizing the property (8.11) and the identity (8.1) we have

$$\begin{aligned}(8.12)\qquad &|\langle f(A)x, g(B)x\rangle - \langle f(A)x,x\rangle\langle x,g(B)x\rangle| \\ &\le \max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle]\,d(g(\mu))\right|\bigvee_m^M(f)\end{aligned}$$

for any $x\in[m,M]$.

The same inequality (8.11) produces the bound

$$\begin{aligned}(8.13)\qquad &\max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle]\,d(g(\mu))\right| \\ &\le \max_{\lambda\in[m,M]}\left[\max_{\mu\in[m,M]}|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle|\right]\bigvee_m^M(f) \\ &= \max_{(\lambda,\mu)\in[m,M]^2}|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle|\bigvee_m^M(f)\end{aligned}$$

for any $x\in[m,M]$.

By making use of (8.12) and (8.13) we deduce the first part of (8.8).

Further, we notice that by the Schwarz inequality in H we have for any $u,v,e\in H$ with $\|e\|=1$ that

$$\begin{aligned}(8.14)\qquad &|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \\ &\le \left(\|u\|^2-|\langle u,e\rangle|^2\right)^{1/2}\left(\|v\|^2-|\langle v,e\rangle|^2\right)^{1/2}.\end{aligned}$$

Indeed, if we write Schwarz's inequality for the vectors $u-\langle u,e\rangle e$ and $v-\langle v,e\rangle e$ we have

$$|\langle u-\langle u,e\rangle e, v-\langle v,e\rangle e\rangle| \le \|u-\langle u,e\rangle e\|\,\|v-\langle v,e\rangle e\|$$

which, by performing the calculations, is equivalent with (8.14).

Now, on utilizing (8.14), we can state that

$$\begin{aligned}(8.15)\qquad &|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle - \langle E_\lambda x,F_\mu x\rangle| \\ &\le \left(\|E_\lambda x\|^2-|\langle E_\lambda x,x\rangle|^2\right)^{1/2}\left(\|F_\mu x\|^2-|\langle F_\mu x,x\rangle|^2\right)^{1/2}\end{aligned}$$

for any $\lambda,\mu\in[m,M]$.

Since E_λ and F_μ are projections and $E_\lambda, F_\mu\ge 0$ then

$$\begin{aligned}(8.16)\qquad \|E_\lambda x\|^2-|\langle E_\lambda x,x\rangle|^2 &= \langle E_\lambda x,x\rangle - \langle E_\lambda x,x\rangle^2 \\ &= \langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle \le \frac{1}{4}\end{aligned}$$

and

$$(8.17)\qquad \|F_\mu x\|^2-|\langle F_\mu x,x\rangle|^2 = \langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle \le \frac{1}{4}$$

for any $\lambda,\mu\in[m,M]$ and $x\in H$ with $\|x\|=1.$

Now, if we use (8.15)-(8.17) then we get the second part of (8.8).

2. Further, recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt. \tag{8.18}$$

If we use the inequality (8.18), then we have in the case when g is Lipschitzian with the constant $K>0$ that

$$\begin{aligned} &\max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right]d(g(\mu))\right| \\ &\le K\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|d\mu\right] \end{aligned} \tag{8.19}$$

for any $x\in H$ with $\|x\|=1$ and the first part of (8.9) is proved.

Further, by employing (8.15)-(8.17) we also get that

$$\begin{aligned} &\max_{\lambda\in[m,M]}\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle\right|d\mu \\ &\le\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\ &\times\int_{m-0}^{M}\left[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}d\mu \end{aligned} \tag{8.20}$$

for any $x\in H$ with $\|x\|=1$.

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation theorem, then we have successively

$$\begin{aligned} &\int_{m-0}^{M}\left(\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle\right)^{1/2}d\mu \\ &\le\left[\int_{m-0}^{M}\langle F_\mu x,x\rangle\,d\mu\right]^{1/2}\left[\int_{m-0}^{M}\langle(1_H-F_\mu)x,x\rangle\,d\mu\right]^{1/2} \\ &=\left[\langle F_\mu x,x\rangle\,\mu\Big|_{m-0}^{M}-\int_{m-0}^{M}\mu d\langle F_\mu x,x\rangle\right]^{1/2} \\ &\times\left[\langle(1_H-F_\mu)x,x\rangle\,\mu\Big|_{m-0}^{M}-\int_{m-0}^{M}\mu d\langle(1_H-F_\mu)x,x\rangle\right] \\ &=\langle(M1_H-B)x,x\rangle^{1/2}\langle(B-m1_H)x,x\rangle^{1/2}, \end{aligned} \tag{8.21}$$

for any $x\in H$ with $\|x\|=1$.

On employing now (8.19)-(8.21) we deduce the second part of (8.9).

The last part of (8.9) follows by the elementary inequality

$$\alpha\beta\le\frac{1}{4}(\alpha+\beta)^2,\ \alpha\beta\ge0 \tag{8.22}$$

for the choice $\alpha=\langle(M1_H-B)x,x\rangle$ and $\beta=\langle(B-m1_H)x,x\rangle$ and the details are omitted.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t). \tag{8.23}$$

Now, if we assume that g is monotonic nondecreasing on $[m,M]$, then by (8.23) we have that

$$\max_{\lambda\in[m,M]}\left|\int_{m-0}^{M}[\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle]\,d(g(\mu))\right| \le \max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle|\,dg(\mu)\right] \tag{8.24}$$

for any $x\in H$ with $\|x\|=1$.

Further, by employing (8.15)-(8.17) we also get that

$$\max_{\lambda\in[m,M]}\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,F_\mu x\rangle-\langle E_\lambda x,F_\mu x\rangle|\,dg(\mu) \le \max_{\lambda\in[m,M]}[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle]^{1/2} \times\int_{m-0}^{M}[\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle]^{1/2}\,dg(\mu) \tag{8.25}$$

for any $x\in H$ with $\|x\|=1$. These prove the first part of (8.10).

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators and the spectral representation theorem, then we have successively

$$\begin{aligned}&\int_{m-0}^{M}(\langle F_\mu x,x\rangle\langle(1_H-F_\mu)x,x\rangle)^{1/2}\,dg(\mu)\\ &\le\left[\int_{m-0}^{M}\langle F_\mu x,x\rangle\,dg(\mu)\right]^{1/2}\left[\int_{m-0}^{M}\langle(1_H-F_\mu)x,x\rangle\,dg(\mu)\right]^{1/2}\\ &=\left[\langle F_\mu x,x\rangle g(\mu)\big|_{m-0}^{M}-\int_{m-0}^{M}g(\mu)\,d\langle F_\mu x,x\rangle\right]^{1/2}\\ &\times\left[\langle(1_H-F_\mu)x,x\rangle g(\mu)\big|_{m-0}^{M}-\int_{m-0}^{M}g(\mu)\,d\langle(1_H-F_\mu)x,x\rangle\right]^{1/2}\\ &=\langle(g(M)1_H-g(B))x,x\rangle^{1/2}\langle(g(B)-g(m)1_H)x,x\rangle^{1/2},\end{aligned} \tag{8.26}$$

for any $x\in H$ with $\|x\|=1$.

Utilising (8.26) we then deduce the last part of (8.10). The details are omitted. ∎

Now, in order to provide other results that are similar to the Grüss' type inequalities stated in the introduction, we can state the following corollary:

COROLLARY 8.3 (Dragomir, 2010, [**36**]). *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Also, assume that $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$.*

1. If $g:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$, then we have the inequality

$$
\begin{aligned}
&\left|\langle f(A)x,g(A)x\rangle-\langle f(A)x,x\rangle\langle x,g(A)x\rangle\right| \\
&\leq \max_{(\lambda,\mu)\in[m,M]^2}\left|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle\right|\bigvee_m^M(g)\bigvee_m^M(f) \\
&\leq \max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]\bigvee_m^M(g)\bigvee_m^M(f) \\
&\leq \frac{1}{4}\bigvee_m^M(g)\bigvee_m^M(f)
\end{aligned}
\tag{8.27}
$$

for any $x\in H$ with $\|x\|=1$.

2. If $g:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$
\begin{aligned}
&\left|\langle f(A)x,g(A)x\rangle-\langle f(A)x,x\rangle\langle x,g(A)x\rangle\right| \\
&\leq K\max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}\left|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle\right|d\mu\right]\bigvee_m^M(f) \\
&\leq K\bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\
&\times\int_{m-0}^{M}\left[\langle E_\mu x,x\rangle\langle(1_H-E_\mu)x,x\rangle\right]^{1/2}d\mu \\
&\leq \frac{1}{2}K\bigvee_m^M(f)\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2} \\
&\leq \frac{1}{4}K(M-m)\bigvee_m^M(f)
\end{aligned}
\tag{8.28}
$$

for any $x\in H$ with $\|x\|=1$.

3. If $g:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then we have the inequality

$$\begin{aligned}(8.29)\quad & |\langle f(A)x, g(A)x\rangle - \langle f(A)x,x\rangle\langle x, g(A)x\rangle| \\ & \le \max_{\lambda\in[m,M]}\left[\int_{m-0}^{M}|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle|\,dg(\mu)\right]\bigvee_m^M(f) \\ & \le \bigvee_m^M(f)\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]^{1/2} \\ & \times\int_{m-0}^{M}\left[\langle E_\mu x,x\rangle\langle(1_H-E_\mu)x,x\rangle\right]^{1/2}dg(\mu) \\ & \le \frac{1}{2}\bigvee_m^M(f)\langle(g(M)1_H-g(A))x,x\rangle^{1/2}\langle(g(A)-g(m)1_H)x,x\rangle^{1/2} \\ & \le \frac{1}{4}[g(M)-g(m)]\bigvee_m^M(f)\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

REMARK 8.2. The following inequality for the variance of $f(A)$ under the assumptions that A is a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ is the *spectral family of* A and $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$ can be stated

$$\begin{aligned}(8.30)\quad & 0\le\|f(A)x\|^2-\langle f(A)x,x\rangle^2 \\ & \le \max_{(\lambda,\mu)\in[m,M]^2}|\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle-\langle E_\lambda x,E_\mu x\rangle|\left[\bigvee_m^M(f)\right]^2 \\ & \le \max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right]\left[\bigvee_m^M(f)\right]^2\le\frac{1}{4}\left[\bigvee_m^M(f)\right]^2\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

8.3. Bounds for f Lipschitzian. The case when the first function is Lipschitzian is as follows:

THEOREM 8.4 (Dragomir, 2010, [**36**]). *Let A,B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A),Sp(B)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. Also, assume that $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$.*

1. If $g : [m, M] \to \mathbb{C}$ *is Lipschitzian with the constant* $K > 0$ *on* $[m, M]$ *, then we have the inequality*

$$
\begin{aligned}
(8.31)\qquad & |\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
& \leq LK \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu d\lambda \\
& \leq LK \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda \\
& \times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} \, d\mu \\
& \leq LK [\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle]^{1/2} \\
& \times [\langle (M1_H - B) x, x\rangle \langle (B - m1_H) x, x\rangle]^{1/2} \leq \frac{1}{4} LK (M - m)^2
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

2. If $g : [m, M] \to \mathbb{R}$ *is continuous and monotonic nondecreasing on* $[m, M]$ *, then we have the inequality*

$$
\begin{aligned}
(8.32)\qquad & |\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
& \leq L \int_{m-0}^{M} \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, d\lambda \\
& \leq L \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle]^{1/2} \, d\lambda \\
& \times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle]^{1/2} \, dg(\mu) \\
& \leq L [\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle]^{1/2} \\
& \times [\langle (g(M) 1_H - g(B)) x, x\rangle \langle (g(B) - g(m) 1_H) x, x\rangle]^{1/2} \\
& \leq \frac{1}{4} L (M - m) [g(M) - g(m)]
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

PROOF. 1. We observe that, on utilizing the property (8.18) and the identity (8.1) we have

$$
\begin{aligned}
(8.33)\qquad & |\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
& \leq L \int_{m-0}^{M} \left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| d\lambda
\end{aligned}
$$

for any $x \in H$, $\|x\| = 1$.

By the same property (8.18) we also have

$$
\begin{aligned}
(8.34)\qquad & \left| \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle] \, d(g(\mu)) \right| \\
& \leq K \int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, d\mu
\end{aligned}
$$

for any $x \in H$, $\|x\| = 1$ and $\lambda \in [m, M]$.

Therefore, by (8.33) and (8.34) we get

$$\begin{aligned}(8.35)\qquad &\left|\left\langle f(A)x, g(B)x\right\rangle - \left\langle f(A)x, x\right\rangle\left\langle x, g(B)x\right\rangle\right| \\ &\leq LK\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right| d\mu d\lambda\end{aligned}$$

for any $x \in H, \|x\| = 1,$ which proves the first inequality in (8.31).

From (8.15)-(8.17) we have

$$\begin{aligned}(8.36)\qquad &\left|\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right| \\ &\leq \left[\left\langle E_\lambda x, x\right\rangle\left\langle (1_H - E_\lambda)x, x\right\rangle\right]^{1/2}\left[\left\langle F_\mu x, x\right\rangle\left\langle (1_H - F_\mu)x, x\right\rangle\right]^{1/2}\end{aligned}$$

for any $x \in H, \|x\| = 1$ and $\lambda, \mu \in [m, M]$.

Integrating on $[m, M]^2$ the inequality (8.36) and utilizing the Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann integral we have

$$\begin{aligned}(8.37)\qquad &\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right| d\mu d\lambda \\ &\leq \int_{m-0}^{M}\left[\left\langle E_\lambda x, x\right\rangle\left\langle (1_H - E_\lambda)x, x\right\rangle\right]^{1/2} d\lambda \\ &\times \int_{m-0}^{M}\left[\left\langle F_\mu x, x\right\rangle\left\langle (1_H - F_\mu)x, x\right\rangle\right]^{1/2} d\mu \\ &\leq \left[\int_{m-0}^{M}\left\langle E_\lambda x, x\right\rangle d\lambda\right]^{1/2}\left[\int_{m-0}^{M}\left\langle (1_H - E_\lambda)x, x\right\rangle d\lambda\right]^{1/2} \\ &\times \left[\int_{m-0}^{M}\left\langle F_\mu x, x\right\rangle d\mu\right]^{1/2}\left[\int_{m-0}^{M}\left\langle (1_H - F_\mu)x, x\right\rangle d\mu\right]^{1/2}.\end{aligned}$$

Integrating by parts and utilizing the spectral representation theorem we have

$$\begin{aligned}\int_{m-0}^{M}\left\langle E_\lambda x, x\right\rangle d\lambda &= \left\langle E_\lambda x, x\right\rangle \lambda\Big|_{m-0}^{M} - \int_{m-0}^{M}\lambda d\left\langle E_\lambda x, x\right\rangle \\ &= M - \left\langle Ax, x\right\rangle = \left\langle (M1_H - A)x, x\right\rangle,\end{aligned}$$

$$\int_{m-0}^{M}\left\langle (1_H - E_\lambda)x, x\right\rangle d\lambda = \left\langle (A - m1_H)x, x\right\rangle$$

and the similar equalities for B, providing the second part of (8.31).

The last part follows from (8.22) and we omit the details.

2. Utilising the inequality (8.23) we have

$$\begin{aligned}(8.38)\qquad &\left|\int_{m-0}^{M}\left[\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right] d(g(\mu))\right| \\ &\leq \int_{m-0}^{M}\left|\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right| dg(\mu)\end{aligned}$$

which, together with (8.33), produces the inequality

$$\begin{aligned}(8.39)\qquad &\left|\left\langle f(A)x, g(B)x\right\rangle - \left\langle f(A)x, x\right\rangle\left\langle x, g(B)x\right\rangle\right| \\ &\leq L\int_{m-0}^{M}\int_{m-0}^{M}\left|\left\langle E_\lambda x, x\right\rangle\left\langle x, F_\mu x\right\rangle - \left\langle E_\lambda x, F_\mu x\right\rangle\right| dg(\mu) d\lambda\end{aligned}$$

for any $x \in H, \|x\| = 1.$

Now, by utilizing (8.36) and a similar argument to the one outlined above, we deduce the desired result (8.32) and the details are omitted. ∎

The case of one operator is incorporated in

COROLLARY 8.5 (Dragomir, 2010, [**36**]). *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Also, assume that $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$.*

1. If $g : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$
\begin{aligned}
(8.40)\qquad & |\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle\langle x, g(A)x\rangle| \\
&\leq LK\int_{m-0}^{M}\int_{m-0}^{M}|\langle E_\lambda x, x\rangle\langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle|\,d\mu d\lambda \\
&\leq LK\left(\int_{m-0}^{M}[\langle E_\lambda x, x\rangle\langle(1_H - E_\lambda)x, x\rangle]^{1/2}\,d\lambda\right)^2 \\
&\leq LK[\langle(M1_H - A)x, x\rangle\langle(A - m1_H)x, x\rangle] \leq \frac{1}{4}LK(M-m)^2
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

2. If $g : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then we have the inequality

$$
\begin{aligned}
(8.41)\qquad & |\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle\langle x, g(A)x\rangle| \\
&\leq L\int_{m-0}^{M}\int_{m-0}^{M}|\langle E_\lambda x, x\rangle\langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle|\,dg(\mu)\,d\lambda \\
&\leq L\int_{m-0}^{M}[\langle E_\lambda x, x\rangle\langle(1_H - E_\lambda)x, x\rangle]^{1/2}\,d\lambda \\
&\times\int_{m-0}^{M}[\langle E_\mu x, x\rangle\langle(1_H - E_\mu)x, x\rangle]^{1/2}\,dg(\mu) \\
&\leq L[\langle(M1_H - A)x, x\rangle\langle(A - m1_H)x, x\rangle]^{1/2} \\
&\times[\langle(g(M)1_H - g(A))x, x\rangle\langle(g(A) - g(m)1_H)x, x\rangle]^{1/2} \\
&\leq \frac{1}{4}L(M-m)[g(M) - g(m)]
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

REMARK 8.3. The following inequality for the variance of $f(A)$ under the assumptions that A is a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ is the *spectral family of* A and $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$ can be stated

$$
\begin{aligned}
0 &\leq \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \\
&\leq L^2 \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle| \, d\mu d\lambda \\
&\leq L^2 \left(\int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle]^{1/2} d\lambda \right)^2 \\
&\leq L^2 [\langle (M1_H - A)x, x\rangle \langle (A - m1_H)x, x\rangle] \\
&\leq \frac{1}{4} L^2 (M-m)^2
\end{aligned} \tag{8.42}
$$

for any $x \in H$ with $\|x\| = 1$.

8.4. Bounds for f Monotonic Nondecreasing. Finally, for the case of two monotonic functions we have the following result as well:

THEOREM 8.6 (Dragomir, 2010, [**36**]). *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A and $\{F_\mu\}_\mu$ the spectral family of B. If $f, g : [m, M] \to \mathbb{C}$ are continuous and monotonic nondecreasing on $[m, M]$, then*

$$
\begin{aligned}
&|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle \langle x, g(B)x\rangle| \\
&\leq \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle| \, dg(\mu) \, df(\lambda) \\
&\leq \int_{m-0}^{M} [\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle]^{1/2} df(\lambda) \\
&\times \int_{m-0}^{M} [\langle F_\mu x, x\rangle \langle (1_H - F_\mu)x, x\rangle]^{1/2} dg(\mu) \\
&\leq [\langle (f(M)1_H - f(A))x, x\rangle \langle (f(A) - f(m)1_H)x, x\rangle]^{1/2} \\
&\times [\langle (g(M)1_H - g(B))x, x\rangle \langle (g(B) - g(m)1_H)x, x\rangle]^{1/2} \\
&\leq \frac{1}{4} [f(M) - f(m)] [g(M) - g(m)]
\end{aligned} \tag{8.43}
$$

for any $x \in H, \|x\| = 1$.

The details of the proof are omitted.

In particular we have:

COROLLARY 8.7 (Dragomir, 2010, [**36**]). *Let A be a selfadjoint operators in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. If $f, g : [m, M] \to \mathbb{C}$ are continuous and monotonic nondecreasing on*

$[m,M]$, *then*

$$|\langle f(A)x, g(A)x\rangle - \langle f(A)x,x\rangle\langle x,g(A)x\rangle| \tag{8.44}$$
$$\leq \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle|\, dg(\mu)\, df(\lambda)$$
$$\leq \int_{m-0}^{M} [\langle E_\lambda x,x\rangle\langle (1_H - E_\lambda)x,x\rangle]^{1/2} df(\lambda)$$
$$\times \int_{m-0}^{M} [\langle E_\mu x,x\rangle\langle (1_H - E_\mu)x,x\rangle]^{1/2} dg(\mu)$$
$$\leq [\langle (f(M)1_H - f(A))x,x\rangle\langle (f(A) - f(m)1_H)x,x\rangle]^{1/2}$$
$$\times [\langle (g(M)1_H - g(A))x,x\rangle\langle (g(A) - g(m)1_H)x,x\rangle]^{1/2}$$
$$\leq \frac{1}{4}[f(M)-f(m)][g(M)-g(m)]$$

for any $x \in H, \|x\| = 1.$

In particular, the following inequality for the variance of $f(A)$ *in the case of monotonic nondecreasing functions* f *holds:*

$$0 \leq \|f(A)x\|^2 - \langle f(A)x,x\rangle^2 \tag{8.45}$$
$$\leq \int_{m-0}^{M}\int_{m-0}^{M} |\langle E_\lambda x,x\rangle\langle x,E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle|\, df(\mu)\, df(\lambda)$$
$$\leq \left(\int_{m-0}^{M} [\langle E_\lambda x,x\rangle\langle (1_H - E_\lambda)x,x\rangle]^{1/2} df(\lambda)\right)^2$$
$$\leq [\langle (f(M)1_H - f(A))x,x\rangle\langle (f(A) - f(m)1_H)x,x\rangle]$$
$$\leq \frac{1}{4}[f(M)-f(m)]^2$$

for any $x \in H, \|x\| = 1.$

8.5. Applications. By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f,g:(0,\infty)\to(0,\infty)$ with $f(t)=t^p, g(t)=t^q$ and $p,q>0$, then for any selfadjoint operators A,B with $Sp(A), Sp(B) \subseteq [m,M] \subset (0,\infty)$ we get from (8.43) the inequalities:

$$
\begin{aligned}
(8.46)\qquad & \left|\langle A^{p}x,B^{q}x\rangle-\langle A^{p}x,x\rangle\langle B^{q}x,x\rangle\right| \\
&\le pq\int_{m-0}^{M}\int_{m-0}^{M}\left|\langle E_{\lambda}x,x\rangle\langle x,F_{\mu}x\rangle-\langle E_{\lambda}x,F_{\mu}x\rangle\right|\mu^{q-1}\lambda^{p-1}d\mu d\lambda \\
&\le pq\int_{m-0}^{M}\left[\langle E_{\lambda}x,x\rangle\langle(1_{H}-E_{\lambda})x,x\rangle\right]^{1/2}\lambda^{p-1}d\lambda \\
&\times\int_{m-0}^{M}\left[\langle F_{\mu}x,x\rangle\langle(1_{H}-F_{\mu})x,x\rangle\right]^{1/2}\mu^{q-1}d\mu \\
&\le\left[\langle(M^{p}1_{H}-A^{p})x,x\rangle\langle(A^{p}-m^{p}1_{H})x,x\rangle\right]^{1/2} \\
&\times\left[\langle(M^{q}1_{H}-B^{q})x,x\rangle\langle(B^{q}-m^{q}1_{H})x,x\rangle\right]^{1/2} \\
&\le\frac{1}{4}(M^{p}-m^{p})(M^{q}-m^{q})
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$, where $\{E_{\lambda}\}_{\lambda}$ is the spectral family of A and $\{F_{\mu}\}_{\mu}$ is the spectral family of $B.$

When $B=A$ then by the Čebyšev's inequality for functions of same monotonicity the inequality (8.46) becomes

$$
\begin{aligned}
(8.47)\qquad & 0\le\langle A^{p}x,A^{q}x\rangle-\langle A^{p}x,x\rangle\langle A^{q}x,x\rangle \\
&\le pq\int_{m-0}^{M}\int_{m-0}^{M}\left|\langle E_{\lambda}x,x\rangle\langle x,E_{\mu}x\rangle-\langle E_{\lambda}x,E_{\mu}x\rangle\right|\mu^{q-1}\lambda^{p-1}d\mu d\lambda \\
&\le pq\int_{m-0}^{M}\left[\langle E_{\lambda}x,x\rangle\langle(1_{H}-E_{\lambda})x,x\rangle\right]^{1/2}\lambda^{p-1}d\lambda \\
&\times\int_{m-0}^{M}\left[\langle E_{\mu}x,x\rangle\langle(1_{H}-E_{\mu})x,x\rangle\right]^{1/2}\mu^{q-1}d\mu \\
&\le\left[\langle(M^{p}1_{H}-A^{p})x,x\rangle\langle(A^{p}-m^{p}1_{H})x,x\rangle\right]^{1/2} \\
&\times\left[\langle(M^{q}1_{H}-B^{q})x,x\rangle\langle(B^{q}-m^{q}1_{H})x,x\rangle\right]^{1/2} \\
&\le\frac{1}{4}(M^{p}-m^{p})(M^{q}-m^{q})
\end{aligned}
$$

for any $x\in H$ with $\|x\|=1$ and $p,q>0.$

Now. define the coefficients

$$
(8.48)\qquad \Delta_{p}:=p\times\begin{cases} M^{p-1}-m^{p-1} \text{ if } p\ge 1\\ \frac{M^{1-p}-m^{1-p}}{M^{1-p}m^{1-p}} \text{ if } 0<p<1.\end{cases}
$$

On utilizing the inequality (8.31) for the same power functions considered above, we can state the inequality

$$\begin{aligned}
(8.49) \quad & \left|\langle A^p x, B^q x\rangle - \langle A^p x, x\rangle \langle B^q x, x\rangle\right| \\
&\leq \Delta_p \Delta_q \int_{m-0}^{M} \int_{m-0}^{M} \left|\langle E_\lambda x, x\rangle \langle x, F_\mu x\rangle - \langle E_\lambda x, F_\mu x\rangle\right| d\mu d\lambda \\
&\leq \Delta_p \Delta_q \int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right]^{1/2} d\lambda \\
&\times \int_{m-0}^{M} \left[\langle F_\mu x, x\rangle \langle (1_H - F_\mu) x, x\rangle\right]^{1/2} d\mu \\
&\leq \Delta_p \Delta_q \left[\langle (M 1_H - A) x, x\rangle \langle (A - m 1_H) x, x\rangle\right]^{1/2} \\
&\times \left[\langle (M 1_H - B) x, x\rangle \langle (B - m 1_H) x, x\rangle\right]^{1/2} \leq \frac{1}{4} \Delta_p \Delta_q (M-m)^2
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

In particular, for $B = A$ we have

$$\begin{aligned}
(8.50) \quad & 0 \leq \langle A^p x, A^q x\rangle - \langle A^p x, x\rangle \langle A^q x, x\rangle \\
&\leq \Delta_p \Delta_q \int_{m-0}^{M} \int_{m-0}^{M} \left|\langle E_\lambda x, x\rangle \langle x, E_\mu x\rangle - \langle E_\lambda x, E_\mu x\rangle\right| d\mu d\lambda \\
&\leq \Delta_p \Delta_q \left(\int_{m-0}^{M} \left[\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right]^{1/2} d\lambda\right)^2 \\
&\leq \Delta_p \Delta_q \left[\langle (M 1_H - A) x, x\rangle \langle (A - m 1_H) x, x\rangle\right] \leq \frac{1}{4} \Delta_p \Delta_q (M-m)^2
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

Similar results can be stated if $p < 0$ or $q < 0$. However the details are left to the interest reader.

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CHAPTER 4

Inequalities of Ostrowski Type

1. Introduction

Ostrowski's type inequalities provide sharp error estimates in approximating the value of a function by its integral mean. They can be utilized to obtain a priory error bounds for different quadrature rules in approximating the Riemann integral by different Riemann sums. They also shows, in general, that the mid-point rule provides the best approximation in the class of all Riemann sums sampled in the interior points of a given partition.

As revealed by a simple search in the data base *MathSciNet* of the *American Mathematical Society* with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 280 papers that can be found by performing the above search. Numerous extensions, generalisations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In the present chapter we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. As far as we know, the obtained results are new with no previous similar results ever obtained in the literature.

Applications for mid-point inequalities and some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

2. Scalar Ostrowski's Type Inequalities

In the scalar case, comparison between functions and integral means are incorporated in Ostrowski type inequalities as mentioned below.

The first result in this direction is known in the literature as Ostrowski's inequality [**44**].

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{2.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity..

The following Ostrowski type result for absolutely continuous functions holds (see [**34**] – [**36**]).

THEOREM 2.2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.2}$$
$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f'\in L_q[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases}$$

where $\|\cdot\|_r$ ($r\in[1,\infty]$) are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty:=ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 2.1.

The above inequalities can also be obtained from the Fink result in [**39**] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [**32**] and the references therein for earlier contributions):

THEOREM 2.3. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)|\leq H|x-y|^r,\ \textit{for all}\ \ x,y\in[a,b], \tag{2.3}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then, for all $x\in[a,b]$, we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{2.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [**24**])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{2.5}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [**23**]).

THEOREM 2.4. *Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{2.6}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (2.6) may be improved in the following manner [**12**] (see also the monograph [**33**]).

THEOREM 2.5. *Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\} \\ &\le \frac{1}{b-a} \left\{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \right\} \\ &\le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)] . \end{aligned} \tag{2.7}$$

All the inequalities in (2.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other scalar Ostrowski's type inequalities, see [**2**]-[**4**] and [**25**].

3. Ostrowski's type Inequalities for Hölder Continuous Functions

3.1. Introduction. Let U be a selfadjoint operator on the Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation* theorem in terms of the *Riemann-Stieltjes integral*:

$$\langle f(U) x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle) , \tag{3.1}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m-0) = 0 \text{ and } g_x(M) = 1 \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the representation (3.1) and the following Ostrowski's type inequality for the Riemann-Stieltjes integral obtained by the author in [**28**]:

$$\begin{aligned} &\left| f(s)[u(b) - u(a)] - \int_a^b f(t)\, du(t) \right| \\ &\le L \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (u) \end{aligned} \tag{3.3}$$

for any $s \in [a, b]$, provided that f is of $r - L-$Hölder type on $[a, b]$ (see (3.4) below), u is of *bounded variation* on $[a, b]$ and $\bigvee_a^b (u)$ denotes the *total variation* of u on $[a, b]$, we obtained the following inequality of Ostrowski type for selfadjoint operators:

THEOREM 3.1 (Dragomir, 2008, [**29**]). *Let A and B be selfadjoint operators with $Sp(A)$, $Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \leq L |s - t|^r \text{ for any } s, t \in [m, M], \tag{3.4}$$

then we have the inequality:

$$|f(s) - \langle f(A) x, x\rangle| \leq L \left[\frac{1}{2}(M - m) + \left|s - \frac{m + M}{2}\right|\right]^r, \tag{3.5}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Moreover, we have

$$\begin{aligned} &|\langle f(B) y, y\rangle - \langle f(A) x, x\rangle| \\ &\leq \langle |f(B) - \langle f(A) x, x\rangle \cdot 1_H| y, y\rangle \\ &\leq L \left[\frac{1}{2}(M - m) + \left\langle \left|B - \frac{m + M}{2} \cdot 1_H\right| y, y\right\rangle\right]^r, \end{aligned} \tag{3.6}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

With the above assumptions for f, A and B we have the following particular inequalities of interest:

$$\left|f\left(\frac{m + M}{2}\right) - \langle f(A) x, x\rangle\right| \leq \frac{1}{2^r} L (M - m)^r \tag{3.7}$$

and

$$\begin{aligned} &|f(\langle Ax, x\rangle) - \langle f(A) x, x\rangle| \\ \leq \;\; & L \left[\frac{1}{2}(M - m) + \left|\langle Ax, x\rangle - \frac{m + M}{2}\right|\right]^r, \end{aligned} \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequalities:

$$\begin{aligned} &|\langle f(A) y, y\rangle - \langle f(A) x, x\rangle| \\ &\leq \langle |f(A) - \langle f(A) x, x\rangle \cdot 1_H| y, y\rangle \\ &\leq L \left[\frac{1}{2}(M - m) + \left\langle \left|A - \frac{m + M}{2} \cdot 1_H\right| y, y\right\rangle\right]^r, \end{aligned} \tag{3.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$,

$$\begin{aligned} &|\langle [f(B) - f(A)] x, x\rangle| \\ &\leq \langle |f(B) - \langle f(A) x, x\rangle \cdot 1_H| x, x\rangle \\ &\leq L \left[\frac{1}{2}(M - m) + \left\langle \left|B - \frac{m + M}{2} \cdot 1_H\right| x, x\right\rangle\right]^r \end{aligned} \tag{3.10}$$

and, more particularly,

$$\text{(3.11)}\qquad \begin{aligned}&\langle\left|f(A)-\langle f(A)x,x\rangle\cdot 1_H\right|x,x\rangle\\ &\le L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}$$

for any $x\in H$ with $\|x\|=1.$

We also have the norm inequality

$$\text{(3.12)}\qquad \|f(B)-f(A)\|\le L\left[\frac{1}{2}(M-m)+\left\|B-\frac{m+M}{2}\cdot 1_H\right\|\right]^r.$$

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [**31**] and the references therein.

3.2. More Inequalities of Ostrowski's Type.

The following result holds:

THEOREM 3.2 (Dragomir, 2010, [**30**]). *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\longrightarrow\mathbb{R}$ is of $r-L-$Hölder type with $r\in(0,1]$, then we have the inequality:*

$$\text{(3.13)}\qquad \begin{aligned}|f(s)-\langle f(A)x,x\rangle|&\le L\langle|s\cdot 1_H-A|x,x\rangle^r\\ &\le L\left[(s-\langle Ax,x\rangle)^2+D^2(A;x)\right]^{r/2},\end{aligned}$$

for any $s\in[m,M]$ and any $x\in H$ with $\|x\|=1$, where $D(A;x)$ is the variance of the selfadjoint operator A in x and is defined by

$$D(A;x):=\left(\|Ax\|^2-\langle Ax,x\rangle^2\right)^{1/2},$$

where $x\in H$ with $\|x\|=1$.

PROOF. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [**40**, p. 5]) applied for the modulus, we can state that

$$\text{(M)}\qquad |\langle h(A)x,x\rangle|\le\langle|h(A)|x,x\rangle$$

for any $x\in H$ with $\|x\|=1$, where h is a continuous function on $[m,M]$.

Utilising the property (M) we then get

$$\text{(3.14)}\qquad \begin{aligned}|f(s)-\langle f(A)x,x\rangle|&=|\langle f(s)\cdot 1_H-f(A)x,x\rangle|\\ &\le\langle|f(s)\cdot 1_H-f(A)|x,x\rangle\end{aligned}$$

for any $x\in H$ with $\|x\|=1$ and any $s\in[m,M]$.

Since f is of $r-L-$Hölder type, then for any $t,s\in[m,M]$ we have

$$\text{(3.15)}\qquad |f(s)-f(t)|\le L|s-t|^r.$$

If we fix $s\in[m,M]$ and apply the property (P) for the inequality (3.15) and the operator A we get

$$\text{(3.16)}\qquad \begin{aligned}\langle|f(s)\cdot 1_H-f(A)|x,x\rangle&\le L\langle|s\cdot 1_H-A|^r x,x\rangle\\ &\le L\langle|s\cdot 1_H-A|x,x\rangle^r\end{aligned}$$

for any $x\in H$ with $\|x\|=1$ and any $s\in[m,M]$, where, for the last inequality we have used the fact that if P is a positive operator and $r\in(0,1)$ then, by the Hölder-McCarthy inequality [**42**],

$$\text{(HM)}\qquad \langle P^r x,x\rangle\le\langle Px,x\rangle^r$$

for any $x\in H$ with $\|x\|=1$. This proves the fist inequality in (3.13).

Now, observe that for any bounded linear operator T we have

$$\langle |T|\, x, x\rangle = \left\langle (T^*T)^{1/2}\, x, x\right\rangle \le \langle (T^*T)\, x, x\rangle^{1/2} = \|Tx\|$$

for any $x \in H$ with $\|x\| = 1$ which implies that

$$\begin{aligned}\langle |s \cdot 1_H - A|\, x, x\rangle^r &\le \|sx - Ax\|^r \\ &= \left(s^2 - 2s\,\langle Ax, x\rangle + \|Ax\|^2\right)^{r/2} \\ &= \left[(s - \langle Ax, x\rangle)^2 + \|Ax\|^2 - \langle Ax, x\rangle^2\right]^{r/2}\end{aligned} \tag{3.17}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$.

Finally, on making use of (3.14), (3.16) and (3.17) we deduce the desired result (3.13). ∎

REMARK 3.1. If we choose in (3.13) $s = \frac{m+M}{2}$, then we get the sequence of inequalities

$$\begin{aligned}&\left| f\left(\frac{m+M}{2}\right) - \langle f(A)\, x, x\rangle\right| \\ &\le L\left\langle \left|\frac{m+M}{2} \cdot 1_H - A\right| x, x\right\rangle^r \\ &\le L\left[\left(\frac{m+M}{2} - \langle Ax, x\rangle\right)^2 + D^2(A; x)\right]^{r/2} \\ &\le L\left[\frac{1}{4}(M-m)^2 + D^2(A; x)\right]^{r/2} \le \frac{1}{2^r} L\,(M-m)^r\end{aligned} \tag{3.18}$$

for any $x \in H$ with $\|x\| = 1$, since, obviously,

$$\left(\frac{m+M}{2} - \langle Ax, x\rangle\right)^2 \le \frac{1}{4}(M-m)^2$$

and

$$D^2(A; x) \le \frac{1}{4}(M-m)^2$$

for any $x \in H$ with $\|x\| = 1$.

We notice that the inequality (3.18) provides a refinement for the result (3.7) above.

The best inequality we can get from (3.13) is incorporated in the following:

COROLLARY 3.3 (Dragomir, 2010, [**30**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality*

$$\begin{aligned}|f(\langle Ax, x\rangle) - \langle f(A)\, x, x\rangle| &\le L\,\langle |\langle Ax, x\rangle \cdot 1_H - A|\, x, x\rangle^r \\ &\le LD^r(A; x),\end{aligned} \tag{3.19}$$

for any $x \in H$ with $\|x\| = 1$.

The inequality (3.13) may be used to obtain other inequalities for two selfadjoint operators as follows:

COROLLARY 3.4 (Dragomir, 2010, [**30**]). *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with*

$r \in (0,1]$*, then we have the inequality*

$$|\langle f(B)y,y\rangle - \langle f(A)x,x\rangle| \le L\left[(\langle By,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(B;y)\right]^{r/2} \tag{3.20}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1.$

PROOF. If we apply the property (P) to the inequality (3.13) and for the operator B, then we get

$$\langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H| y,y\rangle \le L\left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right]^{r/2} y,y\right\rangle \tag{3.21}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

Utilising the inequality (M) we also have that

$$|f(\langle By,y\rangle) - \langle f(A)x,x\rangle| \le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H| y,y\rangle \tag{3.22}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

Now, by the Hölder-McCarthy inequality (HM) we also have

$$\begin{aligned}\left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right]^{r/2} y,y\right\rangle &\le \left\langle \left[(B - \langle Ax,x\rangle \cdot 1_H)^2 + D^2(A;x)\cdot 1_H\right] y,y\right\rangle^{r/2} \\ &= \left((\langle By,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(B;y)\right)^{r/2}\end{aligned} \tag{3.23}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1.$

On making use of (3.21)-(3.23) we deduce the desired result (3.20). ∎

REMARK 3.2. Since

$$D^2(A;x) \le \frac{1}{4}(M-m)^2, \tag{3.24}$$

then we obtain from (3.20) the following vector inequalities

$$\begin{aligned}|\langle f(A)y,y\rangle - \langle f(A)x,x\rangle| &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(A;y)\right]^{r/2} \\ &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + \frac{1}{2}(M-m)^2\right]^{r/2},\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}|\langle [f(B) - f(A)]x,x\rangle| &\le L\left[\langle (B-A)x,x\rangle^2 + D^2(A;x) + D^2(B;x)\right]^{r/2} \\ &\le L\left[\langle (B-A)x,x\rangle^2 + \frac{1}{2}(M-m)^2\right]^{r/2}.\end{aligned} \tag{3.26}$$

In particular, we have the norm inequality

$$\|f(B) - f(A)\| \le L\left[\|B-A\|^2 + \frac{1}{2}(M-m)^2\right]^{r/2}. \tag{3.27}$$

The following result provides convenient examples for applications:

COROLLARY 3.5 (Dragomir, 2010, [**30**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m, M]$, then we have the inequality:*

$$(3.28)\quad \begin{aligned} &|f(s) - \langle f(A)x, x\rangle| \\ &\leq \begin{cases} \langle |s\cdot 1_H - A| x, x\rangle \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M], \\ \langle |s\cdot 1_H - A| x, x\rangle^{1/q} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \\ &\leq \begin{cases} \left[(s - \langle Ax, x\rangle)^2 + D^2(A; x)\right]^{1/2} \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M], \\ \left[(s - \langle Ax, x\rangle)^2 + D^2(A; x)\right]^{\frac{1}{2q}} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \end{aligned}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$, where $\|f'\|_{[m,M],\ell}$ are the Lebesgue norms, i.e.,

$$\|f'\|_{[m,M],\ell} := \begin{cases} ess\sup_{t\in[m,M]} |f'(t)| & \text{if } \ell = \infty \\ \left(\int_m^M |f'(t)|^p dt\right)^{1/p} & \text{if } \ell = p \geq 1. \end{cases}$$

PROOF. Follows from Theorem 3.2 and on tacking into account that if $f : [m, M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m, M]$, then for any $s, t \in [m, M]$ we have

$$\begin{aligned} &|f(s) - f(t)| \\ &= \left|\int_t^s f'(u)\, du\right| \\ &\leq \begin{cases} |s - t|\, ess\sup_{t\in[m,M]} |f'(t)| & \text{if } f' \in L_\infty[m, M] \\ |s - t|^{1/q} \left(\int_m^M |f'(t)|^p dt\right)^{1/p} & \text{if } f' \in L_p[m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned}$$

∎

REMARK 3.3. It is clear that all the inequalities from Corollaries 3.3, 3.4 and Remark 3.2 may be stated for absolutely continuous functions. However, we mention here only one, namely

$$(3.29)\quad \begin{aligned} &|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \\ &\leq \begin{cases} \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M] \\ \langle |\langle Ax, x\rangle \cdot 1_H - A| x, x\rangle^{1/q} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \\ &\leq \begin{cases} D(A; x) \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m, M] \\ D^{1/q}(A; x) \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned}$$

3.3. The Case of (φ, Φ) −Lipschitzian Functions.

The following result can be stated:

PROPOSITION 3.6 (Dragomir, 2010, [**30**]). *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is (γ, Γ) −Lipschitzian on $[m, M]$,* *then we have the inequality*

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \le \frac{1}{2}(\Gamma-\gamma)\langle|\langle Ax,x\rangle\cdot 1_H - A|x,x\rangle \le \frac{1}{2}(\Gamma-\gamma)D(A;x), \tag{3.30}$$

for any $x\in H$ *with* $\|x\|=1$.

PROOF. Follows by Corollary 3.3 on taking into account that in this case we have $r=1$ and $L=\frac{1}{2}(\Gamma-\gamma)$. ∎

We can use the result (3.30) for the particular case of convex functions to provide an interesting reverse inequality for the Jensen's type operator inequality due to Mond and Pečarić [**43**] (see also [**40**, p. 5]):

THEOREM 3.7 (Mond-Pečarić, 1993, [**43**]). *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A)\subseteq[m,M]$ *for some scalars* m,M *with* $m<M$. *If* f *is a convex function on* $[m,M]$, *then*

$$f(\langle Ax,x\rangle)\le\langle f(A)x,x\rangle \tag{MP}$$

for each $x\in H$ *with* $\|x\|=1$.

COROLLARY 3.8 (Dragomir, 2010, [**30**]). *With the assumptions of Theorem 3.7 we have the inequality*

$$\begin{aligned}(0\le)\,&\langle f(A)x,x\rangle - f(\langle Ax,x\rangle)\\ &\le\frac{1}{2}\left(f'_-(M)-f'_+(m)\right)\langle|\langle Ax,x\rangle\cdot 1_H-A|x,x\rangle\\ &\le\frac{1}{2}\left(f'_-(M)-f'_+(m)\right)D(A;x)\le\frac{1}{4}\left(f'_-(M)-f'_+(m)\right)(M-m)\end{aligned} \tag{3.31}$$

for each $x\in H$ *with* $\|x\|=1$.

PROOF. Follows by Proposition 3.6 on taking into account that

$$f'_+(m)(t-s)\le f(t)-f(s)\le f'_-(M)(t-s)$$

for each s,t with the property that $M>t>s>m$. ∎

The following result may be stated as well:

PROPOSITION 3.9 (Dragomir, 2010, [**30**]). *Let* A *be a selfadjoint operator with* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$. *If* $f:[m,M]\longrightarrow\mathbb{R}$ *is* (γ,Γ) *−Lipschitzian on* $[m,M]$, *then we have the inequality*

$$\begin{aligned}&|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle|\\ &\le\frac{1}{2}(\Gamma-\gamma)\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]\end{aligned} \tag{3.32}$$

for any $x\in H$ *with* $\|x\|=1$.

The following particular case for convex functions holds:

COROLLARY 3.10 (Dragomir, 2010, [**30**]). *With the assumptions of Theorem 3.7 we have the inequality*

$$(0 \leq) \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \leq \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)\left[\frac{1}{2}(M-m) + \left|\langle Ax,x\rangle - \frac{m+M}{2}\right|\right] \tag{3.33}$$

for each $x \in H$ *with* $\|x\| = 1$.

3.4. Related Results. In the previous sections we have compared amongst other the following quantities

$$f\left(\frac{m+M}{2}\right) \text{ and } f(\langle Ax,x\rangle)$$

with $\langle f(A)x,x\rangle$ for a selfadjoint operator A on the Hilbert space H with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $f : [m,M] \longrightarrow \mathbb{R}$ a function of $r-L-$Hölder type with $r \in (0,1]$ and $x \in H$ with $\|x\| = 1$.

Since, obviously,

$$m \leq \frac{1}{M-m}\int_m^M f(t)\,dt \leq M,$$

then is also natural to compare $\frac{1}{M-m}\int_m^M f(t)\,dt$ with $\langle f(A)x,x\rangle$ under the same assumptions for f, A and x.

The following result holds:

THEOREM 3.11 (Dragomir, 2010, [**30**]). *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m,M]$ *for some real numbers* $m < M$. *If* $f : [m,M] \longrightarrow \mathbb{R}$ *is of* $r-L-$*Hölder type with* $r \in (0,1]$, *then we have the inequality:*

$$\begin{aligned} &\left|\frac{1}{M-m}\int_m^M f(s)\,dt - \langle f(A)x,x\rangle\right| \\ &\leq \frac{1}{r+1}L(M-m)^r \\ &\times\left[\left\langle\left(\frac{M\cdot 1_H - A}{M-m}\right)^{r+1}x,x\right\rangle + \left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\right] \\ &\leq \frac{1}{r+1}L(M-m)^r, \end{aligned} \tag{3.34}$$

for any $x \in H$ *with* $\|x\| = 1$.

In particular, if $f : [m,M] \longrightarrow \mathbb{R}$ *is Lipschitzian with a constant* K, *then*

$$\begin{aligned} &\left|\frac{1}{M-m}\int_m^M f(s)\,dt - \langle f(A)x,x\rangle\right| \\ &\leq K(M-m)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left(D^2(A;x) + \left(\langle Ax,x\rangle - \frac{m+M}{2}\right)^2\right)\right] \\ &\leq \frac{1}{2}K(M-m) \end{aligned} \tag{3.35}$$

for any $x \in H$ *with* $\|x\| = 1$.

PROOF. We use the following Ostrowski's type result (see for instance [**31**, p. 3]) written for the function f that is of $r-L-$Hölder type on the interval $[m,M]$:

$$\left|\frac{1}{M-m}\int_m^M f(s)\,dt-f(t)\right| \leq \frac{L}{r+1}(M-m)^r\left[\left(\frac{M-t}{M-m}\right)^{r+1}+\left(\frac{t-m}{M-m}\right)^{r+1}\right] \tag{3.36}$$

for any $t\in[m,M]$.

If we apply the properties (P) and (M) then we have successively

$$\begin{aligned}\left|\frac{1}{M-m}\int_m^M f(s)\,dt-\langle f(A)x,x\rangle\right| &\leq \left\langle\left|\frac{1}{M-m}\int_m^M f(s)\,dt-f(A)\right|x,x\right\rangle \\ &\leq \frac{L}{r+1}(M-m)^r \\ &\times\left[\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{r+1}x,x\right\rangle+\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\right]\end{aligned} \tag{3.37}$$

which proves the first inequality in (3.34).

Utilising the Lah-Ribarić inequality version for selfadjoint operators A with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and convex functions $g:[m,M]\to\mathbb{R}$, namely (see for instance [**40**, p. 57]):

$$\langle g(A)x,x\rangle\leq\frac{M-\langle Ax,x\rangle}{M-m}g(m)+\frac{\langle Ax,x\rangle-m}{M-m}g(M)$$

for any $x\in H$ with $\|x\|=1$, then we get for the convex function $g(t):=\left(\frac{M-t}{M-m}\right)^{r+1}$,

$$\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{r+1}x,x\right\rangle\leq\frac{M-\langle Ax,x\rangle}{M-m}$$

and for the convex function $g(t):=\left(\frac{t-m}{M-m}\right)^{r+1}$,

$$\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}x,x\right\rangle\leq\frac{\langle Ax,x\rangle-m}{M-m}$$

for any $x\in H$ with $\|x\|=1$.

Now, on making use of the last two inequalities, we deduce the second part of (3.34).

Since

$$\begin{aligned}&\frac{1}{2}\left\langle\left(\frac{M\cdot 1_H-A}{M-m}\right)^{2}x,x\right\rangle+\left\langle\left(\frac{A-m\cdot 1_H}{M-m}\right)^{2}x,x\right\rangle \\ =\;&\frac{1}{4}+\frac{1}{(M-m)^2}\left(D^2(A;x)+\left(\langle Ax,x\rangle-\frac{m+M}{2}\right)^2\right)\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, then on choosing $r=1$ in (3.34) we deduce the desired result (3.35). ∎

REMARK 3.4. We should notice from the proof of the above theorem, we also have the following inequalities in the operator order of $B(H)$

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(s)\,dt\right)\cdot 1_H\right| \le \frac{L}{r+1}(M-m)^r\left[\left(\frac{M\cdot 1_H-A}{M-m}\right)^{r+1}+\left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}\right] \le \frac{1}{r+1}L(M-m)^r\cdot 1_H. \tag{3.38}$$

The following particular case is of interest:

COROLLARY 3.12 (Dragomir, 2010, [**30**]). *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\longrightarrow\mathbb{R}$ is $(\gamma,\Gamma)-$Lipschitzian on $[m,M]$, then we have the inequality*

$$\left|\langle f(A)x,x\rangle-\frac{\Gamma+\gamma}{2}-\frac{1}{M-m}\int_m^M f(s)\,dt+\frac{\Gamma+\gamma}{2}\cdot\frac{m+M}{2}\right| \le \frac{1}{2}(\Gamma-\gamma)(M-m) \times\left[\frac{1}{4}+\frac{1}{(M-m)^2}\left(D^2(A;x)+\left(\langle Ax,x\rangle-\frac{m+M}{2}\right)^2\right)\right] \le \frac{1}{4}(\Gamma-\gamma)(M-m). \tag{3.39}$$

PROOF. Follows by (3.35) applied for the $\frac{1}{2}(\Gamma-\gamma)$-Lipshitzian function $f-\frac{\Gamma+\gamma}{2}\cdot e.$ ∎

3.5. Applications for Some Particular Functions. 1. We have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

THEOREM 3.13 (Hölder-McCarthy, 1967, [**42**]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then*
(i) $\langle A^r x,x\rangle\ge\langle Ax,x\rangle^r$ for all $r>1$ and $x\in H$ with $\|x\|=1$;
(ii) $\langle A^r x,x\rangle\le\langle Ax,x\rangle^r$ for all $0<r<1$ and $x\in H$ with $\|x\|=1$;
(iii) If A is invertible, then $\langle A^{-r}x,x\rangle\ge\langle Ax,x\rangle^{-r}$ for all $r>0$ and $x\in H$ with $\|x\|=1$.

We can provide the following reverse inequalities:

PROPOSITION 3.14. *Let A be a selfadjoint positive operator on a Hilbert space H and $0<r<1$. Then*

$$(0\le)\,\langle Ax,x\rangle^r-\langle A^r x,x\rangle\le\langle|\langle Ax,x\rangle\cdot 1_H-A|\,x,x\rangle^r\le D^r(A;x) \tag{3.40}$$

for all $x\in H$ with $\|x\|=1$.

PROOF. Follows from Corollary 3.3 by taking into account that the function $f(t)=t^r$ is of $r-L-$Hölder type with $L=1$ on any compact interval of $(0,\infty)$. ∎

On making use of Corollary 3.8 we can state the following result as well:

PROPOSITION 3.15. *Let A be a selfadjoint positive operator on a Hilbert space H. Assume that $Sp(A)\subseteq[m,M]\subseteq[0,\infty)$.*

(i) We have

$$0 \le \langle A^r x, x\rangle - \langle Ax, x\rangle^r \le \frac{1}{2} r \left(M^{r-1} - m^{r-1}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| \, x, x\rangle \le \frac{1}{2} r \left(M^{r-1} - m^{r-1}\right) D(A; x) \le \frac{1}{4} r \left(M^{r-1} - m^{r-1}\right)(M - m) \tag{3.41}$$

for all $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;

(ii) We also have

$$0 \le \langle Ax, x\rangle^r - \langle A^r x, x\rangle \le \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| \, x, x\rangle \le \frac{1}{2} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right) D(A; x) \le \frac{1}{4} r \left(\frac{M^{1-r} - m^{1-r}}{m^{1-r} M^{1-r}}\right)(M - m) \tag{3.42}$$

for all $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;

(iii) If A *is invertible, then*

$$0 \le \left\langle A^{-r} x, x\right\rangle - \langle Ax, x\rangle^{-r} \le \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right) \langle |\langle Ax, x\rangle \cdot 1_H - A| \, x, x\rangle \le \frac{1}{2} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right) D(A; x) \le \frac{1}{4} r \left(\frac{M^{r+1} - m^{r+1}}{M^{r+1} m^{r+1}}\right)(M - m) \tag{3.43}$$

for all $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

2. Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. On utilizing the inequality (3.31), we can state the following result:

PROPOSITION 3.16. *For any positive definite operator* A *on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subseteq [0, \infty)$ *we have the inequality*

$$(0 \le) \ln(\langle Ax, x\rangle) - \langle \ln(A) x, x\rangle \le \frac{1}{2} \cdot \frac{M - m}{mM} \langle |\langle Ax, x\rangle \cdot 1_H - A| \, x, x\rangle \le \frac{1}{2} \cdot \frac{M - m}{mM} D(A; x) \le \frac{1}{4} \cdot \frac{(M - m)^2}{mM} \tag{3.44}$$

for any $x \in H$ *with* $\|x\| = 1$.

Finally, the following result for logarithms also holds:

PROPOSITION 3.17. *Under the assumptions of Proposition 3.16 we have the inequality*

$$(0 \le) \langle A \ln(A) x, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle) \le \ln\sqrt{\frac{M}{m}} \langle |\langle Ax, x\rangle \cdot 1_H - A| \, x, x\rangle \le \ln\sqrt{\frac{M}{m}} \cdot D(A; x) \le \frac{1}{2}(M - m) \ln\sqrt{\frac{M}{m}} \tag{3.45}$$

for any $x \in H$ *with* $\|x\| = 1$.

REMARK 3.5. On utilizing the results from the previous sections for other convex functions of interest such as $f(x) = \ln[(1-x)/x]$, $x \in (0, 1/2)$ or $f(x) = \ln(1 + \exp x)$, $x \in (-\infty, \infty)$ we can get other interesting operator inequalities. However, the details are left to the interested reader.

4. OTHER OSTROWSKI INEQUALITIES FOR CONTINUOUS FUNCTIONS

4.1. Inequalities for Absolutely Continuous Functions of Selfadjoint Operators. We start with the following scalar inequality that is of interest in itself since it provides a generalization of the Ostrowski inequality when upper and lower bounds for the derivative are provided:

LEMMA 4.1 (Dragomir, 2010, [**27**]). *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function whose derivative is bounded above and below on $[a, b]$, i.e., there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property that $\gamma \le f'(s) \le \Gamma$ for almost every $s \in [a, b]$. Then we have the double inequality*

$$-\frac{1}{2} \cdot \frac{\Gamma - \gamma}{b-a}\left[\left(s - \frac{b\Gamma - a\gamma}{\Gamma - \gamma}\right)^2 - \Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right] \le f(s) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2} \cdot \frac{\Gamma - \gamma}{b-a}\left[\left(s - \frac{a\Gamma - b\gamma}{\Gamma - \gamma}\right)^2 - \Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right] \tag{4.1}$$

for any $s \in [a, b]$. The inequalities are sharp.

PROOF. We start with *Montgomery's identity*

$$f(s) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^s (t-a) f'(t)\,dt + \frac{1}{b-a}\int_s^b (t-b) f'(t)\,dt \tag{4.2}$$

that holds for any $s \in [a, b]$.

Since $\gamma \le f'(t) \le \Gamma$ for almost every $t \in [a, b]$, then

$$\frac{\gamma}{b-a}\int_a^s (t-a)\,dt \le \frac{1}{b-a}\int_a^s (t-a) f'(t)\,dt \le \frac{\Gamma}{b-a}\int_a^s (t-a)\,dt$$

and

$$\frac{\Gamma}{b-a}\int_s^b (b-t)\,dt \le \frac{1}{b-a}\int_s^b (b-t) f'(t)\,dt \le \frac{\Gamma}{b-a}\int_s^b (b-t)\,dt$$

for any $s \in [a, b]$.

Now, due to the fact that

$$\int_a^s (t-a)\,dt = \frac{1}{2}(s-a)^2 \text{ and } \int_s^b (b-t)\,dt = \frac{1}{2}(b-s)^2$$

then by (4.2) we deduce the following inequality that is of interest in itself:

$$
\begin{aligned}
(4.3)\qquad & -\frac{1}{2(b-a)}\left[\Gamma(b-s)^2-\gamma(s-a)^2\right] \\
&\leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{2(b-a)}\left[\Gamma(s-a)^2-\gamma(b-s)^2\right]
\end{aligned}
$$

for any $s\in[a,b]$.

Further on, if we denote by

$$A:=\gamma(s-a)^2-\Gamma(b-s)^2 \text{ and } B:=\Gamma(s-a)^2-\gamma(b-s)^2$$

then, after some elementary calculations, we derive that

$$A=-(\Gamma-\gamma)\left(s-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2+\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^2$$

and

$$B=(\Gamma-\gamma)\left(s-\frac{a\Gamma-b\gamma}{\Gamma-\gamma}\right)^2-\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^2$$

which, together with (4.3), produces the desired result (4.1).

The sharpness of the inequalities follow from the sharpness of some particular cases outlined below. The details are omitted. ∎

COROLLARY 4.2. *With the assumptions of Lemma 4.1 we have the inequalities*

$$\frac{1}{2}\gamma(b-a)\leq\frac{1}{b-a}\int_a^b f(t)\,dt-f(a)\leq\frac{1}{2}\Gamma(b-a) \tag{4.4}$$

and

$$\frac{1}{2}\gamma(b-a)\leq f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{2}\Gamma(b-a) \tag{4.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}(\Gamma-\gamma)(b-a) \tag{4.6}$$

respectively. The constant $\frac{1}{8}$ is best possible in (4.6).

The proof is obvious from (4.1) on choosing $s=a, s=b$ and $s=\frac{a+b}{2}$, respectively.

COROLLARY 4.3 (Dragomir, 2010, [**27**]). *With the assumptions of Lemma 4.1 and if, in addition $\gamma=-\alpha$ and $\Gamma=\beta$ with $\alpha,\beta>0$ then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{b\beta+a\alpha}{\beta+\alpha}\right)\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right) \tag{4.7}$$

and

$$f\left(\frac{a\beta+b\alpha}{\beta+\alpha}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right). \tag{4.8}$$

The proof follows from (4.1) on choosing $s=\frac{b\beta+a\alpha}{\beta+\alpha}\in[a,b]$ and $s=\frac{a\beta+b\alpha}{\beta+\alpha}\in[a,b]$, respectively.

REMARK 4.1. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous and

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

then by choosing $\gamma = -\|f'\|_\infty$ and $\Gamma = \|f'\|_\infty$ in (4.1) we deduce the classical Ostrowski's inequality for absolutely continuous functions. The constant $\frac{1}{4}$ in Ostrowski's inequality is best possible.

We are able now to state the following result providing upper and lower bounds for absolutely convex functions of selfadjoint operators in Hilbert spaces whose derivatives are bounded below and above:

THEOREM 4.4 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property that $\gamma \le f'(s) \le \Gamma$ for almost every $s \in [m, M]$, then we have the following double inequality in the operator order of $B(H)$:*

$$\begin{aligned} &-\frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m} \\ &\times\left[\left(A-\frac{M\Gamma-m\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right] \\ &\le f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H \\ &\le \frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m} \\ &\times\left[\left(A-\frac{m\Gamma-M\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2-\Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right]. \end{aligned} \tag{4.9}$$

The proof follows by the property (P) applied for the inequality (4.1) in Lemma 4.1.

THEOREM 4.5 (Dragomir, 2010, [**27**]). *With the assumptions in Theorem 4.4 we have in the operator order the following inequalities*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\right| \le \begin{cases} \left[\frac{1}{4}1_H+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\|f'\|_\infty, \text{ if } f'\in L_\infty[m,M]; \\ \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A-m1_H}{M-m}\right)^{p+1}+\left(\frac{M1_H-A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\|f'\|_q \\ \qquad \text{if } f'\in L_p[m,M],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \\ \left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\|f'\|_1. \end{cases} \tag{4.10}$$

The proof is obvious by the scalar inequalities from Theorem 2.2 and the property (P).

The third inequality in (4.10) can be naturally generalized for functions of bounded variation as follows:

THEOREM 4.6 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left| f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \leq \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M-m} \right| \right] \bigvee_m^M (f) \tag{4.11}$$

where $\bigvee_m^M (f)$ denotes the total variation of f on $[m, M]$. The constant $\frac{1}{2}$ is best possible in (4.11).

PROOF. Follows from the scalar inequality obtained by the author in [**23**], namely

$$\left| f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{4.12}$$

for any $s \in [a, b]$, where f is a function of bounded variation on $[a, b]$. The constant $\frac{1}{2}$ is best possible in (4.12). ∎

4.2. Inequalities for Convex Functions of Selfadjoint Operators. The case of convex functions is important for applications.

We need the following lemma.

LEMMA 4.7 (Dragomir, 2010, [**27**]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function such that the derivative f' is continuous on (a, b) and with the lateral derivative finite and $f'_-(b) \neq f'_+(a)$. Then we have the following double inequality*

$$\begin{aligned} & -\frac{1}{2} \cdot \frac{f'_-(b) - f'_+(a)}{b-a} \\ & \times \left[\left(s - \frac{bf'_-(b) - af'_+(a)}{f'_-(b) - f'_+(a)} \right)^2 - f'_-(b) f'_+(a) \left(\frac{b-a}{f'_-(b) - f'_+(a)} \right)^2 \right] \\ & \leq f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \leq f'(s) \left(s - \frac{a+b}{2} \right) \end{aligned} \tag{4.13}$$

for any $s \in [a, b]$.

PROOF. Since f is convex, then by the fact that f' is monotonic nondecreasing, we have

$$\frac{f'_+(a)}{b-a} \int_a^s (t-a)\, dt \leq \frac{1}{b-a} \int_a^s (t-a) f'(t)\, dt \leq \frac{f'(s)}{b-a} \int_a^s (t-a)\, dt$$

and

$$\frac{f'(s)}{b-a} \int_s^b (b-t)\, dt \leq \frac{1}{b-a} \int_s^b (b-t) f'(t)\, dt \leq \frac{f'_-(b)}{b-a} \int_s^b (b-t)\, dt$$

for any $s \in [a, b]$, where $f'_+(a)$ and $f'_-(b)$ are the lateral derivatives in a and b respectively.

Utilising the Montgomery identity (4.2) we then have

$$\begin{aligned}&\frac{f'_{+}(a)}{b-a}\int_{a}^{s}(t-a)\,dt-\frac{f'_{-}(b)}{b-a}\int_{s}^{b}(b-t)\,dt\\&\leq f(s)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\\&\leq\frac{f'(s)}{b-a}\int_{a}^{s}(t-a)\,dt-\frac{f'(s)}{b-a}\int_{s}^{b}(b-t)\,dt\end{aligned}$$

which is equivalent with the following inequality that is of interest in itself

$$\begin{aligned}&\frac{1}{2(b-a)}\left[f'_{+}(a)(s-a)^{2}-f'_{-}(b)(b-s)^{2}\right]\\&\leq f(s)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq f'(s)\left(s-\frac{a+b}{2}\right)\end{aligned}\tag{4.14}$$

for any $s\in[a,b]$.

A simple calculation reveals now that the left side of (4.14) coincides with the same side of the desired inequality (4.13). ∎

We are able now to sate our result for convex functions of selfadjoint operators:

THEOREM 4.8 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\to\mathbb{R}$ is a differentiable convex function such that the derivative f' is continuous on (m,M) and with the lateral derivative finite and $f'_{-}(M)\neq f'_{+}(m)$, then we have the double inequality in the operator order of $B(H)$*

$$\begin{aligned}&-\frac{1}{2}\cdot\frac{f'_{-}(M)-f'_{+}(m)}{M-m}\\&\times\left[\left(A-\frac{Mf'_{-}(M)-mf'_{+}(m)}{f'_{-}(M)-f'_{+}(m)}\cdot 1_{H}\right)^{2}\right.\\&\left.-f'_{-}(M)f'_{+}(m)\left(\frac{M-m}{f'_{-}(M)-f'_{+}(m)}\right)^{2}\cdot 1_{H}\right]\\&\leq f(A)-\left(\frac{1}{M-m}\int_{m}^{M}f(t)\,dt\right)\cdot 1_{H}\leq\left(A-\frac{m+M}{2}\cdot 1_{H}\right)f'(A).\end{aligned}\tag{4.15}$$

The proof follows from the scalar case in Lemma 4.7.

REMARK 4.2. We observe that one can drop the assumption of differentiability of the convex function and will still have the first inequality in (4.15). This follows from the fact that the class of differentiable convex functions is dense in the class of all convex functions defined on a given interval.

A different lower bound for the quantity

$$f(A)-\left(\frac{1}{M-m}\int_{m}^{M}f(t)\,dt\right)\cdot 1_{H}$$

expressed only in terms of the operator A and not its second power as above, also holds:

THEOREM 4.9 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a convex function on $[m, M]$, then we have the following inequality in the operator order of $B(H)$*

$$f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H \geq \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H - \frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m}. \tag{4.16}$$

PROOF. It suffices to prove for the case of differentiable convex functions defined on (m, M).

So, by the gradient inequality we have that

$$f(t) - f(s) \geq (t-s) f'(s)$$

for any $t, s \in (m, M)$.

Now, if we integrate this inequality over $s \in [m, M]$ we get

$$(M-m) f(t) - \int_m^M f(s)\, ds \geq \int_m^M (t-s) f'(s)\, ds = \int_m^M f(s)\, ds - (M-t) f(M) - (t-m) f(m) \tag{4.17}$$

for each $s \in [m, M]$.

Finally, if we apply to the inequality (4.17) the property (P), we deduce the desired result (4.16). ∎

COROLLARY 4.10 (Dragomir, 2010, [**27**]). *With the assumptions of Theorem 4.9 we have the following double inequality in the operator order*

$$\frac{f(m)+f(M)}{2} \cdot 1_H \geq \frac{1}{2}\left[f(A) + \frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m} \right] \geq \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H. \tag{4.18}$$

PROOF. The second inequality is equivalent with (4.16).

For the first inequality, we observe, by the convexity of f we have that

$$\frac{f(M)(t-m) + f(m)(M-t)}{M-m} \geq f(t)$$

for any $t \in [m, M]$, which produces the operator inequality

$$\frac{f(M)(A - m \cdot 1_H) + f(m)(M \cdot 1_H - A)}{M-m} \geq f(A). \tag{4.19}$$

Now, if in both sides of (4.19) we add the same quantity

$$\frac{f(M)(M \cdot 1_H - A) + f(m)(A - m \cdot 1_H)}{M-m}$$

and perform the calculations, then we obtain the first part of (4.18) and the proof is complete. ∎

4.3. Some Vector Inequalities. The following result holds:

THEOREM 4.11 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function on $[m, M]$, then we have the inequalities*

$$
\begin{aligned}
(4.20)\qquad & |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\
&\leq \bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\times \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \\
&\leq \|x\|\|y\| \\
&\times \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \end{cases}
\end{aligned}
$$

for any $x, y \in H$ and $s \in [m, M]$.

PROOF. Since f is absolutely continuous, then we have

$$
\begin{aligned}
(4.21)\qquad & |f(s) - f(t)| \\
&= \left|\int_s^t f'(u)\,du\right| \leq \left|\int_s^t |f'(u)|\,du\right| \\
&\leq \begin{cases} |t-s|\,\|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ |t-s|^{1/q}\|f'\|_p & \text{if } f' \in L_p[m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}
\end{aligned}
$$

for any $s, t \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous functions and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$
\left|\int_a^b p(t)\,dv(t)\right| \leq \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),
$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Now, by the above property of the Riemann-Stieltjes integral we have from the representation (4.27) that

$$
\begin{aligned}
(4.22)\qquad & \left| f(s)\langle x,y\rangle - \langle f(A)x,y\rangle \right| \\
&= \left| \int_{m-0}^{M} [f(s)-f(t)]\, d\left(\langle E_t x,y\rangle\right) \right| \\
&\le \max_{t\in[m,M]} |f(s)-f(t)| \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\le \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad \times \begin{cases} \max_{t\in[m,M]} |t-s| \, \|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \max_{t\in[m,M]} |t-s|^{1/q} \, \|f'\|_p & \text{if } f'\in L_p[m,M],\, p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \end{cases} \\
&:= F
\end{aligned}
$$

where $\bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the total variation of $\left\langle E_{(\cdot)}x,y\right\rangle$ and $x,y\in H$.

Since, obviously, we have $\max_{t\in[m,M]} |t-s| = \frac{1}{2}(M-m) + \left|s-\frac{m+M}{2}\right|$, then

$$
\begin{aligned}
(4.23)\qquad F &= \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\quad \times \begin{cases} \left[\frac{1}{2}(M-m) + \left|s-\frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \left[\frac{1}{2}(M-m) + \left|s-\frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f'\in L_p[m,M],\, p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \end{cases}
\end{aligned}
$$

for any $x,y\in H$.

The last part follows by the Total Variation Schwarz's inequality and the details are omitted. ∎

COROLLARY 4.12 (Dragomir, 2010, [**27**]). *With the assumptions of Theorem 4.11 we have the following inequalities*

$$
\begin{aligned}
(4.24)\qquad & \left| f\left(\frac{\langle Ax,x\rangle}{\|x\|^2}\right)\langle x,y\rangle - \langle f(A)x,y\rangle \right| \\
&\le \|x\|\,\|y\| \\
&\quad \times \begin{cases} \left[\frac{1}{2}(M-m) + \left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right] \|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \left[\frac{1}{2}(M-m) + \left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right]^{1/q} \|f'\|_p & \text{if } f'\in L_p[m,M],\, p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \end{cases}
\end{aligned}
$$

and

$$\left|f\left(\frac{m+M}{2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \leq \|x\|\|y\| \times \begin{cases} \frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{4.25}$$

for any $x,y\in H$.

REMARK 4.3. In particular, we obtain from (4.8) the following inequalities

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \leq \begin{cases} \left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{4.26}$$

and

$$\left|f\left(\frac{m+M}{2}\right)-\langle f(A)x,x\rangle\right| \leq \begin{cases} \frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M] \\ \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{4.27}$$

for any $x\in H$ with $\|x\|=1$.

THEOREM 4.13 (Dragomir, 2010, [**27**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is $r-H$-Hölder continuous on $[m,M]$, then we have the inequality*

$$\begin{aligned} &|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\ &\leq H\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r \\ &\leq H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r \end{aligned} \tag{4.28}$$

for any $x,y\in H$ and $s\in[m,M]$.

In particular, we have the inequalities

$$\begin{aligned} &\left|f\left(\frac{\langle Ax,x\rangle}{\|x\|^2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \\ &\leq H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right]^r \end{aligned} \tag{4.29}$$

and

$$\left|f\left(\frac{m+M}{2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right|\leq\frac{1}{2^r}H\|x\|\|y\|(M-m)^r \tag{4.30}$$

for any $x, y \in H$.

PROOF. Utilising the inequality (4.22) and the fact that f is $r - H$-Hölder continuous we have successively

$$
\begin{aligned}
(4.31) \qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
&= \left|\int_{m-0}^{M}[f(s)-f(t)]\,d(\langle E_t x,y\rangle)\right| \\
&\le \max_{t\in[m,M]}|f(s)-f(t)|\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&\le H\max_{t\in[m,M]}|s-t|^r\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&= H\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)
\end{aligned}
$$

for any $x, y \in H$ and $s \in [m, M]$.

The argument follows now as in the proof of Theorem 4.11 and the details are omitted. ∎

4.4. Logarithmic Inequalities. Consider *the identric mean*

$$
I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \qquad a, b > 0;
$$

and observe that

$$
\frac{1}{b-a}\int_a^b \ln t\,dt = \ln[I(a,b)].
$$

If we apply Theorem 4.8 for the convex function $f(t) = -\ln t, t > 0$, then we can state:

PROPOSITION 4.14. *Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some positive numbers $0 < m < M$. Then we have the double inequality in the operator order of $B(H)$*

$$
(4.32) \qquad -\frac{1}{2mM}\left(A^2 - mM\right) \le \ln I(m,M)\cdot 1_H - \ln A \le \frac{m+M}{2}\cdot A^{-1} - 1_H.
$$

If we denote by $G(a,b) := \sqrt{ab}$ the geometric mean of the positive numbers a, b, then we can state the following result as well:

PROPOSITION 4.15. *With the assumptions of Proposition 4.14, we have the inequalities in the operator order of $B(H)$*

$$
\begin{aligned}
(4.33) \qquad & \ln G(m,M)\cdot 1_H \\
&\le \frac{1}{2}\left[\ln A + \frac{\ln M\cdot(M\cdot 1_H - A) + \ln m\cdot(A - m\cdot 1_H)}{M-m}\right] \\
&\le \ln I(m,M)\cdot 1_H.
\end{aligned}
$$

The inequality follows by Corollary 4.10 applied for the convex function $f(t) = -\ln t, t > 0$.

Finally, the following vector inequality may be stated

PROPOSITION 4.16. *With the assumptions of Proposition 4.14, for any* $x, y \in H$ *we have the inequalities*

$$|\langle x, y\rangle \ln s - \langle \ln Ax, y\rangle| \le \|x\| \|y\| \begin{cases} \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right] \frac{1}{m}, \\ \left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^{1/q} \frac{M^{p-1}-m^{p-1}}{(p-1)M^{p-1}m^{p-1}}, \end{cases} \tag{4.34}$$

for any $s \in [m, M]$, *where* $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

5. MORE OSTROWSKI'S TYPE INEQUALITIES

5.1. Some Vector Inequalities for Functions of Bounded Variation. The following result holds:

THEOREM 5.1 (Dragomir, 2010, [**16**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f : [m, M] \to \mathbb{R}$ *is a continuous function of bounded variation on* $[m, M]$, *then we have the inequality*

$$\begin{aligned} &|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\ &\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &+ \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ &\le \|x\| \|y\| \left(\frac{1}{2}\bigvee_m^M (f) + \frac{1}{2}\left|\bigvee_m^s (f) - \bigvee_s^M (f)\right|\right) \left(\le \|x\| \|y\| \bigvee_m^M (f)\right) \end{aligned} \tag{5.1}$$

for any $x, y \in H$ *and for any* $s \in [m, M]$.

PROOF. We use the following identity for the Riemann-Stieltjes integral established by the author in 2000 in [**10**] (see also [**31**, p. 452]):

$$\begin{aligned} &[u(b) - u(a)] f(s) - \int_a^b f(t)\, du(t) \\ &= \int_a^s [u(t) - u(a)]\, df(t) + \int_s^b [u(t) - u(b)]\, df(t), \end{aligned} \tag{5.2}$$

for any $s \in [a, b]$, provided the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists.

A simple proof can be done by utilizing the integration by parts formula and starting from the right hand side of (5.2).

If we choose in (5.2) $a = m, b = M$ and $u(t) = \langle E_t x, y\rangle$, then we have the following identity of interest in itself

$$\begin{aligned} &f(s)\langle x, y\rangle - \langle f(A)x, y\rangle \\ &= \int_{m-0}^s \langle E_t x, y\rangle\, df(t) + \int_s^M \langle (E_t - 1_H)x, y\rangle\, df(t) \end{aligned} \tag{5.3}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v)$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising this property we have from (5.3) that

$$\begin{aligned}(5.4)\qquad &|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle|\\ &\leq\left|\int_{m-0}^{s}\langle E_t x,y\rangle\,df(t)\right|+\left|\int_{s}^{M}\langle(E_t-1_H)x,y\rangle\,df(t)\right|\\ &\leq\max_{t\in[m,s]}|\langle E_t x,y\rangle|\bigvee_m^s(f)+\max_{t\in[s,M]}|\langle(E_t-1_H)x,y\rangle|\bigvee_s^M(f):=T\end{aligned}$$

for any $x,y\in H$ and for any $s\in[m,M]$.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$(5.5)\qquad |\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for any $x,y\in H$.

On applying the inequality (5.5) we have

$$|\langle E_t x,y\rangle|\leq\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$

and

$$|\langle(1_H-E_t)x,y\rangle|\leq\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}$$

for any $x,y\in H$ and $t\in[m,M]$.

Therefore

$$\begin{aligned}(5.6)\qquad T&\leq\max_{t\in[m,s]}\left[\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}\right]\bigvee_m^s(f)\\ &+\max_{t\in[s,M]}\left[\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\right]\bigvee_s^M(f)\\ &\leq\max_{t\in[m,s]}\langle E_t x,x\rangle^{1/2}\max_{t\in[m,s]}\langle E_t y,y\rangle^{1/2}\bigvee_m^s(f)\\ &+\max_{t\in[s,M]}\langle(1_H-E_t)x,x\rangle^{1/2}\max_{t\in[s,M]}\langle(1_H-E_t)y,y\rangle^{1/2}\bigvee_s^M(f)\\ &=\langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}\bigvee_m^s(f)\\ &+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\bigvee_s^M(f)\\ &:=V\end{aligned}$$

for any $x, y \in H$ and for any $s \in [m, M]$, proving the first inequality in (5.1).

Now, observe that

$$V \leq \max\left\{\bigvee_{m}^{s}(f), \bigvee_{s}^{M}(f)\right\}$$
$$\times\left[\langle E_s x, x\rangle^{1/2}\langle E_s y, y\rangle^{1/2} + \langle(1_H - E_s)x, x\rangle^{1/2}\langle(1_H - E_s)y, y\rangle^{1/2}\right].$$

Since

$$\max\left\{\bigvee_{m}^{s}(f), \bigvee_{s}^{M}(f)\right\} = \frac{1}{2}\bigvee_{m}^{M}(f) + \frac{1}{2}\left|\bigvee_{m}^{s}(f) - \bigvee_{s}^{M}(f)\right|$$

and by the Cauchy-Buniakovski-Schwarz inequality for positive real numbers a_1, b_1, a_2, b_2

$$a_1 b_1 + a_2 b_2 \leq \left(a_1^2 + a_2^2\right)^{1/2}\left(b_1^2 + b_2^2\right)^{1/2} \tag{5.7}$$

we have

$$\langle E_s x, x\rangle^{1/2}\langle E_s y, y\rangle^{1/2} + \langle(1_H - E_s)x, x\rangle^{1/2}\langle(1_H - E_s)y, y\rangle^{1/2}$$
$$\leq [\langle E_s x, x\rangle + \langle(1_H - E_s)x, x\rangle]^{1/2}[\langle E_s y, y\rangle + \langle(1_H - E_s)y, y\rangle]^{1/2}$$
$$= \|x\|\,\|y\|$$

for any $x, y \in H$ and $s \in [m, M]$, then the last part of (5.1) is proven as well. ∎

REMARK 5.1. For the continuous function with bounded variation $f : [m, M] \to \mathbb{R}$ if $p \in [m, M]$ is a point with the property that

$$\bigvee_{m}^{p}(f) = \bigvee_{p}^{M}(f)$$

then from (5.1) we get the interesting inequality

$$|f(p)\langle x, y\rangle - \langle f(A)x, y\rangle| \leq \frac{1}{2}\|x\|\,\|y\|\bigvee_{m}^{M}(f) \tag{5.8}$$

for any $x, y \in H$.

If the continuous function $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing and therefore of bounded variation, we get from (5.1) the following inequality as well

$$\begin{aligned} &|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\ &\leq \langle E_s x, x\rangle^{1/2}\langle E_s y, y\rangle^{1/2}(f(s) - f(m)) \\ &+ \langle(1_H - E_s)x, x\rangle^{1/2}\langle(1_H - E_s)y, y\rangle^{1/2}(f(M) - f(s)) \\ &\leq \|x\|\,\|y\|\left(\frac{1}{2}(f(M) - f(m)) + \left|f(s) - \frac{f(m) + f(M)}{2}\right|\right) \\ &(\leq \|x\|\,\|y\|\,f(M) - f(m)) \end{aligned} \tag{5.9}$$

for any $x, y \in H$ and $s \in [m, M]$.

Moreover, if the continuous function $f : [m, M] \to \mathbb{R}$ is nondecreasing on $[m, M]$, then the equation

$$f(s) = \frac{f(m) + f(M)}{2}$$

has got at least a solution in $[m, M]$. In his case we get from (5.9) the following trapezoidal type inequality

$$\left|\frac{f(m)+f(M)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \leq \frac{1}{2}\|x\|\|y\|(f(M)-f(m)) \tag{5.10}$$

for any $x, y \in H$.

5.2. Some Vector Inequalities for Lipshitzian Functions. The following result that incorporates the case of Lipschitzian functions also holds

THEOREM 5.2 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.,*

$$|f(s)-f(t)| \leq L|s-t| \text{ for any } s,t\in[m,M],$$

then we have the inequality

$$\begin{aligned} &|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \\ &\leq L\left[\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt\right)^{1/2}\right. \\ &\left.+\left(\int_{s}^{M}\langle(1_H-E_t)x,x\rangle\,dt\right)^{1/2}\left(\int_{s}^{M}\langle(1_H-E_t)y,y\rangle\,dt\right)^{1/2}\right] \\ &\leq L\langle|A-s1_H|x,x\rangle^{1/2}\langle|A-s1_H|y,y\rangle^{1/2} \\ &\leq L\left[D^2(A;x)+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/4} \\ &\times\left[D^2(A;y)+\left(s\|y\|^2-\langle Ay,y\rangle\right)^2\right]^{1/4} \end{aligned} \tag{5.11}$$

for any $x, y \in H$ and $s \in [m, M]$, where $D(A; x)$ is the variance of the selfadjoint operator A in x and is defined by

$$D(A;x) := \left(\|Ax\|^2\|x\|^2-\langle Ax,x\rangle^2\right)^{1/2}.$$

PROOF. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s)-f(t)| \leq L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (5.3) that

$$\begin{aligned}
&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle|\\
&\le\left|\int_{m-0}^{s}\langle E_t x,y\rangle\,df(t)\right|+\left|\int_{s}^{M}\langle (E_t-1_H)x,y\rangle\,df(t)\right|\\
&\le L\left[\int_{m-0}^{s}|\langle E_t x,y\rangle|\,dt+\int_{s}^{M}|\langle (E_t-1_H)x,y\rangle|\,dt\right]:=LW
\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$.

By utilizing the generalized Schwarz inequality for nonnegative operators (5.5) and the Cauchy-Buniakovski-Schwarz inequality for the Riemann integral we have

$$\begin{aligned}
W&\le\int_{m-0}^{s}\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}\,dt\\
&\quad+\int_{s}^{M}\langle (1_H-E_t)x,x\rangle^{1/2}\langle (1_H-E_t)y,y\rangle^{1/2}\,dt\\
&\le\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt\right)^{1/2}\\
&\quad+\left(\int_{s}^{M}\langle (1_H-E_t)x,x\rangle\,dt\right)^{1/2}\left(\int_{s}^{M}\langle (1_H-E_t)y,y\rangle\,dt\right)^{1/2}\\
&:=Z
\end{aligned}\tag{5.12}$$

for any $x,y\in H$ and $s\in[m,M]$.

On the other hand, by making use of the elementary inequality (5.7) we also have

$$\begin{aligned}
Z&\le\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)x,x\rangle\,dt\right)^{1/2}\\
&\quad\times\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)y,y\rangle\,dt\right)^{1/2}
\end{aligned}\tag{5.13}$$

for any $x,y\in H$ and $s\in[m,M]$.

Now, observe that, by the use of the representation (5.3) for the continuous function $f:[m,M]\to\mathbb{R}$, $f(t)=|t-s|$ where s is fixed in $[m,M]$ we have the following identity that is of interest in itself

$$\langle|A-s\cdot 1_H|x,y\rangle=\int_{m-0}^{s}\langle E_t x,y\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)x,y\rangle\,dt\tag{5.14}$$

for any $x,y\in H$.

On utilizing (5.14) for x and then for y we deduce the second part of (5.11).

Finally, by the well known inequality for the modulus of a bounded linear operator

$$\langle|T|x,x\rangle\le\|Tx\|\,\|x\|\,,x\in H$$

we have

$$\begin{aligned}\langle |A - s\cdot 1_H| \, x, x\rangle^{1/2} &\le \|Ax - sx\|^{1/2} \|x\|^{1/2} \\ &= \left(\|Ax\|^2 - 2s\langle Ax, x\rangle + s^2 \|x\|^2\right)^{1/4} \|x\|^{1/2} \\ &= \left[\|Ax\|^2 \|x\|^2 - \langle Ax, x\rangle^2 + \left(s\|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4} \\ &= \left[D^2(A; x) + \left(s\|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4}\end{aligned}$$

and a similar relation for y. The proof is thus complete. ∎

REMARK 5.2. Since A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, then

$$\left|A - \frac{m+M}{2}\cdot 1_H\right| \le \frac{M-m}{2} 1_H$$

giving from (5.11) that

$$\begin{aligned}&\left|f\left(\frac{m+M}{2}\right)\langle x, y\rangle - \langle f(A)x, y\rangle\right| \\ &\le L\left[\left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t x, x\rangle\, dt\right)^{1/2}\left(\int_{m-0}^{\frac{m+M}{2}} \langle E_t y, y\rangle\, dt\right)^{1/2}\right. \\ &\left.+\left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t)x, x\rangle\, dt\right)^{1/2}\left(\int_{\frac{m+M}{2}}^{M} \langle (1_H - E_t)y, y\rangle\, dt\right)^{1/2}\right] \\ &\le L\left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| x, x\right\rangle^{1/2}\left\langle \left|A - \frac{m+M}{2}\cdot 1_H\right| y, y\right\rangle^{1/2} \\ &\le \frac{1}{2} L(M-m)\|x\|\|y\|\end{aligned} \tag{5.15}$$

for any $x, y \in H$.

The particular case of equal vectors is of interest:

COROLLARY 5.3 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\begin{aligned}&\left|f(s)\|x\|^2 - \langle f(A)x, x\rangle\right| \\ &\le L\langle |A - s\cdot 1_H| \, x, x\rangle \\ &\le L\left[D^2(A; x) + \left(s\|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/2}\end{aligned} \tag{5.16}$$

for any $x \in H$ and $s \in [m, M]$.

REMARK 5.3. An important particular case that can be obtained from (5.16) is the one when $s = \frac{\langle Ax,x\rangle}{\|x\|^2}, x \neq 0$, giving the inequality

$$
\begin{aligned}
&\left| f\left(\frac{\langle Ax, x\rangle}{\|x\|^2}\right)\|x\|^2 - \langle f(A)x, x\rangle \right| \\
&\leq L\left\langle \left| A - \frac{\langle Ax, x\rangle}{\|x\|^2}\cdot 1_H \right| x, x \right\rangle \\
&\leq LD(A;x) \leq \frac{1}{2}L(M-m)\|x\|^2
\end{aligned}
\tag{5.17}
$$

for any $x \in H, x \neq 0.$

We are able now to provide the following corollary:

COROLLARY 5.4 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a (φ, Φ) –Lipschitzian functions on $[m, M]$ with $\Phi > \varphi$, then we have the inequality*

$$
\begin{aligned}
&\left| \langle f(A)x, y\rangle - \frac{\Phi+\varphi}{2}\langle Ax, y\rangle + \frac{\Phi+\varphi}{2}s\langle x, y\rangle - f(s)\langle x, y\rangle \right| \\
&\leq \frac{1}{2}(\Phi-\varphi)\left[\left(\int_{m-0}^{s}\langle E_t x, x\rangle dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y, y\rangle dt\right)^{1/2}\right. \\
&\left. + \left(\int_{s}^{M}\langle (1_H - E_t)x, x\rangle dt\right)^{1/2}\left(\int_{s}^{M}\langle (1_H - E_t)y, y\rangle dt\right)^{1/2}\right] \\
&\leq \frac{1}{2}(\Phi-\varphi)\langle |A - s1_H|x, x\rangle^{1/2}\langle |A - s1_H|y, y\rangle^{1/2} \\
&\leq \frac{1}{2}(\Phi-\varphi)\left[D^2(A;x) + \left(s\|x\|^2 - \langle Ax, x\rangle\right)^2\right]^{1/4} \\
&\times \left[D^2(A;y) + \left(s\|y\|^2 - \langle Ay, y\rangle\right)^2\right]^{1/4}
\end{aligned}
\tag{5.18}
$$

for any $x, y \in H$.

REMARK 5.4. Various particular cases can be stated by utilizing the inequality (5.18), however the details are left to the interested reader.

6. SOME VECTOR INEQUALITIES FOR MONOTONIC FUNCTIONS

The case of monotonic functions is of interest as well. The corresponding result is incorporated in the following

THEOREM 6.1 (Dragomir, 2010, [**16**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on*

$[m,M]$, then we have the inequality

$$
\begin{aligned}
(6.1)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
& \leq \left(\int_{m-0}^{s} \langle E_t x,x\rangle \, df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y,y\rangle \, df(t)\right)^{1/2} \\
& + \left(\int_{s}^{M} \langle (1_H - E_t)x,x\rangle \, df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t)y,y\rangle \, df(t)\right)^{1/2} \\
& \leq \langle |f(A) - f(s)1_H| x,x\rangle^{1/2} \langle |f(A) - f(s)1_H| y,y\rangle^{1/2} \\
& \leq \left[D^2(f(A);x) + \left(f(s)\|x\|^2 - \langle f(A)x,x\rangle\right)^2\right]^{1/4} \\
& \times \left[D^2(f(A);y) + \left(f(s)\|y\|^2 - \langle f(A)y,y\rangle\right)^2\right]^{1/4}
\end{aligned}
$$

for any $x,y \in H$ and $s \in [m,M]$, where, as above $D(f(A);x)$ is the variance of the selfadjoint operator $f(A)$ in x.

PROOF. From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \int_a^b |p(t)|\,dv(t).$$

On utilizing this property and the representation (5.3) we have successively

$$
\begin{aligned}
(6.2)\qquad & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \\
& \leq \left|\int_{m-0}^{s} \langle E_t x,y\rangle \, df(t)\right| + \left|\int_{s}^{M} \langle (E_t - 1_H)x,y\rangle \, df(t)\right| \\
& \leq \int_{m-0}^{s} |\langle E_t x,y\rangle| \, df(t) + \int_{s}^{M} |\langle (E_t - 1_H)x,y\rangle| \, df(t) \\
& \leq \int_{m-0}^{s} \langle E_t x,x\rangle^{1/2} \langle E_t y,y\rangle^{1/2} \, df(t) \\
& + \int_{s}^{M} \langle (1_H - E_t)x,x\rangle^{1/2} \langle (1_H - E_t)y,y\rangle^{1/2} \, df(t) \\
& := Y,
\end{aligned}
$$

for any $x,y\in H$ and $s\in[m,M]$.

We use now the following version of the Cauchy-Buniakovski-Schwarz inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators

$$\left(\int_a^b p(t)\,q(t)\,dv(t)\right)^2 \leq \int_a^b p^2(t)\,dv(t) \int_a^b q^2(t)\,dv(t)$$

to get that

$$
\begin{aligned}
& \int_{m-0}^{s} \langle E_t x,x\rangle^{1/2} \langle E_t y,y\rangle^{1/2} \, df(t) \\
& \leq \left(\int_{m-0}^{s} \langle E_t x,x\rangle \, df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y,y\rangle \, df(t)\right)^{1/2}
\end{aligned}
$$

and

$$\begin{aligned}
&\int_s^M \langle (1_H - E_t)\, x, x\rangle^{1/2} \langle (1_H - E_t)\, y, y\rangle^{1/2} df\,(t) \\
&\le \left(\int_s^M \langle (1_H - E_t)\, x, x\rangle\, df\,(t) \right)^{1/2} \left(\int_s^M \langle (1_H - E_t)\, y, y\rangle\, df\,(t) \right)^{1/2}
\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$.

Therefore

$$\begin{aligned}
Y &\le \left(\int_{m-0}^s \langle E_t x, x\rangle\, df\,(t) \right)^{1/2} \left(\int_{m-0}^s \langle E_t y, y\rangle\, df\,(t) \right)^{1/2} \\
&\quad + \left(\int_s^M \langle (1_H - E_t)\, x, x\rangle\, df\,(t) \right)^{1/2} \left(\int_s^M \langle (1_H - E_t)\, y, y\rangle\, df\,(t) \right)^{1/2} \\
&\le \left(\int_{m-0}^s \langle E_t x, x\rangle\, df\,(t) + \int_s^M \langle (1_H - E_t)\, x, x\rangle\, df\,(t) \right)^{1/2} \\
&\quad \times \left(\int_{m-0}^s \langle E_t y, y\rangle\, df\,(t) + \int_s^M \langle (1_H - E_t)\, y, y\rangle\, df\,(t) \right)^{1/2}
\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$, where, to get the last inequality we have used the elementary inequality (5.7).

Now, since f is monotonic nondecreasing, on applying the representation (5.3) for the function $|f(\cdot) - f(s)|$ with s fixed in $[m, M]$ we deduce the following identity that is of interest in itself as well:

(6.3)
$$\begin{aligned}
&\langle |f(A) - f(s)|\, x, y\rangle \\
&= \int_{m-0}^s \langle E_t x, y\rangle\, df\,(t) + \int_s^M \langle (1_H - E_t)\, x, y\rangle\, df\,(t)
\end{aligned}$$

for any $x, y \in H$.

The second part of (6.1) follows then by writing (6.3) for x then by y and utilizing the relevant inequalities from above.

The last part is similar to the corresponding one from the proof of Theorem 5.2 and the details are omitted. ∎

The following corollary is of interest:

COROLLARY 6.2 (Dragomir, 2010, [**16**]). *With the assumption of Theorem 6.1 we have the inequalities*

(6.4)
$$\begin{aligned}
&\left| \frac{f(m) + f(M)}{2} \langle x, y\rangle - \langle f(A)\, x, y\rangle \right| \\
&\le \left\langle \left| f(A) - \frac{f(m) + f(M)}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \\
&\times \left\langle \left| f(A) - \frac{f(m) + f(M)}{2} \cdot 1_H \right| y, y \right\rangle^{1/2} \\
&\le \frac{1}{2} \left(f(M) - f(m) \right) \|x\| \, \|y\|,
\end{aligned}$$

for any $x, y \in H$.

PROOF. Since f is monotonic nondecreasing, then $f(u) \in [f(m), f(M)]$ for any $u \in [m, M]$. By the continuity of f it follows that there exists at list one $s \in [m, M]$ such that

$$f(s) = \frac{f(m) + f(M)}{2}.$$

Now, on utilizing the inequality (6.1) for this s we deduce the first inequality in (6.4). The second part follows as above and the details are omitted. ∎

6.1. Power Inequalities. We consider the power function $f(t) := t^p$ where $p \in \mathbb{R} \setminus \{0\}$ and $t > 0$. The following mid-point inequalities hold:

PROPOSITION 6.3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$.*

If $p > 0$, then for any $x, y \in H$

$$\left|\left(\frac{m+M}{2}\right)^p \langle x, y\rangle - \langle A^p x, y\rangle\right| \leq B_p \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| y, y\right\rangle^{1/2} \leq \frac{1}{2} B_p (M - m) \|x\| \|y\| \tag{6.5}$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ m^{p-1} & \text{if } 0 < p < 1, m > 0. \end{cases}$$

and

$$\left|\left(\frac{m+M}{2}\right)^{-p} \langle x, y\rangle - \left\langle A^{-p} x, y\right\rangle\right| \leq C_p \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| y, y\right\rangle^{1/2} \leq \frac{1}{2} C_p (M - m) \|x\| \|y\| \tag{6.6}$$

where

$$C_p = pm^{-p-1} \text{ and } m > 0.$$

The proof follows from (5.15).

We can also state the following trapezoidal type inequalities:

PROPOSITION 6.4. *With the assumption of Proposition 6.3 and if $p > 0$ we have the inequalities*

$$\left|\frac{m^p + M^p}{2} \langle x, y\rangle - \langle A^p x, y\rangle\right| \leq \left\langle \left|A^p - \frac{m^p + M^p}{2} \cdot 1_H\right| x, x\right\rangle^{1/2} \left\langle \left|A^p - \frac{m^p + M^p}{2} \cdot 1_H\right| y, y\right\rangle^{1/2} \leq \frac{1}{2} (M^p - m^p) \|x\| \|y\|, \tag{6.7}$$

and, for $m > 0$,

$$\left| \frac{m^p + M^p}{2m^p M^p} \langle x, y \rangle - \left\langle A^{-p} x, y \right\rangle \right| \tag{6.8}$$
$$\leq \left\langle \left| A^{-p} - \frac{m^p + M^p}{2m^p M^p} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A^{-p} - \frac{m^p + M^p}{2m^p M^p} \cdot 1_H \right| y, y \right\rangle^{1/2}$$
$$\leq \frac{1}{2} \left(\frac{M^p - m^p}{M^p m^p} \right) \|x\| \|y\| ,$$

for any $x, y \in H$.

The proof follows from Corollary 6.2.

6.2. Logarithmic Inequalities. Consider the function $f(t) = \ln t, t > 0$. Denote by $A(a,b) := \frac{a+b}{2}$ the arithmetic mean of $a, b > 0$ and $G(a,b) := \sqrt{ab}$ the geometric mean of these numbers. We have the following result:

PROPOSITION 6.5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. For any $x, y \in H$ we have*

$$|\ln A(m, M) \cdot \langle x, y \rangle - \langle \ln Ax, y \rangle| \tag{6.9}$$
$$\leq \frac{1}{m} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x \right\rangle^{1/2} \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| y, y \right\rangle^{1/2}$$
$$\leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \|x\| \|y\|$$

and

$$|\ln G(m, M) \cdot \langle x, y \rangle - \langle \ln Ax, y \rangle| \tag{6.10}$$
$$\leq \langle |\ln A - \ln G(m, M) \cdot 1_H| \, x, x \rangle^{1/2} \langle |\ln A - \ln G(m, M) \cdot 1_H| \, y, y \rangle^{1/2}$$
$$\leq \ln \sqrt{\frac{M}{m}} \cdot \|x\| \|y\| .$$

The proof follows by (5.15) and (6.4).

7. OSTROWSKI'S TYPE VECTOR INEQUALITIES

7.1. Some Vector Inequalities. The following result holds:

THEOREM 7.1 (Dragomir, 2010, [**26**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left| \langle x, y \rangle \frac{1}{M-m} \int_m^M f(s) \, ds - \langle f(A) x, y \rangle \right| \tag{7.1}$$
$$\leq \frac{1}{M-m} \bigvee_m^M (f) \max_{t \in [m,M]} \Big[(M-t) \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2}$$
$$+ (t-m) \langle (1_H - E_t) x, x \rangle^{1/2} \langle (1_H - E_t) y, y \rangle^{1/2} \Big]$$
$$\leq \|x\| \|y\| \bigvee_m^M (f)$$

for any $x, y \in H$.

PROOF. Assume that $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$. Then under the assumptions of the theorem for A and $\{E_\lambda\}_\lambda$, we have the following representation

$$\begin{aligned}(7.2) \quad & \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x, y\rangle \\ &= \frac{1}{M-m}\int_{m-0}^M \langle [(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle\, df(t)\end{aligned}$$

for any $x, y \in H$.

Indeed, integrating by parts in the Riemann-Stieltjes integral and using the spectral representation theorem we have

$$\begin{aligned}&\frac{1}{M-m}\int_{m-0}^M \langle [(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle\, df(t) \\ &= \int_{m-0}^M \left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right) df(t) \\ &= \left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right) f(t)\Big|_{m-0}^M \\ &- \int_{m-0}^M f(t)\, d\left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right) \\ &= -\int_{m-0}^M f(t)\, d\langle E_t x, y\rangle + \langle x, y\rangle \frac{1}{M-m}\int_m^M f(t)\,dt \\ &= \langle x, y\rangle \frac{1}{M-m}\int_m^M f(t)\,dt - \langle f(A)x, y\rangle\end{aligned}$$

for any $x, y \in H$ and the equality (7.2) is proved.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property we have from (7.2) that

$$\begin{aligned}(7.3) \quad & \left|\langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x, y\rangle\right| \\ &\le \frac{1}{M-m}\max_{t\in[m,M]} |\langle [(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle| \bigvee_m^M (f)\end{aligned}$$

for any $x, y \in H$.

Now observe that

$$\begin{aligned} &\left|\left\langle\left[(M-t) E_t+(t-m)\left(E_t-1_H\right)\right] x, y\right\rangle\right| \\ &=\left|(M-t)\left\langle E_t x, y\right\rangle+(t-m)\left\langle\left(E_t-1_H\right) x, y\right\rangle\right| \\ &\leq(M-t)\left|\left\langle E_t x, y\right\rangle\right|+(t-m)\left|\left\langle\left(E_t-1_H\right) x, y\right\rangle\right| \end{aligned} \tag{7.4}$$

for any $x, y \in H$ and $t \in [m, M]$.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \tag{7.5}$$

for any $x, y \in H$.

On applying the inequality (7.5) we have

$$\begin{aligned} &(M-t)\left|\left\langle E_t x, y\right\rangle\right|+(t-m)\left|\left\langle\left(E_t-1_H\right) x, y\right\rangle\right| \\ &\leq(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2} \\ &+(t-m)\left\langle\left(1_H-E_t\right) x, x\right\rangle^{1/2}\left\langle\left(1_H-E_t\right) y, y\right\rangle^{1/2} \\ &\leq \max\{M-t, t-m\} \\ &\times\left[\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}+\left\langle\left(1_H-E_t\right) x, x\right\rangle^{1/2}\left\langle\left(1_H-E_t\right) y, y\right\rangle^{1/2}\right] \\ &\leq \max\{M-t, t-m\} \\ &\times\left[\left\langle E_s x, x\right\rangle+\left\langle\left(1_H-E_s\right) x, x\right\rangle\right]^{1/2}\left[\left\langle E_s y, y\right\rangle+\left\langle\left(1_H-E_s\right) y, y\right\rangle\right]^{1/2} \\ &=\max\{M-t, t-m\}\|x\|\|y\|, \end{aligned} \tag{7.6}$$

where for the last inequality we used the elementary fact

$$a_1 b_1 + a_2 b_2 \leq \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2} \tag{7.7}$$

that holds for a_1, b_1, a_2, b_2 positive real numbers.

Utilising the inequalities (7.3), (7.4) and (7.6) we deduce the desired result (7.1). ∎

The case of Lipschitzian functions is embodied in the following result:

THEOREM 7.2 (Dragomir, 2010, [**26**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\begin{aligned} &\left|\langle x, y\rangle \frac{1}{M-m} \int_m^M f(s)\, ds-\langle f(A) x, y\rangle\right| \\ &\leq \frac{L}{M-m} \int_m^M\left[(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}\right. \\ &\left.+(t-m)\left\langle\left(1_H-E_t\right) x, x\right\rangle^{1/2}\left\langle\left(1_H-E_t\right) y, y\right\rangle^{1/2}\right] dt \\ &\leq \frac{3}{4} L(M-m)\|x\|\|y\| \end{aligned} \tag{7.8}$$

for any $x, y \in H$.

PROOF. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (7.2) that

$$\begin{aligned}(7.9)\qquad &\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \\ &\le \frac{L}{M-m}\int_{m-0}^M |\langle[(M-t)E_t + (t-m)(E_t - 1_H)]x,y\rangle|\,dt.\end{aligned}$$

Since, from the proof of Theorem 7.1, we have

$$\begin{aligned}(7.10)\qquad &|\langle[(M-t)E_t + (t-m)(E_t - 1_H)]x,y\rangle| \\ &\le (M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\ &+ (t-m)\langle(1_H - E_t)x,x\rangle^{1/2}\langle(1_H - E_t)y,y\rangle^{1/2} \\ &\le \max\{M-t, t-m\}\|x\|\|y\| \\ &= \left[\frac{1}{2}(M-m) + \left|t - \frac{m+M}{2}\right|\right]\|x\|\|y\|\end{aligned}$$

for any $x,y \in H$ and $t \in [m,M]$, then integrating (7.10) and taking into account that

$$\int_m^M \left|t - \frac{m+M}{2}\right| dt = \frac{1}{4}(M-m)^2$$

we deduce the desired result (7.8). ∎

Finally for the section, we provide here the case of monotonic nondecreasing functions as well:

THEOREM 7.3 (Dragomir, 2010, [**26**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m,M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m,M]$, then we have the inequality*

$$\begin{aligned}(7.11)\qquad &\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \\ &\le \frac{1}{M-m}\int_m^M \Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\ &+ (t-m)\langle(1_H - E_t)x,x\rangle^{1/2}\langle(1_H - E_t)y,y\rangle^{1/2}\Big]\,df(t) \\ &\le \left[f(M) - f(m) - \frac{1}{M-m}\int_m^M sgn\left(t - \frac{m+M}{2}\right) f(t)\,dt\right]\|x\|\|y\| \\ &\le [f(M) - f(m)]\|x\|\|y\|\end{aligned}$$

for any $x,y \in H$.

PROOF. From the theory of Riemann-Stieltjes integral is well known that if $p : [a,b] \to \mathbb{C}$ is of bounded variation and $v : [a,b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the

Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (7.2) that

$$\begin{aligned}(7.12)\qquad &\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x,y\rangle\right| \\ &\le \frac{1}{M-m}\int_{m-0}^M |\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,df(t).\end{aligned}$$

Further on, by utilizing the inequality (7.10) we also have that

$$\begin{aligned}(7.13)\qquad &\int_{m-0}^M |\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,df(t) \\ &\le \int_m^M \Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\ &+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big]\,df(t) \\ &\le \left[\frac{1}{2}(M-m)[f(M)-f(m)]+\int_m^M\left|t-\frac{m+M}{2}\right|df(t)\right]\|x\|\,\|y\|.\end{aligned}$$

Now, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}&\int_m^M\left|t-\frac{m+M}{2}\right|df(t) \\ &=\int_m^{\frac{M+m}{2}}\left(\frac{m+M}{2}-t\right)df(t)+\int_{\frac{m+M}{2}}^M\left(t-\frac{m+M}{2}\right)df(t) \\ &=\left(\frac{m+M}{2}-t\right)f(t)\Big|_m^{\frac{M+m}{2}}+\int_m^{\frac{M+m}{2}}f(t)\,dt \\ &+\left(t-\frac{m+M}{2}\right)f(t)\Big|_{\frac{m+M}{2}}^M-\int_{\frac{m+M}{2}}^M f(t)\,dt \\ &=\frac{1}{2}(M-m)[f(M)-f(m)]-\int_m^M sgn\left(t-\frac{m+M}{2}\right)f(t)\,dt,\end{aligned}$$

which together with (7.13) produces the second inequality in (7.11).

Since the functions $sgn\left(\cdot-\frac{m+M}{2}\right)$ and $f(\cdot)$ have the same monotonicity, then by the Čebyšev inequality we have

$$\begin{aligned}&\int_m^M sgn\left(t-\frac{m+M}{2}\right)f(t)\,dt \\ &\ge \frac{1}{M-m}\int_m^M sgn\left(t-\frac{m+M}{2}\right)dt\int_m^M f(t)\,dt=0\end{aligned}$$

and the last part of (7.11) is proved. ∎

7.2. Applications for Particular Functions. It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p > 0$. This function is monotonic increasing on $(0, \infty)$ and applying Theorem 7.3 we can state the following proposition:

PROPOSITION 7.4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\begin{aligned}
(7.14)\qquad & \left|\langle A^p x, y\rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)}\langle x, y\rangle\right| \\
& \leq \frac{p}{M-m}\int_m^M \Big[(M-t)\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2} \\
& + (t-m)\langle (1_H - E_t)x, x\rangle^{1/2}\langle (1_H - E_t)y, y\rangle^{1/2}\Big] t^{p-1}dt \\
& \leq \left[M^p - m^p - \frac{M^{p+1} + m^{p+1} - 2^p(M+m)^{p+1}}{(p+1)(M-m)}\right]\|x\|\|y\|.
\end{aligned}$$

On applying now Theorem 7.2 to the same power function, then we can state the following result as well:

PROPOSITION 7.5. *With the same assumptions from Proposition 7.4 we have*

$$\begin{aligned}
(7.15)\qquad & \left|\langle A^p x, y\rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)}\langle x, y\rangle\right| \\
& \leq \frac{B_p}{M-m}\int_m^M \Big[(M-t)\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2} \\
& + (t-m)\langle (1_H - E_t)x, x\rangle^{1/2}\langle (1_H - E_t)y, y\rangle^{1/2}\Big] dt \\
& \leq \frac{3}{4}B_p(M-m)\|x\|\|y\|
\end{aligned}$$

for any $x, y \in H$, where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ m^{p-1} & \text{if } 0 < p < 1, m > 0. \end{cases}$$

The case of negative powers except $p = -1$ goes likewise and we omit the details.

Now, if we apply Theorem 7.3 and 7.2 for the increasing function $f(t) = -\frac{1}{t}$ with $t > 0$, then we can state the following proposition:

PROPOSITION 7.6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

Then for any $x, y \in H$ *we have the inequalities*

$$\left|\left\langle A^{-1}x, y\right\rangle - \frac{\ln M - \ln m}{M - m}\left\langle x, y\right\rangle\right| \tag{7.16}$$
$$\leq \frac{1}{M-m}\int_m^M \left[(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}\right.$$
$$\left. + (t-m)\left\langle (1_H - E_t)x, x\right\rangle^{1/2}\left\langle (1_H - E_t)y, y\right\rangle^{1/2}\right] t^2 dt$$
$$\leq \left[\frac{M-m}{mM} - \frac{\ln\left[\left(\frac{m+M}{2}\right)^2\right] - \ln(mM)}{M-m}\right]\|x\|\,\|y\|$$

and

$$\left|\left\langle A^{-1}x, y\right\rangle - \frac{\ln M - \ln m}{M - m}\left\langle x, y\right\rangle\right| \tag{7.17}$$
$$\leq \frac{1}{m^2(M-m)}\int_m^M \left[(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}\right.$$
$$\left. + (t-m)\left\langle (1_H - E_t)x, x\right\rangle^{1/2}\left\langle (1_H - E_t)y, y\right\rangle^{1/2}\right] dt$$
$$\leq \frac{3}{4}\frac{M-m}{m^2}\|x\|\,\|y\|.$$

2. Now, if we apply Theorems 7.3 and 7.2 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

PROPOSITION 7.7. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $0 < m < M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. Then for any* $x, y \in H$ *we have the inequalities*

$$|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \tag{7.18}$$
$$\leq \frac{1}{M-m}\int_m^M \left[(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}\right.$$
$$\left. + (t-m)\left\langle (1_H - E_t)x, x\right\rangle^{1/2}\left\langle (1_H - E_t)y, y\right\rangle^{1/2}\right] t dt$$
$$\leq \left[\ln\left(\frac{M}{m}\right) - \ln\left(\sqrt{\frac{I\left(\frac{m+M}{2}, M\right)}{I\left(m, \frac{m+M}{2}\right)}}\right)\right]\|x\|\,\|y\|$$

and

$$|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \tag{7.19}$$
$$\leq \frac{1}{m(M-m)}\int_m^M \left[(M-t)\left\langle E_t x, x\right\rangle^{1/2}\left\langle E_t y, y\right\rangle^{1/2}\right.$$
$$\left. + (t-m)\left\langle (1_H - E_t)x, x\right\rangle^{1/2}\left\langle (1_H - E_t)y, y\right\rangle^{1/2}\right] dt$$
$$\leq \frac{3}{4}\left(\frac{M}{m} - 1\right)\|x\|\,\|y\|,$$

where $I(m, M)$ *is the identric mean of* m *and* M *and is defined by*

$$I(m, M) = \frac{1}{e}\left(\frac{M^M}{m^m}\right)^{1/(M-m)}.$$

8. BOUNDS FOR THE DIFFERENCE BETWEEN FUNCTIONS AND INTEGRAL MEANS

8.1. Vector Inequalities Via Ostrowski's Type Bounds. The following result holds:

THEOREM 8.1 (Dragomir, 2010, [**22**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous function on $[m, M]$, then we have the inequality*

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \max_{t\in[m,M]}\left|f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right| \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \max_{t\in[m,M]}\left|f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right| \|x\| \|y\| \tag{8.1}$$

for any $x, y \in H$.

PROOF. Utilising the spectral representation theorem we have the following equality of interest

$$\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds = \int_{m-0}^M \left[f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right] d\left(\langle E_t x, y\rangle\right) \tag{8.2}$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v), \tag{8.3}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising these two facts we get the first part of (8.1).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

For particular classes of continuous functions $f : [m, M] \to \mathbb{C}$ we are able to provide simpler bounds as incorporated in the following corollary:

COROLLARY 8.2 (Dragomir, 2010, [**22**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$.*

1. If f is of bounded variation on $[m, M]$, then

$$\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \leq \bigvee_m^M (f) \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \|x\| \|y\| \bigvee_m^M (f) \tag{8.4}$$

for any $x, y \in H$.

2. If $f : [m, M] \longrightarrow \mathbb{C}$ *is of* $r - H-$*Hölder type, i.e., for a given* $r \in (0, 1]$ *and* $H > 0$ *we have*

$$|f(s) - f(t)| \leq H |s - t|^r \text{ for any } s, t \in [m, M], \tag{8.5}$$

then we have the inequality:

$$\begin{aligned} &\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds \right| \\ &\leq \frac{1}{r+1} H (M - m)^r \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{r+1} H (M - m)^r \|x\| \|y\| \end{aligned} \tag{8.6}$$

for any $x, y \in H$.

In particular, if $f : [m, M] \longrightarrow \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$, *then*

$$\begin{aligned} &\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds \right| \\ &\leq \frac{1}{2} L (M - m) \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \leq \frac{1}{2} L (M - m) \|x\| \|y\| \end{aligned} \tag{8.7}$$

for any $x, y \in H$.

3. If $f : [m, M] \longrightarrow \mathbb{C}$ *is absolutely continuous, then*

$$\begin{aligned} &\left| \langle f(A) x, y \rangle - \langle x, y \rangle \frac{1}{M - m} \int_m^M f(s)\, ds \right| \\ &\leq \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\quad \times \begin{cases} \frac{1}{2} (M - m) \|f'\|_\infty & \text{if } f' \in L_\infty [m, M] \\ \\ \frac{1}{(q+1)^{1/q}} (M - m)^{1/q} \|f'\|_p & \begin{array}{c} \text{if } f' \in L_p [m, M] \\ p > 1, 1/p + 1/q = 1; \end{array} \\ \\ \|f'\|_1 \end{cases} \\ &\leq \|x\| \|y\| \\ &\quad \times \begin{cases} \frac{1}{2} (M - m) \|f'\|_\infty & \text{if } f' \in L_\infty [m, M] \\ \\ \frac{1}{(q+1)^{1/q}} (M - m)^{1/q} \|f'\|_p & \begin{array}{c} \text{if } f' \in L_p [m, M] \\ p > 1, 1/p + 1/q = 1; \end{array} \\ \\ \|f'\|_1 \end{cases} \end{aligned} \tag{8.8}$$

for any $x, y \in H$, *where* $\|f'\|_p$ *are the Lebesgue norms, i.e., we recall that*

$$\|f'\|_p := \begin{cases} ess \sup_{s \in [m, M]} |f'(s)| & \text{if } p = \infty; \\ \\ \left(\int_m^M |f(s)|^p\, ds \right)^{1/p} & \text{if } p \geq 1. \end{cases}$$

PROOF. We use the Ostrowski type inequalities in order to provide upper bounds for the quantity

$$\max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|$$

where $f:[m,M]\longrightarrow\mathbb{C}$ is a continuous function.

The following result may be stated (see [**23**]) for functions of bounded variation:

LEMMA 8.3. *Assume that $f:[m,M]\to\mathbb{C}$ is of bounded variation and denote by $\bigvee_m^M(f)$ its total variation. Then*

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \le\left[\frac{1}{2}+\left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\bigvee_m^M(f) \tag{8.9}$$

for all $t\in[m,M]$. The constant $\frac{1}{2}$ is the best possible.

Now, taking the maximum over $x\in[m,M]$ in (8.9) we deduce (8.4).

If f is Hölder continuous, then one may state the result:

LEMMA 8.4. *Let $f:[m,M]\to\mathbb{C}$ be of $r-H-$Hölder type, where $r\in(0,1]$ and $H>0$ are fixed, then, for all $x\in[m,M]$, we have the inequality:*

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \le\frac{H}{r+1}\left[\left(\frac{M-t}{M-m}\right)^{r+1}+\left(\frac{t-m}{M-m}\right)^{r+1}\right](M-m)^r. \tag{8.10}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [**17**])

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \le\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)L, \tag{8.11}$$

for any $x\in[m,M]$. Here the constant $\frac{1}{4}$ is also best.

Taking the maximum over $x\in[m,M]$ in (8.10) we deduce (8.6) and the second part of the corollary is proved.

The following Ostrowski type result for absolutely continuous functions holds.

LEMMA 8.5. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $t \in [a,b]$, we have:*

$$\left| f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds \right| \tag{8.12}$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)\|f'\|_\infty & \text{if } f' \in L_\infty[m,M]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{t-m}{M-m}\right)^{q+1} + \left(\frac{M-t}{M-m}\right)^{q+1}\right]^{\frac{1}{q}}(M-m)^{\frac{1}{q}}\|f'\|_p & \text{if } f' \in L_p[m,M], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\|f'\|_1. & \end{cases}$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented above.

The above inequalities can also be obtained from the Fink result in [**39**] on choosing $n = 1$ and performing some appropriate computations.

Taking the maximum in these inequalities we deduce (8.8). ∎

For other scalar Ostrowski's type inequalities, see [**1**] and [**18**].

8.2. Other Vector Inequalities. In [**37**], the authors have considered the following functional

$$D(f;u) := \int_a^b f(s)\,du(s) - [u(b) - u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{8.13}$$

provided that the Stieltjes integral $\int_a^b f(s)\,du(s)$ exists.

This functional plays an important role in approximating the Stieltjes integral $\int_a^b f(s)\,du(s)$ in terms of the Riemann integral $\int_a^b f(t)\,dt$ and the divided difference of the integrator u.

In [**37**], the following result in estimating the above functional $D(f;u)$ has been obtained:

$$|D(f;u)| \leq \frac{1}{2}L(M-m)(b-a), \tag{8.14}$$

provided u is $L-$*Lipschitzian* and f is *Riemann integrable* and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$m \leq f(t) \leq M \quad \text{for any} \quad t \in [a,b]. \tag{8.15}$$

The constant $\frac{1}{2}$ is best possible in (8.14) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of *bounded variation* and f is $K-$*Lipschitzian*, then $D(f,u)$ satisfies the inequality [**38**]

$$|D(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{8.16}$$

Here the constant $\frac{1}{2}$ is also best possible.

Now, for the function $u:[a,b]\to\mathbb{C}$, consider the following auxiliary mappings Φ,Γ and Δ [**19**]:

$$\Phi(t):=\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t),\qquad t\in[a,b],$$
$$\Gamma(t):=(t-a)[u(b)-u(t)]-(b-t)[u(t)-u(a)],\qquad t\in[a,b],$$
$$\Delta(t):=[u;b,t]-[u;t,a],\qquad t\in(a,b),$$

where $[u;\alpha,\beta]$ is the *divided difference* of u in α,β, i.e.,

$$[u;\alpha,\beta]:=\frac{u(\alpha)-u(\beta)}{\alpha-\beta}.$$

The following representation of $D(f,u)$ may be stated, see [**19**] and [**20**]. Due to its importance in proving our new results we present here a short proof as well.

LEMMA 8.6. *Let $f,u:[a,b]\to\mathbb{C}$ be such that the Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. Then*

$$D(f,u)=\int_a^b\Phi(t)\,df(t)=\frac{1}{b-a}\int_a^b\Gamma(t)\,df(t)=\frac{1}{b-a}\int_a^b(t-a)(b-t)\Delta(t)\,df(t).\tag{8.17}$$

PROOF. Since $\int_a^b f(t)\,du(t)$ exists, hence $\int_a^b\Phi(t)\,df(t)$ also exists, and the integration by parts formula for Riemann-Stieltjes integrals gives that

$$\begin{aligned}\int_a^b\Phi(t)\,df(t)&=\int_a^b\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right]df(t)\\&=\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right]f(t)\bigg|_a^b\\&\quad-\int_a^b f(t)\,d\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right]\\&=-\int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}dt-du(t)\right]=D(f,u),\end{aligned}$$

proving the required identity. ∎

For recent inequalities related to $D(f;u)$ for various pairs of functions (f,u), see [**21**].

The following representation for a continuous function of selfadjoint operator may be stated:

LEMMA 8.7 (Dragomir, 2010, [**22**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f:[m,M]\to\mathbb{C}$ a continuous function on $[m,M]$. If $x,y\in H$, then we have the representation*

$$\langle f(A)x,y\rangle=\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds+\frac{1}{M-m}\int_{m-0}^M\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\,df(t).\tag{8.18}$$

PROOF. Utilising Lemma 8.6 we have

$$\int_m^M f(t)\,du(t) = [u(M) - u(m)] \cdot \frac{1}{M-m}\int_m^M f(s)\,ds + \int_m^M \left[\frac{(t-m)\,u(M) + (M-t)\,u(m)}{M-m} - u(t)\right] df(t)\,, \tag{8.19}$$

for any continuous function $f : [m, M] \to \mathbb{C}$ and any function of bounded variation $u : [m, M] \to \mathbb{C}$.

Now, if we write the equality (8.19) for $u(t) = \langle E_t x, y\rangle$ with $x, y \in H$, then we get

$$\int_{m-0}^M f(t)\,d\langle E_t x, y\rangle = \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\,ds + \int_{m-0}^M \left[\frac{(t-m)\langle x, y\rangle}{M-m} - \langle E_t x, y\rangle\right] df(t)\,, \tag{8.20}$$

which, by the spectral representation theorem, produces the desired result (8.18). ∎

The following result may be stated:

THEOREM 8.8 (Dragomir, 2010, [**22**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$.*

1. If f is of bounded variation, then

$$\begin{aligned}\left|\langle f(A)\,x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| &\le \|y\| \bigvee_m^M (f) \\ &\times \max_{t\in[m,M]}\left[\left(\frac{t-m}{M-m}\right)^2 \|(1_H - E_t)\,x\|^2 + \left(\frac{M-t}{M-m}\right)^2 \|E_t x\|^2\right]^{1/2} \\ &\le \|x\|\,\|y\| \bigvee_m^M (f)\end{aligned} \tag{8.21}$$

for any $x, y \in H$.

2. If f is Lipschitzian with the constant $L > 0$, then

$$\begin{aligned}\left|\langle f(A)\,x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| &\le \frac{L\,\|y\|}{M-m}\int_{m-0}^M \left[(t-m)^2 \|(1_H - E_t)\,x\|^2 + (M-t)^2 \|E_t x\|^2\right]^{1/2} dt \\ &\le \frac{1}{2}\left[1 + \frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right](M-m)\,L\,\|y\|\,\|x\|\end{aligned} \tag{8.22}$$

for any $x, y \in H$.

3. If $f:[m,M]\to\mathbb{R}$ is monotonic nondecreasing, then

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \\
&\le \frac{\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}df(t) \\
&\le \|y\|\,\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}df(t) \\
&\le \|y\|\,\|x\|\,[f(M)-f(m)]^{1/2} \\
&\times\left[f(M)-f(m)-\frac{4}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\right]^{1/2} \\
&\le \|y\|\,\|x\|\,[f(M)-f(m)]
\end{aligned}\tag{8.23}$$

for any $x,y\in H$.

PROOF. If we assume that f is of bounded variation, then on applying the property (8.3) to the representation (8.18) we get

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \\
&\le\frac{1}{M-m}\max_{t\in[m,M]}\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|\bigvee_m^M(f).
\end{aligned}\tag{8.24}$$

Now, on utilizing the Schwarz inequality and the fact that E_t is a projector for any $t\in[m,M]$, then we have

$$\begin{aligned}
&\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right| \\
&\le\|[(t-m)(1_H-E_t)-(M-t)E_t]x\|\,\|y\| \\
&=\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}\|y\| \\
&\le\left[(t-m)^2+(M-t)^2\right]^{1/2}\|x\|\,\|y\|
\end{aligned}\tag{8.25}$$

for any $x,y\in H$ and for any $t\in[m,M]$.

Taking the maximum in (8.25) we deduce the desired inequality (8.21).

It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t|\ \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral to the representation (8.18), we get

$$\begin{aligned}(8.26)\qquad &\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right|\\&\le\frac{L}{M-m}\int_{m-0}^M\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|dt\\&\le\frac{L\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}dt\\&\le L\|y\|\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}dt,\end{aligned}$$

for any $x,y\in H$.

Now, if we change the variable in the integral by choosing $u=\frac{t-m}{M-m}$ then we get

$$\begin{aligned}&\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}dt\\&=(M-m)\int_0^1\left[u^2+(1-u)^2\right]^{1/2}du\\&=\frac{1}{2}(M-m)\left[1+\frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right],\end{aligned}$$

which together with (8.26) produces the desired result (8.22).

From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b|p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (8.18)

$$\begin{aligned}(8.27)\qquad &\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right|\\&\le\frac{1}{M-m}\int_{m-0}^M\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|df(t)\\&\le\frac{\|y\|}{M-m}\int_{m-0}^M\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}df(t)\\&\le\|y\|\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}df(t),\end{aligned}$$

for any $x,y\in H$ and the proof of the first and second inequality in (8.23) is completed.

For the last part we use the following Cauchy-Buniakowski-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrator v

$$\left|\int_a^b p(t)\,q(t)\,dv(t)\right| \le \left[\int_a^b |p(t)|^2\,dv(t)\right]^{1/2}\left[\int_a^b |q(t)|^2\,dv(t)\right]^{1/2}$$

where $p,q:[a,b]\to\mathbb{C}$ are continuous on $[a,b]$.

By applying this inequality we conclude that

$$\begin{aligned}(8.28)\qquad &\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}df(t)\\ &\le\left[\int_m^M df(t)\right]^{1/2}\left[\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]df(t)\right]^{1/2}.\end{aligned}$$

Further, integrating by parts in the Riemann-Stieltjes integral we also have that

$$\begin{aligned}(8.29)\qquad &\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]df(t)\\ &=f(M)-f(m)-\frac{4}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\\ &\le f(M)-f(m)\end{aligned}$$

where for the last part we used the fact that by the Čebyšev integral inequality for monotonic functions with the same monotonicity we have that

$$\begin{aligned}&\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\\ &\ge\frac{1}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)dt\int_m^M f(t)\,dt=0.\end{aligned}$$

∎

8.3. Some Applications for Particular Functions. 1. Consider the function $f:(0,\infty)\to\mathbb{R}$ given by $f(t)=t^r$ with $r\in(0,1]$. This function is r-Hölder continuous with the constant $H>0$. Then, by applying Corollary 8.2 we can state the following result

PROPOSITION 8.9. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $0<m<M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then for all r with $r\in(0,1]$ we have the inequality*

$$\begin{aligned}(8.30)\qquad &\left|\langle A^r x,y\rangle-\langle x,y\rangle\frac{M^{r+1}-m^{r+1}}{(r+1)(M-m)}\right|\\ &\le\frac{1}{r+1}(M-m)^r\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\le\frac{1}{r+1}(M-m)^r\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$.

The case of $p>1$ is incorporated in the following proposition:

PROPOSITION 8.10. *With the same assumptions from Proposition 8.9 and if $p > 1$, then we have*

$$\left|\langle A^p x, y\rangle - \frac{M^{p+1} - m^{p+1}}{(p+1)(M-m)}\langle x, y\rangle\right| \le \frac{1}{2}pM^{p-1}(M-m)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}pM^{p-1}(M-m)\|x\|\|y\| \tag{8.31}$$

for any $x, y \in H$.

The case of negative powers except $p = -1$ goes likewise and we omit the details.

Now, if we apply Corollary 8.2 for the function $f(t) = -\frac{1}{t}$ with $t > 0$, then we can state the following proposition:

PROPOSITION 8.11. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left|\left\langle A^{-1}x, y\right\rangle - \frac{\ln M - \ln m}{M - m}\langle x, y\rangle\right| \le \frac{1}{2}\frac{M-m}{m^2}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\frac{M-m}{m^2}\|x\|\|y\|. \tag{8.32}$$

2. Now, if we apply Corollary 8.2 to the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state

PROPOSITION 8.12. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln I(m, M)| \le \frac{1}{2}\left(\frac{M}{m} - 1\right)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\left(\frac{M}{m} - 1\right)\|x\|\|y\|, \tag{8.33}$$

where $I(m, M)$ is the identric mean of m and M and is defined by

$$I(m, M) = \frac{1}{e}\left(\frac{M^M}{m^m}\right)^{1/(M-m)}.$$

9. OSTROWSKI'S TYPE INEQUALITIES FOR n-TIME DIFFERENTIABLE FUNCTIONS

9.1. Some Identities. In [**6**], the authors have pointed out the following integral identity:

LEMMA 9.1 (Cerone-Dragomir-Roumeliotis, 1999, [**6**]). *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the $(n-1)$-derivative $f^{(n-1)}$ (where $n \ge 1$) is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$, we have the identity:*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) + (-1)^n \int_a^b K_n(x, t) f^{(n)}(t)\,dt \tag{9.1}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & a \le t \le x \le b \\ \\ \frac{(t-b)^n}{n!}, & a \le x < t \le b. \end{cases} \tag{9.2}$$

The identity (9.2) can be written in the following equivalent form as:

$$\begin{aligned} f(z) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &- \frac{1}{b-a}\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[(b-z)^{k+1} + (-1)^k (z-a)^{k+1}\right] f^{(k)}(z) \\ &+ \frac{(-1)^{n-1}}{(b-a)\,n!}\left[\int_a^z (t-a)^n f^{(n)}(t)\,dt + \int_z^b (t-b)^n f^{(n)}(t)\,dt\right] \end{aligned} \tag{9.3}$$

for all $z \in [a,b]$.

Note that for $n = 1$, the sum $\sum_{k=1}^{n-1}$ is empty and we obtain the well known *Montgomery's identity* (see for example [**3**])

$$\begin{aligned} f(z) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &+ \frac{1}{b-a}\left[\int_a^z (t-a) f^{(1)}(t)\,dt + \int_z^b (t-b) f^{(1)}(t)\,dt\right], \end{aligned} \tag{9.4}$$

for any $z \in [a,b]$.

In a slightly more general setting, by the use of the identity (9.3), we can state the following result as well:

LEMMA 9.2 (Dragomir, 2010, [**8**]). *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping such that the* n*-derivative* $f^{(n)}$ *(where* $n \ge 1$*) is of bounded variation on* $[a,b]$*. Then for all* $\lambda \in [a,b]$*, we have the identity:*

$$\begin{aligned} f(\lambda) &= \frac{1}{b-a}\int_a^b f(t)\,dt \\ &- \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(b-\lambda)^{k+1} + (-1)^k (\lambda-a)^{k+1}\right] f^{(k)}(\lambda) \\ &+ \frac{(-1)^n}{(b-a)(n+1)!} \\ &\times \left[\int_a^\lambda (t-a)^{n+1} d\left(f^{(n)}(t)\right) + \int_\lambda^b (t-b)^{n+1} d\left(f^{(n)}(t)\right)\right]. \end{aligned} \tag{9.5}$$

Now we can state the following representation result for functions of selfadjoint operators:

THEOREM 9.3 (Dragomir, 2010, [**8**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m,M]$ *for some real numbers* $m < M$*,* $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m,M] \subset \mathring{I}$ *(the interior of* I*) and let* n *be an integer with* $n \ge 1$*. If* $f : I \to \mathbb{C}$ *is such that the* n*-th derivative* $f^{(n)}$ *is of bounded variation on* *the interval* $[m, M]$*, then we have the representation*

$$f(A) = \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H - \frac{1}{M-m} \times \sum_{k=1}^{n} \frac{1}{(k+1)!}\left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}(A) + T_n(A,m,M) \tag{9.6}$$

where the remainder is given by

$$T_n(A,m,M) := \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_{m-0}^{M}\left(\int_m^{\lambda}(t-m)^{n+1}d\left(f^{(n)}(t)\right)\right)dE_\lambda + \int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-M)^{n+1}d\left(f^{(n)}(t)\right)\right)dE_\lambda\right]. \tag{9.7}$$

In particular, if the n*-th derivative* $f^{(n)}$ *is absolutely continuous on* $[m, M]$*, then the remainder can be represented as*

$$T_n(A,m,M) = \frac{(-1)^n}{(M-m)(n+1)!} \times \int_{m-0}^{M}\left[(\lambda-m)^{n+1}(1_H - E_\lambda) + (\lambda - M)^{n+1}E_\lambda\right] f^{(n+1)}(\lambda)\,d\lambda. \tag{9.8}$$

PROOF. By Lemma 9.2 we have

$$f(\lambda) = \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{1}{M-m} \times \sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(M-\lambda)^{k+1} + (-1)^k(\lambda-m)^{k+1}\right]f^{(k)}(\lambda) + \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_m^{\lambda}(t-m)^{n+1}d\left(f^{(n)}(t)\right) + \int_{\lambda}^{M}(t-M)^{n+1}d\left(f^{(n)}(t)\right)\right] \tag{9.9}$$

for any $\lambda \in [m, M]$.

Integrating the identity (9.9) in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\int_m^M f(\lambda)\,dE_\lambda = \frac{1}{M-m}\int_m^M f(t)\,dt\int_m^M dE_\lambda - \frac{1}{M-m} \times \sum_{k=1}^{n}\frac{1}{(k+1)!}\int_m^M\left[(M-\lambda)^{k+1} + (-1)^k(\lambda-m)^{k+1}\right]f^{(k)}(\lambda)\,dE_\lambda + T_n(A,m,M). \tag{9.10}$$

Since, by the spectral representation theorem we have

$$\int_{m-0}^{M} f(\lambda)\, dE_\lambda = f(A), \int_{m-0}^{M} dE_\lambda = 1_H$$

and

$$\int_{m-0}^{M} \left[(M-\lambda)^{k+1} + (-1)^k (\lambda - m)^{k+1}\right] f^{(k)}(\lambda)\, dE_\lambda$$
$$= \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}(A),$$

then by (9.10) we deduce the representation (9.6).

Now, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then

$$\int_{m}^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) = \int_{m}^{\lambda} (t-m)^{n+1} f^{(n+1)}(t)\, dt$$

and

$$\int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right) = \int_{\lambda}^{M} (t-M)^{n+1} f^{(n+1)}(t)\, dt$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Utilising the integration by parts formula for the Riemann-Stieltjes integral and the differentiation rule for the Stieltjes integral we have successively

$$\int_{m-0}^{M} \left(\int_{m}^{\lambda} (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) dE_\lambda$$
$$= \left(\int_{m}^{\lambda} (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) E_\lambda \Big|_{m-0}^{M}$$
$$- \int_{m-0}^{M} (\lambda - m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$
$$= \left(\int_{m}^{M} (t-m)^{n+1} f^{(n+1)}(t)\, dt\right) 1_H$$
$$- \int_{m-0}^{M} (\lambda - m)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$
$$= \int_{m-0}^{M} (\lambda - m)^{n+1} f^{(n+1)}(\lambda)\, (1_H - E_\lambda)\, d\lambda$$

and

$$\int_{m-0}^{M} \left(\int_{\lambda}^{M} (t-M)^{n+1} f^{(n+1)}(t)\, dt\right) dE_\lambda$$
$$= \left(\int_{\lambda}^{M} (t-M)^{n+1} f^{(n+1)}(t)\, dt\right) E_\lambda \Big|_{m-0}^{M}$$
$$+ \int_{m-0}^{M} (\lambda - M)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$
$$= \int_{m-0}^{M} (\lambda - M)^{n+1} f^{(n+1)}(\lambda)\, E_\lambda d\lambda$$

and the representation (9.8) is thus obtained. ∎

REMARK 9.1. Let A be a positive selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some positive real numbers $0 < m < M$ and $\{E_\lambda\}_\lambda$ be its *spectral family. Then, for* $n \geq 1,$ *we have the equality*

$$\ln A = [\ln I(m, M)] 1_H + \frac{1}{M-m} \times \sum_{k=1}^{n} \frac{1}{k(k+1)} \left[(A - m1_H)^{k+1} + (-1)^k (M1_H - A)^{k+1} \right] A^{-k} + \frac{1}{(M-m)(n+1)} \times \left[\int_{m-0}^{M} \left[(\lambda - m)^{n+1} (1_H - E_\lambda) + (\lambda - M)^{n+1} E_\lambda \right] \lambda^{-n-1} d\lambda \right], \tag{9.11}$$

where $I(m, M)$ is the identric mean and is defined by

$$I(m, M) = \begin{cases} \frac{1}{e} \left(\frac{M^M}{m^m} \right)^{1/(M-m)} & \text{if } M \neq m; \\ M & \text{if } M = m. \end{cases}$$

REMARK 9.2. If we introduce the exponential mean by

$$E(m, M) = \begin{cases} \frac{\exp M - \exp m}{M-m} & \text{if } M \neq m; \\ M & \text{if } M = m \end{cases}$$

and applying the identity (9.6) for the exponential function, we have

$$\left[1_H + \frac{1}{M-m} \sum_{k=1}^{n} \frac{1}{(k+1)!} \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1} \right] \right] \times \exp A - E(m, M) 1_H = \frac{(-1)^n}{(M-m)(n+1)!} \int_{m-0}^{M} \left[(\lambda - m)^{n+1} (1_H - E_\lambda) + (\lambda - M)^{n+1} E_\lambda \right] e^\lambda d\lambda \tag{9.12}$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ is its *spectral family.*

9.2. Error Bounds for $f^{(n)}$ of Bounded Variation. From the identity (9.6), we define for any $x, y \in H$

$$T_n(A, m, M; x, y) := \langle f(A) x, y \rangle + \frac{1}{M-m} \sum_{k=1}^{n} \frac{1}{(k+1)!} \times \left[\left\langle (M1_H - A)^{k+1} f^{(k)}(A) x, y \right\rangle + (-1)^k \left\langle (A - m1_H)^{k+1} f^{(k)}(A) x, y \right\rangle \right] - \left(\frac{1}{M-m} \int_m^M f(t) dt \right) \langle x, y \rangle . \tag{9.13}$$

We have the following result concerning bounds for the absolute value of $T_n(A, m, M; x, y)$ when the n-th derivative $f^{(n)}$ is of bounded variation:

THEOREM 9.4 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$.*

1. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
&|T_n(A, m, M; x, y)| \\
&\leq \frac{1}{(M-m)(n+1)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \\
&\times \max_{\lambda \in [m,M]} \left[(\lambda - m)^{n+1} \bigvee_{m}^{\lambda} \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right)\right] \\
&\leq \frac{(M-m)^n}{(n+1)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \bigvee_{m}^{M} \left(f^{(n)}\right) \leq \frac{(M-m)^n}{(n+1)!} \bigvee_{m}^{M} \left(f^{(n)}\right) \|x\| \|y\|
\end{aligned} \tag{9.14}
$$

for any $x, y \in H$.

2. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
|T_n(A, m, M; x, y)| &\leq \frac{L_n (M-m)^{n+1}}{(n+2)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \\
&\leq \frac{L_n (M-m)^{n+1}}{(n+2)!} \|x\| \|y\|
\end{aligned} \tag{9.15}
$$

for any $x, y \in H$.

3. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is monotonic nondecreasing on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
&|T_n(A, m, M; x, y)| \\
&\leq \frac{1}{(M-m)(n+1)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \\
&\times \max_{\lambda \in [m,M]} \Big[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M-\lambda)^{n+1}\right) \\
&+ (n+1) \left[\int_{\lambda}^{M} (M-t)^n f^{(n)}(t)\, dt - \int_{m}^{\lambda} (t-m)^n f^{(n)}(t)\, dt\right]\Big] \\
&\leq \frac{1}{(M-m)(n+1)!} \max_{\lambda \in [m,M]} \Big[(\lambda - m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m)\right] \\
&+ (M-\lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\Big] \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \\
&\leq \frac{(M-m)^n}{(n+1)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \left[f^{(n)}(M) - f^{(n)}(m)\right] \\
&\leq \frac{(M-m)^n}{(n+1)!} \left[f^{(n)}(M) - f^{(n)}(m)\right] \|x\| \|y\|
\end{aligned} \tag{9.16}
$$

for any $x, y \in H$.

PROOF. 1. By the identity (9.7) we have for any $x, y \in H$ that

$$T_n(A, m, M; x, y) := \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_{m-0}^{M} \left(\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) \right) d\langle E_\lambda x, y\rangle + \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right) \right) d\langle E_\lambda x, y\rangle \right]. \tag{9.17}$$

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v), \tag{9.18}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (9.17) and utilizing the property (9.18), we have successively that

$$\begin{aligned} &|T_n(A, m, M; x, y)| \\ &= \frac{1}{(M-m)(n+1)!} \\ &\times \left| \int_{m-0}^{M} \left[\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) \right.\right. \\ &\left.\left. + \int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right) \right] d\langle E_\lambda x, y\rangle \right| \\ &\le \frac{1}{(M-m)(n+1)!} \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \\ &\times \max_{\lambda\in[m,M]} \left| \int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right) \right| \end{aligned} \tag{9.19}$$

for any $x, y \in H$.

By the same property (9.18) we have for $\lambda \in (m, M)$ that

$$\begin{aligned} \left| \int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) \right| &\le \max_{t\in[m,\lambda]} (t-m)^{n+1} \bigvee_m^{\lambda} \left(f^{(n)}\right) \\ &= (\lambda - m)^{n+1} \bigvee_m^{\lambda} \left(f^{(n)}\right) \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right) \right| &\le \max_{t\in[\lambda,M]} (M-t)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right) \\ &= (M - \lambda)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right) \end{aligned}$$

which produce the inequality

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \le (\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right). \tag{9.20}$$

Taking the maximum over $\lambda \in [m, M]$ in (9.20) and utilizing (9.19) we deduce the first inequality in (9.14).

Now observe that

$$\begin{aligned}
&(\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right) \\
&\le \max\left\{(\lambda-m)^{n+1}, (M-\lambda)^{n+1}\right\} \left[\bigvee_m^\lambda \left(f^{(n)}\right) + \bigvee_\lambda^M \left(f^{(n)}\right)\right] \\
&= \max\left\{(\lambda-m)^{n+1}, (M-\lambda)^{n+1}\right\} \bigvee_m^M \left(f^{(n)}\right) \\
&= \left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n+1} \bigvee_m^M \left(f^{(n)}\right)
\end{aligned}$$

giving that

$$\max_{\lambda\in[m,M]} \left[(\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right)\right] \le (M-m)^{n+1} \bigvee_m^M \left(f^{(n)}\right)$$

and the second inequality in (9.14) is proved.

The last part of (9.14) follows by the Total Variation Schwarz's inequality and we omit the details.

2. Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \le L|s-t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L \int_a^b |p(t)|\,dt. \tag{9.21}$$

By the property (9.21) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right)\right| \le L_n \int_m^\lambda (t-m)^{n+1} d(t) = \frac{L_n}{n+2}(\lambda-m)^{n+2}$$

and

$$\left|\int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \le L_n \int_\lambda^M (M-t)^{n+1} dt = \frac{L_n}{n+2}(M-\lambda)^{n+2}.$$

By the inequality (9.19) we then have

$$\begin{aligned}(9.22)\qquad &|T_n(A,m,M;x,y)| \\ &\le \frac{1}{(M-m)(n+1)!}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\times \max_{\lambda\in[m,M]}\left[\frac{L_n}{n+2}(\lambda-m)^{n+2}+\frac{L_n}{n+2}(M-\lambda)^{n+2}\right] \\ &= \frac{L_n(M-m)^{n+1}}{(n+2)!}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le \frac{L_n(M-m)^{n+1}}{(n+2)!}\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$ and the inequality (9.15) is proved.

3. Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$(9.23)\qquad \left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t) \le \max_{t\in[a,b]}|p(t)|\,[v(b)-v(a)].$$

On making use of (9.23) we have

$$\begin{aligned}(9.24)\qquad \left|\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right)\right| &\le \int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &\le (\lambda-m)^{n+1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]\end{aligned}$$

and

$$\begin{aligned}(9.25)\qquad \left|\int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right| &\le \int_\lambda^M (M-t)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &\le (M-\lambda)^{n+1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\end{aligned}$$

for any $\lambda\in(m,M)$.

Integrating by parts in the Riemann-Stieltjes integral, we also have

$$\begin{aligned}&\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &= (\lambda-m)^{n+1}f^{(n)}(\lambda)-(n+1)\int_m^\lambda (t-m)^n f^{(n)}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_\lambda^M (M-t)^{n+1}\,d\left(f^{(n)}(t)\right) \\ &= (n+1)\int_\lambda^M (M-t)^n f^{(n)}(t)\,dt-(M-\lambda)^{n+1}f^{(n)}(\lambda)\end{aligned}$$

for any $\lambda\in(m,M)$.

Therefore, by adding (9.24) with (9.25) we get

$$\begin{aligned}&\left|\int_m^\lambda (t-m)^{n+1}\, d\left(f^{(n)}(t)\right)\right| + \left|\int_\lambda^M (t-M)^{n+1}\, d\left(f^{(n)}(t)\right)\right| \\ &\le \left[f^{(n)}(\lambda)\left((\lambda-m)^{n+1}-(M-\lambda)^{n+1}\right)\right] \\ &+ (n+1)\left[\int_\lambda^M (M-t)^n f^{(n)}(t)\, dt - \int_m^\lambda (t-m)^n f^{(n)}(t)\, dt\right] \\ &\le (\lambda-m)^{n+1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right] + (M-\lambda)^{n+1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\end{aligned}$$

for any $\lambda \in (m, M)$.

Now, on making use of the inequality (9.19) we deduce (9.16). ∎

REMARK 9.3. If we use the inequality (9.14) for the function $\ln$, then we get the inequality

$$\begin{aligned}&|L_n(A,m,M;x,y)| \\ &\le \frac{1}{(M-m)\,n\,(n+1)} \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\times \max_{\lambda\in[m,M]}\left[(\lambda-m)^{n+1}\frac{\lambda^n-m^n}{\lambda^n m^n} + (M-\lambda)^{n+1}\frac{M^n-\lambda^n}{M^n\lambda^n}\right] \\ &\le \frac{(M-m)^n(M^n-m^n)}{n(n+1)M^n m^n} \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{(M-m)^n(M^n-m^n)}{n(n+1)M^n m^n}\|x\|\,\|y\|\end{aligned} \tag{9.26}$$

for any $x, y \in H$, where

$$\begin{aligned}&L_n(A,m,M;x,y) \\ &:= \langle \ln Ax, y\rangle - [\ln I(m,M)]\langle x,y\rangle \\ &- \frac{1}{M-m}\sum_{k=1}^{n}\frac{1}{k(k+1)} \\ &\times\left[\left\langle (A-m1_H)^{k+1}A^{-k}x,y\right\rangle + (-1)^k\left\langle (M1_H-A)^{k+1}A^{-k}x,y\right\rangle\right].\end{aligned} \tag{9.27}$$

If we use the inequality (9.15) for the function $\ln$ we get the following bound as well

$$\begin{aligned}&|L_n(A,m,M;x,y)| \\ &\le \frac{1}{(n+1)(n+2)}\left(\frac{M}{m}-1\right)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{(n+1)(n+2)}\left(\frac{M}{m}-1\right)^{n+1}\|x\|\,\|y\|\end{aligned} \tag{9.28}$$

for any $x, y \in H$.

REMARK 9.4. If we define

$$
\begin{aligned}
&E_n(A,m,M;x,y) \\
&:= \left\langle \left[1_H + \frac{1}{M-m} \right.\right. \\
&\times \left.\left. \sum_{k=1}^{n} \frac{1}{(k+1)!} \left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1} \right] \exp A \right] x, y \right\rangle \\
&- E(m,M)\langle x,y\rangle ,
\end{aligned} \tag{9.29}
$$

then by the inequality (9.14) we have

$$
\begin{aligned}
&|E_n(A,m,M;x,y)| \\
&\le \frac{1}{(M-m)(n+1)!} \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y \right\rangle \right) \\
&\times \max_{\lambda\in[m,M]} \left[(\lambda-m)^{n+1} \left(e^{\lambda}-e^{m}\right) + (M-\lambda)^{n+1}\left(e^{M}-e^{\lambda}\right) \right] \\
&\le \frac{(M-m)^n}{(n+1)!} \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y \right\rangle \right) \left(e^{M}-e^{m}\right) \\
&\le \frac{(M-m)^n}{(n+1)!} \left(e^{M}-e^{m}\right) \|x\| \|y\|
\end{aligned} \tag{9.30}
$$

for any $x, y \in H$.

If we use the inequality (9.15) for the function $\exp$ we get the following bound as well

$$
\begin{aligned}
|E_n(A,m,M;x,y)| &\le \frac{e^M (M-m)^{n+1}}{(n+2)!} \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x,y \right\rangle \right) \\
&\le \frac{e^M (M-m)^{n+1}}{(n+2)!} \|x\| \|y\|
\end{aligned} \tag{9.31}
$$

for any $x, y \in H$.

9.3. Error Bounds for $f^{(n)}$ Absolutely Continuous. We consider the Lebesgue norms defined by

$$\|g\|_{[a,b],\infty} := ess \sup_{t\in[a,b]} |g(t)| \text{ if } g \in L_\infty [a,b]$$

and

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p \, dt \right)^{1/p} \text{ if } g \in L_p [a,b], p \ge 1.$$

THEOREM 9.5 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ and let n be an integer with $n \ge 1$. If* *the n-th derivative $f^{(n)}$ is absolutely continuous on $[m,M]$, then*

$$(9.32)\quad \begin{aligned} &\left|T_n(A,m,M;x,y)\right| \\ &\leq \frac{1}{(M-m)(n+1)!} \\ &\times \int_{m-0}^{M} \left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|\left|f^{(n+1)}(\lambda)\right|d\lambda. \\ &\leq \frac{1}{(M-m)(n+1)!} \\ &\times \begin{cases} B_{n,1}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],\infty} \text{ if } f^{(n)}\in L_\infty[m,M], \\ \\ B_{n,p}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],q} \text{ if } f^{(n)}\in L_q[m,M], p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ B_{n,\infty}(A,m,M;x,y)\left\|f^{(n)}\right\|_{[m,M],1}, \end{cases} \end{aligned}$$

for any $x,y\in H$, where

$$\begin{aligned} &B_{n,p}(A,m,M;x,y) \\ &:= \left(\int_{m-0}^{M}\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|^p d\lambda\right)^{1/p}, p\geq 1 \end{aligned}$$

and

$$\begin{aligned} &B_{n,\infty}(A,m,M;x,y) \\ &:= \sup_{t\in[m,M]}\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|. \end{aligned}$$

PROOF. Follows from the representation

$$\begin{aligned} &T_n(A,m,M;x,y) \\ &= \frac{(-1)^n}{(M-m)(n+1)!} \\ &\times \int_{m-0}^{M}\left[(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right]f^{(n+1)}(\lambda)\,d\lambda \end{aligned}$$

for any $x,y\in H$, by taking the modulus and utilizing the Hölder integral inequality.
The details are omitted. ∎

The bounds provided by $B_{n,p}(A,m,M;x,y)$ are not useful for applications, therefore we will establish in the following some simpler, however coarser bounds.

PROPOSITION 9.6 (Dragomir, 2010, [**8**]). *With the above notations, we have*

$$(9.33)\quad B_{n,\infty}(A,m,M;x,y)\leq (M-m)^{n+1}\|x\|\,\|y\|,$$

$$(9.34)\quad B_{n,1}(A,m,M;x,y)\leq \frac{(2^{n+2}-1)}{(n+2)\,2^{n+1}}(M-m)^{n+2}\|x\|\,\|y\|$$

and for $p > 1$

$$B_{n,p}(A,m,M;x,y) \le \frac{\left(2^{(n+1)p+1}-1\right)^{1/p}}{2^{n+1}\left[(n+1)p+1\right]^{1/p}}(M-m)^{n+1+1/p}\|x\|\|y\| \tag{9.35}$$

for any $x, y \in H$.

PROOF. Utilising the triangle inequality for the modulus we have

$$\begin{aligned} &\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right| \\ &\le(\lambda-m)^{n+1}|\langle(1_H-E_\lambda)x,y\rangle|+(M-\lambda)^{n+1}|\langle E_\lambda x,y\rangle| \\ &\le\max\left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}[|\langle(1_H-E_\lambda)x,y\rangle|+|\langle E_\lambda x,y\rangle|] \end{aligned} \tag{9.36}$$

for any $x, y \in H$.

Utilising the generalization of Schwarz's inequality for nonnegative selfadjoint operators we have

$$|\langle(1_H-E_\lambda)x,y\rangle|\le\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}$$

and

$$|\langle E_\lambda x,y\rangle|\le\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Further, by making use of the elementary inequality

$$ac+bd\le\left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2}, a,b,c,d\ge 0$$

we have

$$\begin{aligned} &|\langle(1_H-E_\lambda)x,y\rangle|+|\langle E_\lambda x,y\rangle| \\ &\le\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}+\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2} \\ &\le(\langle(1_H-E_\lambda)x,x\rangle+\langle E_\lambda x,x\rangle)^{1/2}(\langle(1_H-E_\lambda)y,y\rangle+\langle E_\lambda y,y\rangle)^{1/2} \\ &=\|x\|\|y\| \end{aligned} \tag{9.37}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Combining (9.36) with (9.37) we deduce that

$$\begin{aligned} &\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right| \\ &\le\max\left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}\|x\|\|y\| \end{aligned} \tag{9.38}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Taking the supremum over $\lambda \in [m, M]$ in (9.38) we deduce the inequality (9.33).

Now, if we take the power $r \ge 1$ in (9.38) and integrate, then we get

$$\begin{aligned} &\int_{m-0}^{M}\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle+(\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|^r d\lambda \\ &\le\|x\|^r\|y\|^r\int_m^M\max\left\{(\lambda-m)^{(n+1)r},(M-\lambda)^{(n+1)r}\right\}d\lambda \\ &=\|x\|^r\|y\|^r\left[\int_m^{\frac{M+m}{2}}(M-\lambda)^{(n+1)r}d\lambda+\int_{\frac{M+m}{2}}^M(\lambda-m)^{(n+1)r}d\lambda\right] \\ &=\frac{\left(2^{(n+1)r+1}-1\right)}{[(n+1)r+1]\,2^{(n+1)r}}(M-m)^{(n+1)r+1}\|x\|^r\|y\|^r \end{aligned} \tag{9.39}$$

for any $x, y \in H$.

Utilizing (9.39) for $r = 1$ we deduce the bound (9.34). Also, by making $r = p$ and then taking the power $1/p$, we deduce the last inequality (9.35). ∎

The following result provides refinements of the inequalities in Proposition 9.6:

PROPOSITION 9.7 (Dragomir, 2010, [**8**]). *With the above notations, we have*

$$\begin{aligned} &B_{n,\infty}(A,m,M;x,y) \\ &\leq \|y\| \\ &\times \max_{\lambda\in[m,M]}\left[(\lambda-m)^{2(n+1)}\langle(1_H-E_\lambda)x,x\rangle+(M-\lambda)^{2(n+1)}\langle E_\lambda x,x\rangle\right]^{1/2} \\ &\leq (M-m)^{n+1}\|x\|\|y\|, \end{aligned} \tag{9.40}$$

$$\begin{aligned} &B_{n,1}(A,m,M;x,y) \\ &\leq \|y\| \\ &\times \int_{m-0}^{M}\left[(\lambda-m)^{2(n+1)}\langle(1_H-E_\lambda)x,x\rangle+(M-\lambda)^{2(n+1)}\langle E_\lambda x,x\rangle\right]^{1/2}d\lambda \\ &\leq \frac{(2^{n+2}-1)}{(n+2)2^{n+1}}(M-m)^{n+2}\|x\|\|y\| \end{aligned} \tag{9.41}$$

and for $p > 1$

$$\begin{aligned} &B_{n,p}(A,m,M;x,y) \\ &\leq \|y\| \\ &\times \left(\int_{m-0}^{M}\left[(\lambda-m)^{2(n+1)}\langle(1_H-E_\lambda)x,x\rangle+(M-\lambda)^{2(n+1)}\langle E_\lambda x,x\rangle\right]^{p/2}d\lambda\right)^{1/p} \\ &\leq \frac{\left(2^{(n+1)p+1}-1\right)^{1/p}}{2^{n+1}\left[(n+1)p+1\right]^{1/p}}(M-m)^{n+1+1/p}\|x\|\|y\| \end{aligned} \tag{9.42}$$

for any $x, y \in H$.

PROOF. Utilising the Schwarz inequality in H, we have

$$\begin{aligned} &\left|\left\langle(\lambda-m)^{n+1}(1_H-E_\lambda)x+(\lambda-M)^{n+1}E_\lambda x,y\right\rangle\right| \\ &\leq \|y\|\left\|(\lambda-m)^{n+1}(1_H-E_\lambda)x+(\lambda-M)^{n+1}E_\lambda x\right\| \end{aligned} \tag{9.43}$$

for any $x, y \in H$.

Since E_λ are projectors for each $\lambda \in [m, M]$, then we have

$$\begin{aligned} &\left\|(\lambda-m)^{n+1}(1_H-E_\lambda)x+(\lambda-M)^{n+1}E_\lambda x\right\|^2 \\ &=(\lambda-m)^{2(n+1)}\|(1_H-E_\lambda)x\|^2 \\ &+2(\lambda-m)^{n+1}(\lambda-M)^{n+1}\operatorname{Re}\langle(1_H-E_\lambda)x,E_\lambda x\rangle \\ &+(M-\lambda)^{2(n+1)}\|E_\lambda x\|^2 \\ &=(\lambda-m)^{2(n+1)}\|(1_H-E_\lambda)x\|^2+(M-\lambda)^{2(n+1)}\|E_\lambda x\|^2 \\ &=(\lambda-m)^{2(n+1)}\langle(1_H-E_\lambda)x,x\rangle+(M-\lambda)^{2(n+1)}\langle E_\lambda x,x\rangle \\ &\leq \|x\|^2\max\left\{(\lambda-m)^{2(n+1)},(M-\lambda)^{2(n+1)}\right\} \end{aligned} \tag{9.44}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

On making use of (9.43) and (9.44) we obtain the following refinement of (9.38)

$$\begin{aligned} &\left|\left\langle (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x, y \right\rangle\right| \\ &\le \|y\| \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{1/2} \\ &\le \max\left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \|x\| \|y\| \end{aligned} \tag{9.45}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

The proof now follows the lines of the proof from Proposition 9.6 and we omit the details. ∎

REMARK 9.5. One can apply Theorem 9.5 and Proposition 9.6 for particular functions including the exponential and logarithmic function. However the details are left to the interested reader.

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CHAPTER 5

Inequalities of Trapezoidal Type

1. INTRODUCTION

From a complementary viewpoint to Ostrowski/mid-point inequalities, trapezoidal type inequality provide a priory error bounds in approximating the Riemann integral by a (generalized) trapezoidal formula.

Just like in the case of Ostrowski's inequality the development of these kind of results have registered a sharp growth in the last decade with more than 50 papers published, as one can easily asses this by performing a search with the key word "trapezoid" and "inequality" in the title of the papers reviewed by MathSciNet data base of the American Mathematical Society.

Numerous extensions, generalisations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

In the present chapter we present some recent results obtained by the author in extending trapezoidal type inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. As far as we know, the obtained results are new with no previous similar results ever obtained in the literature.

Applications for some elementary functions of operators such as the power function, the logarithmic and exponential functions are provided as well.

2. SCALAR TRAPEZOIDAL TYPE INEQUALITIES

In Classical Analysis a *trapezoidal type inequality* is an inequality that provides upper and/or lower bounds for the quantity

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt,$$

that is the error in approximating the integral by a trapezoidal rule, for various classes of integrable functions f defined on the compact interval $[a,b]$.

In order to introduce the reader to some of the well known results and prepare the background for considering a similar problem for functions of selfadjoint operators in Hilbert spaces, we mention the following inequalities.

The case of functions of bounded variation was obtained in [**2**] (see also [**1**, p. 68]):

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation. We have the inequality*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f), \tag{2.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is the best possible one.

This result may be improved if one assumes the monotonicity of f as follows (see [**1**, p. 76]):

THEOREM 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. Then we have the inequalities:*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \tag{2.2}$$
$$\leq \frac{1}{2}(b-a)[f(b)-f(a)] - \int_a^b sgn\left(t - \frac{a+b}{2}\right) f(t)\,dt$$
$$\leq \frac{1}{2}(b-a)[f(b)-f(a)].$$

The above inequalities are sharp.

If the mapping is Lipschitzian, then the following result holds as well [**3**] (see also [**1**, p. 82]).

THEOREM 2.3. *Let $f : [a, b] \to \mathbb{C}$ be an $L-$Lipschitzian function on $[a, b]$, i.e., f satisfies the condition:*

$$|f(s) - f(t)| \leq L|s-t| \quad \textit{for any } s, t \in [a, b] \quad (L > 0 \textit{ is given}). \tag{L}$$

Then we have the inequality:

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \leq \frac{1}{4}(b-a)^2 L. \tag{2.3}$$

The constant $\frac{1}{4}$ is best in (2.3).

If we would assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [**1**, p. 93].

THEOREM 2.4. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$. Then we have*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \tag{2.4}$$
$$\leq \begin{cases} \frac{1}{4}(b-a)^2 \|f'\|_\infty & \textit{if} \quad f' \in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{1+1/q}\|f'\|_p & \textit{if} \quad f' \in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a)\|f'\|_1, & \end{cases}$$

where $\|\cdot\|_p$ $(p \in [1, \infty])$ are the Lebesgue norms, i.e.,

$$\|f'\|_\infty = ess \sup_{s\in[a,b]} |f'(s)|$$

and

$$\|f'\|_p := \left(\int_a^b |f'(s)|\,ds\right)^{\frac{1}{p}}, \quad p \geq 1.$$

The case of convex functions is as follows [**4**]:

THEOREM 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}\frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]&\\ \le\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt&\\ \le\frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right].&\end{aligned}\tag{2.5}$$

The constant $\frac{1}{8}$ is sharp in both sides of (2.5).

For other scalar trapezoidal type inequalities, see [**1**].

3. TRAPEZOIDAL VECTOR INEQUALITIES

3.1. Some General Results. With the notations introduced above, we consider in this paper the problem of bounding the error

$$\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle$$

in approximating $\langle f(A)x,y\rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle$, where x,y are vectors in the Hilbert space H, f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m,M]$. Applications for some particular elementary functions are also provided. The following result holds:

THEOREM 3.1 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\begin{aligned}&\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right|\\ &\le\frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.\\ &\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\bigvee_m^M(f)\\ &\le\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)\end{aligned}\tag{3.1}$$

for any $x,y\in H$.

PROOF. If $f,u:[m,M]\to\mathbb{C}$ are such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then a simple integration by parts reveals the identity

$$\begin{aligned}\int_a^b f(t)\,du(t)=&\frac{f(a)+f(b)}{2}[u(b)-u(a)]\\ &-\int_a^b\left[u(t)-\frac{u(a)+u(b)}{2}\right]df(t).\end{aligned}\tag{3.2}$$

If we write the identity (3.2) for $u(\lambda) = \langle E_\lambda x, y\rangle$, then we get

$$\int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle) = \frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle - \int_{m-0}^{M}\left(\langle E_\lambda x, y\rangle - \frac{1}{2}\langle x,y\rangle\right) df(\lambda)$$

which gives the following identity of interest in itself

$$\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle - \langle f(A)x,y\rangle = \frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x, y\rangle + \langle (E_\lambda - 1_H)x,y\rangle\right] df(\lambda), \tag{3.3}$$

for any $x, y \in H$.

It is well known that if $p : [a,b] \to \mathbb{C}$ is a continuous function and $v : [a,b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]}|p(t)| \bigvee_a^b(v) \tag{3.4}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising the property (3.4), we have from (3.3) that

$$\begin{aligned}\left|\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle - \langle f(A)x,y\rangle\right| &\le \frac{1}{2}\max_{\lambda\in[m,M]}\left|\langle E_\lambda x,y\rangle + \langle (E_\lambda - 1_H)x,y\rangle\right| \bigvee_m^M(f)\\ &\le \frac{1}{2}\left[\max_{\lambda\in[m,M]}\left[\left|\langle E_\lambda x,y\rangle\right| + \left|\langle (1_H - E_\lambda)x,y\rangle\right|\right]\right] \bigvee_m^M(f).\end{aligned} \tag{3.5}$$

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in the Hilbert space H

$$|\langle Px,y\rangle|^2 \le \langle Px,x\rangle\langle Py,y\rangle, \tag{3.6}$$

for any $x, y \in H$.

On applying the inequality (3.6) we have

$$|\langle E_\lambda x,y\rangle| \le \langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$

and

$$|\langle (1_H - E_\lambda)x,y\rangle| \le \langle (1_H - E_\lambda)x,x\rangle^{1/2}\langle (1_H - E_\lambda)y,y\rangle^{1/2},$$

which, together with the elementary inequality for $a, b, c, d \ge 0$

$$ab + cd \le \left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2}$$

produce the inequalities

$$
\begin{aligned}
(3.7)\qquad & |\langle E_\lambda x, y\rangle| + |\langle (1_H - E_\lambda)\, x, y\rangle| \\
& \leq \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} + \langle (1_H - E_\lambda)\, x, x\rangle^{1/2} \langle (1_H - E_\lambda)\, y, y\rangle^{1/2} \\
& \leq (\langle E_\lambda x, x\rangle + \langle (1_H - E_\lambda)\, x, x\rangle)(\langle E_\lambda y, y\rangle + \langle (1_H - E_\lambda)\, y, y\rangle) \\
& = \|x\| \, \|y\|
\end{aligned}
$$

for any $x, y \in H$.

On utilizing (3.5) and taking the maximum in (3.7) we deduce the desired result (3.1). ∎

The case of Lipschitzian functions may be useful for applications:

THEOREM 3.2 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$
\begin{aligned}
(3.8)\qquad & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y\rangle - \langle f(A)\, x, y\rangle \right| \\
& \leq \frac{1}{2} L \int_{m-0}^{M} \Big[\langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \\
& + \langle (1_H - E_\lambda)\, x, x\rangle^{1/2} \langle (1_H - E_\lambda)\, y, y\rangle^{1/2} \Big] d\lambda \\
& \leq \frac{1}{2} (M - m)\, L \, \|x\| \, \|y\|
\end{aligned}
$$

for any $x, y \in H$.

PROOF. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$
|f(s) - f(t)| \leq L\, |s - t| \text{ for any } t, s \in [a, b],
$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$
\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.
$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (3.3) that

$$
\begin{aligned}
(3.9)\qquad & \left| \frac{f(m) + f(M)}{2} \cdot \langle x, y\rangle - \langle f(A)\, x, y\rangle \right| \\
& \leq \frac{1}{2} L \int_{m-0}^{M} |\langle E_\lambda x, y\rangle + \langle (E_\lambda - 1_H)\, x, y\rangle|\, d\lambda, \\
& \leq \frac{1}{2} L \int_{m-0}^{M} [|\langle E_\lambda x, y\rangle| + |\langle (1_H - E_\lambda)\, x, y\rangle|]\, d\lambda,
\end{aligned}
$$

for any $x, y \in H$.

Further, integrating (3.7) on $[m, M]$ we have

$$\begin{aligned}
&\int_{m-0}^{M} \left[\left| \langle E_\lambda x, y \rangle \right| + \left| \langle (1_H - E_\lambda) x, y \rangle \right| \right] d\lambda \\
&\leq \int_{m-0}^{M} \Big[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \\
&+ \langle (1_H - E_\lambda) x, x \rangle^{1/2} \langle (1_H - E_\lambda) y, y \rangle^{1/2} \Big] d\lambda \\
&\leq (M - m) \|x\| \|y\|
\end{aligned} \tag{3.10}$$

which together with (3.9) produces the desired result (3.8). ∎

3.2. Other Trapezoidal Vector Inequalities. The following result provides a different perspective in bounding the error in the trapezoidal approximation:

THEOREM 3.3 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$. Then we have the inequalities*

$$\begin{aligned}
&\left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\
&\leq \begin{cases} \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\ L \int_{m-0}^{M} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| d\lambda & \text{if } f \text{ is } L \text{ Lipschitzian} \\ \int_{m-0}^{M} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| df(\lambda) & \text{if } f \text{ is nondecreasing} \end{cases} \\
&\leq \frac{1}{2} \|x\| \|y\| \begin{cases} \bigvee_m^M (f) & \text{if } f \text{ is of bounded variation} \\ L(M - m) & \text{if } f \text{ is } L \text{ Lipschitzian} \\ (f(M) - f(m)) & \text{if } f \text{ is nondecreasing} \end{cases}
\end{aligned} \tag{3.11}$$

for any $x, y \in H$.

PROOF. From (3.5) we have that

$$\begin{aligned}
&\left| \frac{f(m) + f(M)}{2} \cdot \langle x, y \rangle - \langle f(A) x, y \rangle \right| \\
&\leq \frac{1}{2} \max_{\lambda \in [m,M]} \left| \langle E_\lambda x, y \rangle + \langle (E_\lambda - 1_H) x, y \rangle \right| \bigvee_m^M (f) \\
&= \max_{\lambda \in [m,M]} \left| \left\langle E_\lambda x - \frac{1}{2} x, y \right\rangle \right| \bigvee_m^M (f)
\end{aligned} \tag{3.12}$$

for any $x, y \in H$.

Utilising the Schwarz inequality in H and the fact that E_λ are projectors we have successively

$$\left|\left\langle E_\lambda x-\frac{1}{2}x,y\right\rangle\right| \le \left\|E_\lambda x-\frac{1}{2}x\right\|\|y\| \tag{3.13}$$
$$=\left[\langle E_\lambda x,E_\lambda x\rangle-\langle E_\lambda x,x\rangle+\frac{1}{4}\|x\|^2\right]^{1/2}\|y\|$$
$$=\frac{1}{2}\|x\|\|y\|$$

for any $x,y\in H$, which proves the first branch in (3.11).

The second inequality follows from (3.9).

From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t). \tag{3.14}$$

From the representation (3.3) we then have

$$\left|\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{3.15}$$
$$\le\frac{1}{2}\int_{m-0}^{M}|\langle E_\lambda x,y\rangle+\langle(E_\lambda-1_H)x,y\rangle|\,df(\lambda)$$
$$=\int_{m-0}^{M}\left|\left\langle E_\lambda x-\frac{1}{2}x,y\right\rangle\right|df(\lambda)$$

for any $x,y\in H$, from which we obtain the last branch in (3.11). ∎

We recall that a function $f:[a,b]\to\mathbb{C}$ is called $r-H$-Hölder continuous with fixed $r\in(0,1]$ and $H>0$ if

$$|f(t)-f(s)|\le H|t-s|^r \text{ for any } t,s\in[a,b].$$

We have the following result concerning this class of functions.

THEOREM 3.4 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is $r-H$-Hölder continuous on $[m,M]$, then we have the inequality*

$$\left|\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{3.16}$$
$$\le\frac{1}{2^r}H(M-m)^r\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\le\frac{1}{2^r}H(M-m)^r\|x\|\|y\|$$

for any $x,y\in H$.

PROOF. We start with the equality

$$\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle = \int_{m-0}^{M}\left[\frac{f(M)+f(m)}{2}-f(\lambda)\right]d(\langle E_\lambda x,y\rangle) \tag{3.17}$$

for any $x, y \in H$, that follows from the spectral representation theorem.

Since the function $\left\langle E_{(\cdot)}x, y\right\rangle$ is of bounded variation for any vector $x, y \in H$, by applying the inequality (3.4) we conclude that

$$\left|\frac{f(m)+f(M)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \le \max_{\lambda\in[m,M]}\left|\frac{f(M)+f(m)}{2}-f(\lambda)\right|\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \tag{3.18}$$

for any $x, y \in H$.

As $f : [m, M] \to \mathbb{C}$ is $r - H$-Hölder continuous on $[m, M]$, then we have

$$\left|\frac{f(M)+f(m)}{2}-f(\lambda)\right| \le \frac{1}{2}|f(M)-f(\lambda)|+\frac{1}{2}|f(\lambda)-f(m)| \le \frac{1}{2}H\left[(M-\lambda)^r+(\lambda-m)^r\right] \tag{3.19}$$

for any $\lambda \in [m, M]$.

Since, obviously, the function $g_r(\lambda) := (M-\lambda)^r + (\lambda-m)^r$, $r \in (0,1)$ has the property that

$$\max_{\lambda\in[m,M]} g_r(\lambda) = g_r\left(\frac{m+M}{2}\right) = 2^{1-r}(M-m)^r,$$

then by (3.18) we deduce the first part of (3.16).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

3.3. Applications for Some Particular Functions. It is obvious that the results established above can be applied for various particular functions of selfadjoint operators. We restrict ourselves here to only two examples, namely the logarithm and the power functions.

1. If we consider the logarithmic function $f : (0,\infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state the following result:

PROPOSITION 3.5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have*

$$\left|\langle x,y\rangle\ln\sqrt{mM}-\langle \ln Ax,y\rangle\right| \le \ln\left(\frac{M}{m}\right)\times\begin{cases}\frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.\\ \quad\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\\ \\ \max_{\lambda\in[m,M]}\left|\left\langle E_\lambda x-\frac{1}{2}x,y\right\rangle\right|\end{cases} \le \frac{1}{2}\|x\|\,\|y\|\ln\left(\frac{M}{m}\right) \tag{3.20}$$

and

$$\left|\langle x,y\rangle \ln\sqrt{mM} - \langle \ln Ax, y\rangle\right| \tag{3.21}$$
$$\leq \frac{1}{m}\times\begin{cases} \frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2} \right. \\ \left. +\langle (1_H - E_\lambda)x,x\rangle^{1/2}\langle (1_H - E_\lambda)y,y\rangle^{1/2}\right] d\lambda \\ \\ \int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| d\lambda \end{cases}$$
$$\leq \frac{1}{2}\|x\|\,\|y\|\left(\frac{M}{m}-1\right)$$

and

$$\left|\langle x,y\rangle \ln\sqrt{mM} - \langle \ln Ax, y\rangle\right| \leq \int_{m-0}^{M}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right|\lambda^{-1} d\lambda \leq \frac{1}{2}\|x\|\,\|y\|\ln\left(\frac{M}{m}\right) \tag{3.22}$$

respectively.

The proof is obvious from Theorems 3.1, 3.2 and 3.3 applied for the logarithmic function. The details are omitted.

2. Consider now the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^p$ with $p\in(-\infty,0)\cup(0,\infty)$. In the case when $p\in(0,1)$, the function is $p-H$-Hölder continuous with $H=1$ on any subinterval $[m,M]$ of $[0,\infty)$. By making use of Theorem 3.4 we can state the following result:

PROPOSITION 3.6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0\leq m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p\in(0,1)$ we have*

$$\left|\frac{m^p+M^p}{2}\cdot\langle x,y\rangle - \langle A^p x,y\rangle\right| \leq \frac{1}{2^p}(M-m)^p \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2^p}(M-m)^p\|x\|\,\|y\|, \tag{3.23}$$

for any $x,y\in H$.

The case of powers $p\geq 1$ is embodied in the following:

PROPOSITION 3.7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0\leq m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for $p\geq 1$ and for any $x,y\in H$ we have*

$$\left|\frac{m^p+M^p}{2}\cdot\langle x,y\rangle - \langle A^p x,y\rangle\right| \tag{3.24}$$
$$\leq (M^p-m^p)\times\begin{cases} \frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right. \\ \left. +\langle (1_H - E_\lambda)x,x\rangle^{1/2}\langle (1_H - E_\lambda)y,y\rangle^{1/2}\right] \\ \\ \max_{\lambda\in[m,M]}\left|\left\langle E_\lambda x - \frac{1}{2}x, y\right\rangle\right| \end{cases}$$
$$\leq \frac{1}{2}\|x\|\,\|y\|\,(M^p-m^p)$$

and

$$\left|\frac{m^p+M^p}{2}\cdot\langle x,y\rangle-\langle A^p x,y\rangle\right| \tag{3.25}$$
$$\leq pM^{p-1}\times\begin{cases}\frac{1}{2}\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.\\ \quad\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]d\lambda\\ \\ \int_{m-0}^{M}\left|\left\langle E_\lambda x-\frac{1}{2}x,y\right\rangle\right|d\lambda\end{cases}$$
$$\leq\frac{1}{2}p\|x\|\|y\|M^{p-1}$$

and

$$\left|\frac{m^p+M^p}{2}\cdot\langle x,y\rangle-\langle A^p x,y\rangle\right|\leq p\int_{m-0}^{M}\left|\left\langle E_\lambda x-\frac{1}{2}x,y\right\rangle\right|\lambda^{p-1}d\lambda \tag{3.26}$$
$$\leq\frac{1}{2}\|x\|\|y\|(M^p-m^p)$$

respectively.

The proof is obvious from Theorems 3.1, 3.2 and 3.3 applied for the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^p$ with $p\geq 1$. The details are omitted.

The case of negative powers is similar. The details are left to the interested reader.

4. GENERALISED TRAPEZOIDAL INEQUALITIES

4.1. Some Vector Inequalities. In the present section we are interested in providing error bounds for approximating $\langle f(A)x,y\rangle$ with the quantity

$$\frac{1}{M-m}\left[f(m)(M\langle x,y\rangle-\langle Ax,y\rangle)+f(M)(\langle Ax,y\rangle-m\langle x,y\rangle)\right] \tag{4.1}$$

where $x,y\in H$, which is a generalized trapezoid formula. Applications for some particular functions are provided as well. The following representation is of interest in itself and will be useful in deriving our inequalities later as well:

LEMMA 4.1 (Dragomir, 2010, [**6**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function on $[m,M]$, then we have the representation*

$$\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle \tag{4.2}$$
$$=\int_{m-0}^{M}\langle E_t x,y\rangle\,df(t)-\frac{f(M)-f(m)}{M-m}\int_{m-0}^{M}\langle E_t x,y\rangle\,dt$$
$$=\int_{m-0}^{M}\left[\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right]df(t)$$

for any $x,y\in H$.

PROOF. Integrating by parts and utilizing the spectral representation theorem we have

$$\int_{m-0}^{M}\langle E_t x,y\rangle\,df(t)=f(M)\langle x,y\rangle-\int_{m-0}^{M}f(t)\,d\langle E_t x,y\rangle$$
$$=f(M)\langle x,y\rangle-\langle f(A)x,y\rangle$$

and

$$\int_{m-0}^{M} \langle E_t x, y \rangle \, dt = M \langle x, y \rangle - \langle Ax, y \rangle$$

for any $x, y \in H$.

Therefore

$$\begin{aligned}
&\int_{m-0}^{M} \langle E_t x, y \rangle \, df(t) - \frac{f(M) - f(m)}{M - m} \int_{m-0}^{M} \langle E_t x, y \rangle \, dt \\
&= f(M) \langle x, y \rangle - \langle f(A) x, y \rangle - \frac{f(M) - f(m)}{M - m} (M \langle x, y \rangle - \langle Ax, y \rangle) \\
&= \frac{1}{M - m} [f(m) (M \langle x, y \rangle - \langle Ax, y \rangle) + f(M) (\langle Ax, y \rangle - m \langle x, y \rangle)] \\
&- \langle f(A) x, y \rangle
\end{aligned}$$

for any $x, y \in H$, which proves the first equality in (4.2).

The second equality is obvious. ∎

The following result provides error bounds in approximating $\langle f(A) x, y \rangle$ by the generalized trapezoidal rule (4.1):

THEOREM 4.2 (Dragomir, 2010, [**6**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

1. If $f : [m, M] \to \mathbb{C}$ is of bounded variation on $[m, M]$, then

$$\begin{aligned}
(4.3) \quad &\left| \left\langle \left[\frac{f(m)(M 1_H - A) + f(M)(A - m 1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\
&\leq \sup_{t \in [m, M]} \left[\frac{t - m}{M - m} \bigvee_{m-0}^{t} \left(\langle E_{(\cdot)} x, y \rangle \right) + \frac{M - t}{M - m} \bigvee_{t}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \right] \bigvee_{m}^{M} (f) \\
&\leq \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \bigvee_{m}^{M} (f) \leq \|x\| \|y\| \bigvee_{m}^{M} (f)
\end{aligned}$$

for any $x, y \in H$.

2. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\begin{aligned}
(4.4) \quad &\left| \left\langle \left[\frac{f(m)(M 1_H - A) + f(M)(A - m 1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\
&\leq L \int_{m}^{M} \left[\frac{t - m}{M - m} \bigvee_{m-0}^{t} \left(\langle E_{(\cdot)} x, y \rangle \right) + \frac{M - t}{M - m} \bigvee_{t}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \right] dt \\
&\leq L (M - m) \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right) \leq L (M - m) \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$.

3. If $f:[m,M]\to\mathbb{R}$ *is monotonic nondecreasing on* $[m,M]$*, then*

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{4.5}$$
$$\le\int_m^M\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]df(t)$$
$$\le\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)[f(M)-f(m)]\le\|x\|\|y\|[f(M)-f(m)]$$

for any $x,y\in H$.

PROOF. It is well known that if $p:[a,b]\to\mathbb{C}$ is a bounded function, $v:[a,b]\to\mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, then the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{4.6}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Applying this property to the equality (4.2), we have

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{4.7}$$
$$\le\sup_{t\in[m,M]}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right|\bigvee_m^M(f)$$

for any $x,y\in H$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds \tag{4.8}$$
$$=\frac{1}{M-m}\left[\int_{m-0}^{t}(s-m)\,d\langle E_sx,y\rangle+\int_t^M(s-M)\,d\langle E_sx,y\rangle\right]$$

that holds for any $t\in[m,M]$ and for any $x,y\in H$.

Since the function $v(s):=\langle E_sx,y\rangle$ is of bounded variation on $[m,M]$ for any $x,y\in H$, then on applying the inequality (4.6) once more, we get

$$\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right| \tag{4.9}$$
$$\le\frac{1}{M-m}\left[\left|\int_{m-0}^{t}(s-m)\,d\langle E_sx,y\rangle\right|+\left|\int_t^M(s-M)\,d\langle E_sx,y\rangle\right|\right]$$
$$\le\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$

that holds for any $t\in[m,M]$ and for any $x,y\in H$.

Now, taking the supremum in (4.9) and taking into account that

$$\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right),\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$

for any $t\in[m,M]$ and for any $x,y\in H,$ we deduce the first and the second inequality in (4.3).

The last part of (4.3) follows by the Total Variation Schwarz's inequality and we omit the details.

Now, recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\leq L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (4.2) that

$$\begin{aligned}(4.10)\qquad &\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right|\\ &\leq L\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right|dt\end{aligned}$$

for any $x,y\in H.$

Further on, by utilizing (4.8) we can state that

$$\begin{aligned}&\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right|dt\\ &\leq\frac{1}{M-m}\int_{m-0}^{M}\left[\left|\int_{m-0}^{t}(s-m)\,d\langle E_sx,y\rangle\right|+\left|\int_{t}^{M}(s-M)\,d\langle E_sx,y\rangle\right|\right]dt\\ &\leq\int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]dt\\ &\leq(M-m)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

for any $x,y\in H,$ which proves the desired result (4.4).

From the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b|p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\int_a^b|p(t)|\,dv(t).$$

From the representation (4.2) we then have

$$\begin{aligned}(4.11)\qquad &\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right|\\ &\leq\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right|df(t)\end{aligned}$$

for any $x, y \in H$.

Further on, by utilizing (4.8) we can state that

$$\begin{aligned}
&\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle\, ds\right| df(t) \\
&\leq \frac{1}{M-m}\int_{m-0}^{M}\left[\left|\int_{m-0}^{t}(s-m)\, d\langle E_s x, y\rangle\right| + \left|\int_{t}^{M}(s-M)\, d\langle E_s x, y\rangle\right|\right] df(t) \\
&\leq \int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right] df(t) \\
&\leq (f(M) - f(m))\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)
\end{aligned}$$

for any $x, y \in H$, which proves the desired result (4.5). ∎

A different approach for Lipschitzian functions is incorporated in:

THEOREM 4.3 (Dragomir, 2010, [**6**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then*

$$\begin{aligned}
&\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \\
&\leq L\|y\|\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \leq \frac{1}{2}L(M-m)\|x\|\|y\|
\end{aligned} \tag{4.12}$$

for any $x, y \in H$.

PROOF. We will use the inequality (4.10) for which a different upper bound will be provided.

By the Schwarz inequality in H we have that

$$\begin{aligned}
&\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle\, ds\right| dt \\
&= \int_{m-0}^{M}\left|\left\langle\left[E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right], y\right\rangle\right| dt \\
&\leq \|y\|\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt
\end{aligned} \tag{4.13}$$

for any $x, y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$\begin{aligned}
&\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \\
&\leq (M-m)^{1/2}\left(\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt\right)^{1/2}
\end{aligned} \tag{4.14}$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$\frac{1}{M-m}\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt = \frac{1}{M-m}\int_{m-0}^{M}\|E_t x\|^2 dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 \tag{4.15}$$

and

$$\frac{1}{M-m}\int_{m-0}^{M}\|E_t x\|^2 dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 = \frac{1}{M-m}\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt \tag{4.16}$$

for any $x\in H$.

By (4.14), (4.15) and (4.16) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \le (M-m)^{1/2}\left(\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt\right)^{1/2} \tag{4.17}$$

for any $x\in H$.

On making use of the Schwarz inequality in H we also have

$$\begin{aligned}\int_{m-0}^{M}&\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt \\ &\le \int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|\left\|E_t x-\frac{1}{2}x\right\| dt \\ &= \frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt,\end{aligned} \tag{4.18}$$

where we used the fact that E_t are projectors, and in this case we have

$$\begin{aligned}\left\|E_t x-\frac{1}{2}x\right\|^2 &= \|E_t x\|^2-\langle E_t x, x\rangle+\frac{1}{4}\|x\|^2 \\ &= \left\langle E_t^2 x, x\right\rangle-\langle E_t x, x\rangle+\frac{1}{4}\|x\|^2=\frac{1}{4}\|x\|^2\end{aligned}$$

for any $t\in[m,M]$ for any $x\in H$.

From (4.17) and (4.18) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \le (M-m)^{1/2}\left(\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt\right)^{1/2} \tag{4.19}$$

which is clearly equivalent with the following inequality of interest in itself

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \le \frac{1}{2}\|x\|(M-m) \tag{4.20}$$

for any $x \in H$.

This proves the last part of (4.12). ∎

4.2. Applications for Particular Functions. It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^p$ with $p\neq 0$. On applying Theorem 4.3 we can state the following proposition:

PROPOSITION 4.4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $0\leq m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x,y\in H$ we have the inequalities*

$$
\begin{aligned}
(4.21)\qquad & \left|\left\langle\left[\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle A^p x,y\rangle\right| \\
&\leq B_p\|y\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt \\
&\leq \frac{1}{2}B_p(M-m)\|x\|\|y\|
\end{aligned}
$$

where

$$
B_p=p\times\begin{cases} M^{p-1} & \textit{if } p\geq 1\\ m^{p-1} & \textit{if } 0<p<1, m>0\end{cases}
$$

and

$$
B_p=(-p)\,m^{p-1}\ \textit{if } p<0, m>0.
$$

2. The case of logarithmic function is as follows:

PROPOSITION 4.5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $0<m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x,y\in H$ we have the inequalities*

$$
\begin{aligned}
(4.22)\qquad & \left|\left\langle\left[\frac{(M1_H-A)\ln m+(A-m1_H)\ln M}{M-m}\right]x,y\right\rangle-\langle \ln A x,y\rangle\right| \\
&\leq \frac{1}{m}\|y\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt \\
&\leq \frac{1}{2}\left(\frac{M}{m}-1\right)\|x\|\|y\|.
\end{aligned}
$$

5. MORE GENERALISED TRAPEZOIDAL INEQUALITIES

5.1. Other Vector Inequalities. The following result for general continuous functions holds:

THEOREM 5.1 (Dragomir, 2010, [**7**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its*

spectral family. If $f:[m,M]\to\mathbb{R}$ *is continuous on* $[m,M]$ *, then we have the inequalities:*

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{5.1}$$
$$\leq\left[\max_{t\in[m,M]}f(t)-\min_{t\in[m,M]}f(t)\right]\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq\left[\max_{t\in[m,M]}f(t)-\min_{t\in[m,M]}f(t)\right]\|x\|\,\|y\|$$

for any $x,y\in H$.

PROOF. We observe that, by the spectral representation theorem, we have the equality

$$\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle \tag{5.2}$$
$$=\int_{m-0}^{M}\Phi_f(t)\,d(\langle E_t x,y\rangle)$$

for any $x,y\in H$, where $\Phi_f:[m,M]\to\mathbb{R}$ is given by

$$\Phi_f(t)=\frac{1}{M-m}[(M-t)f(m)+(t-m)f(M)]-f(t).$$

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{5.3}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Now, if we denote by $\gamma:=\min_{t\in[m,M]}f(t)$ and by $\Gamma:=\max_{t\in[m,M]}f(t)$ then we have

$$\gamma(M-t)\leq(M-t)f(m)\leq\Gamma(M-t),$$
$$\gamma(t-m)\leq(t-m)f(M)\leq\Gamma(t-m)$$

and

$$-(M-m)\Gamma\leq-(M-m)f(t)\leq-\gamma(M-m)$$

for any $t\in[m,M]$. If we add these three inequalities, then we get

$$-(M-m)(\Gamma-\gamma)\leq(M-m)\Phi_f(t)\leq(M-m)(\Gamma-\gamma)$$

for any $t\in[m,M]$, which shows that

$$|\Phi_f(t)|\leq\Gamma-\gamma \text{ for any } t\in[m,M]. \tag{5.4}$$

On applying the inequality (5.3) for the representation (5.2) we have from (5.4) that

$$\left|\int_{m-0}^{M}\Phi_f(t)\,d(\langle E_t x,y\rangle)\right|\leq(\Gamma-\gamma)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$

for any $x,y\in H$, which proves the first part of (5.1).

The last part of (5.1) follows by the Total Variation Schwarz's inequality and we omit the details. ∎

When the generating function is of bounded variation, we have the following result.

THEOREM 5.2 (Dragomir, 2010, [7]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then we have the inequalities:*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \tag{5.5}$$
$$\leq \max_{t \in [m,M]} \left[\frac{M - t}{M - m} \bigvee_m^t (f) + \frac{t - m}{M - m} \bigvee_t^M (f) \right] \bigvee_{m-0}^M \left(\langle E_{(\cdot)} x, y \rangle \right)$$
$$\leq \bigvee_{m-0}^M \left(\langle E_{(\cdot)} x, y \rangle \right) \bigvee_m^M (f) \leq \bigvee_m^M (f) \|x\| \|y\|$$

for any $x, y \in H$.

PROOF. First of all, observe that

$$(M - m) \Phi_f(t) = (t - M) [f(t) - f(m)] + (t - m) [f(M) - f(t)] \tag{5.6}$$
$$= (t - M) \int_m^t df(s) + (t - m) \int_t^M df(s)$$

for any $t \in [m, M]$.

Therefore

$$|\Phi_f(t)| \leq \frac{M - t}{M - m} \left| \int_m^t df(s) \right| + \frac{t - m}{M - m} \left| \int_t^M df(s) \right| \tag{5.7}$$
$$\leq \frac{M - t}{M - m} \bigvee_m^t (f) + \frac{t - m}{M - m} \bigvee_t^M (f)$$
$$\leq \max \left\{ \frac{M - t}{M - m}, \frac{t - m}{M - m} \right\} \left[\bigvee_m^t (f) + \bigvee_t^M (f) \right]$$
$$= \left[\frac{1}{2} + \frac{\left| t - \frac{m+M}{2} \right|}{M - m} \right] \bigvee_m^M (f)$$

for any $t \in [m, M]$, which implies that

$$\max_{t \in [m,M]} |\Phi_f(t)| \leq \max_{t \in [m,M]} \left[\frac{M - t}{M - m} \bigvee_m^t (f) + \frac{t - m}{M - m} \bigvee_t^M (f) \right] \tag{5.8}$$
$$\leq \max_{t \in [m,M]} \left[\frac{1}{2} + \frac{\left| t - \frac{m+M}{2} \right|}{M - m} \right] \bigvee_m^M (f) = \bigvee_m^M (f).$$

On applying the inequality (5.3) for the representation (5.2) we have from (5.8) that

$$\begin{aligned}&\left|\int_{m-0}^{M}\Phi_f(t)\,d(\langle E_t x,y\rangle)\right|\\&\le\max_{t\in[m,M]}\left[\frac{M-t}{M-m}\bigvee_m^t(f)+\frac{t-m}{M-m}\bigvee_t^M(f)\right]\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\&\le\bigvee_m^M(f)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

for any $x, y \in H$, which produces the desired result (5.5). ∎

The case of Lipschitzian functions is as follows:

THEOREM 5.3 (Dragomir, 2010, [7]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then we have the inequalities:*

$$\begin{aligned}(5.9)\quad&\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right|\\&\le\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\&\times\max_{t\in[m,M]}\left[\frac{M-t}{M-m}|f(t)-f(m)|+\frac{t-m}{M-m}|f(M)-f(t)|\right]\\&\le\frac{1}{2}(M-m)L\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{2}(M-m)L\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

PROOF. We have from the first part of the equality (5.6) that

$$\begin{aligned}(5.10)\quad|\Phi_f(t)|&\le\frac{M-t}{M-m}|f(t)-f(m)|+\frac{t-m}{M-m}|f(M)-f(t)|\\&\le\frac{2L}{M-m}(M-t)(t-m)\le\frac{1}{2}(M-m)L\end{aligned}$$

for any $t \in [m, M]$, which, by a similar argument to the one from the above Theorem 5.2, produces the desired result (5.9). The details are omitted. ∎

The following corollary holds:

COROLLARY 5.4 (Dragomir, 2010, [7]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $l, L \in \mathbb{R}$ are such that $L > l$ and $f : [m, M] \to \mathbb{R}$ is (l, L) $-$Lipschitzian on $[m, M]$, then we have the inequalities:*

$$\begin{aligned}(5.11)\quad&\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right|\\&\le\frac{1}{4}(M-m)(L-l)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{4}(M-m)(L-l)\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

PROOF. Follows by applying the inequality (5.9) to the $\frac{1}{2}(L-l)$-Lipschitzian function $f-\frac{1}{2}(l+L)e$, where $e(t)=t$, $t\in[m,M]$. The details are omitted. ∎

When the generating function is continuous convex, we can state the following result as well:

THEOREM 5.5 (Dragomir, 2010, [7]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is continuous convex on $[m,M]$ with finite lateral derivatives $f'_-(M)$ and $f'_+(m)$, then we have the inequalities:*

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{5.12}$$
$$\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]\|x\|\,\|y\|$$

for any $x,y\in H$.

PROOF. By the convexity of f on $[m,M]$ we have

$$f(t)-f(M)\geq f'_-(M)(t-M)$$

for any $t\in[m,M]$. If we multiply this inequality with $t-m\geq0$ we deduce

$$(t-m)f(t)-(t-m)f(M)\geq f'_-(M)(t-M)(t-m) \tag{5.13}$$

for any $t\in[m,M]$.

Similarly, we get

$$(M-t)f(t)-(M-t)f(m)\geq f'_+(m)(M-t)(t-m) \tag{5.14}$$

for any $t\in[m,M]$.

Summing the above inequalities and dividing by $M-m$ we deduce the inequality

$$\Phi_f(t)\leq\frac{(M-t)(t-m)}{M-m}\left[f'_-(M)-f'_+(m)\right] \tag{5.15}$$
$$\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]$$

for any $t\in[m,M]$.

By the convexity of f we also have that

$$\frac{1}{M-m}\left[(M-t)f(m)+(t-m)f(M)\right] \tag{5.16}$$
$$\geq f\left(\frac{(M-t)m+(t-m)M}{M-m}\right)$$
$$=f(t)$$

giving that

$$\Phi_f(t)\geq0\text{ for any }t\in[m,M]. \tag{5.17}$$

Utilising (5.3) for the representation (5.2) we deduce from (5.15) and (5.17) the desired result (5.12). ∎

5.2. Inequalities in the Operator Order. The following result providing some inequalities in the operator order may be stated:

THEOREM 5.6 (Dragomir, 2010, [7]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

1. If $f : [m, M] \to \mathbb{R}$ is continuous on $[m, M]$, then

$$\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A)\right| \leq \left[\max_{t\in[m,M]} f(t) - \min_{t\in[m,M]} f(t)\right] 1_H. \tag{5.18}$$

2. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\begin{aligned} &\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A)\right| \\ &\leq \frac{M1_H - A}{M - m} \bigvee_m^A (f) + \frac{A - m1_H}{M - m} \bigvee_A^M (f) \\ &\leq \left[\frac{1}{2} + \frac{\left|A - \frac{m+M}{2} 1_H\right|}{M - m}\right] \bigvee_m^M (f), \end{aligned} \tag{5.19}$$

where $\bigvee_m^A (f)$ denotes the operator generated by the scalar function $[m, M] \ni t \longmapsto \bigvee_m^t (f) \in \mathbb{R}$. The same notation applies for $\bigvee_A^M (f)$.

3. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\begin{aligned} &\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A)\right| \\ &\leq \frac{M1_H - A}{M - m} |f(A) - f(m) 1_H| + \frac{A - m1_H}{M - m} |f(M) 1_H - f(A)| \\ &\leq \frac{1}{2} (M - m) L1_H. \end{aligned} \tag{5.20}$$

4. If $f : [m, M] \to \mathbb{R}$ is continuous convex on $[m, M]$ with finite lateral derivatives $f'_-(M)$ and $f'_+(m)$, then we have the inequalities:

$$\begin{aligned} 0 &\leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \\ &\leq \frac{(M1_H - A)(A - m1_H)}{M - m} \left[f'_-(M) - f'_+(m)\right] \\ &\leq \frac{1}{4} (M - m) \left[f'_-(M) - f'_+(m)\right] 1_H. \end{aligned} \tag{5.21}$$

PROOF. Follows by applying the property (P) to the scalar inequalities (5.4), (5.7), (5.10), (5.15) and (5.17). The details are omitted. ∎

The following particular case is perhaps more useful for applications:

COROLLARY 5.7 (Dragomir, 2010, [**7**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $l, L \in \mathbb{R}$ with $L > l$ and $f : [m, M] \to \mathbb{R}$ is (l, L) $-$Lipschitzian on $[m, M]$, then we have the inequalities:*

$$\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A)\right| \leq \frac{1}{4}(M - m)(L - l)\,1_H. \tag{5.22}$$

5.3. More Inequalities for Differentiable Functions.

The following result holds:

THEOREM 5.8 (Dragomir, 2010, [**7**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned} &\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \\ &\leq \frac{1}{4}(M - m)\bigvee_m^M(f')\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\leq \frac{1}{4}(M - m)\bigvee_m^M(f')\,\|x\|\,\|y\| \end{aligned} \tag{5.23}$$

for any $x, y \in H$.

2. If the derivative f' is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$\begin{aligned} &\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \\ &\leq \frac{1}{8}(M - m)^2 K\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{8}(M - m)^2 K\,\|x\|\,\|y\| \end{aligned} \tag{5.24}$$

for any $x, y \in H$.

PROOF. First of all we notice that if $f : [m, M] \to \mathbb{C}$ is absolutely continuous on $[m, M]$ and such that the derivative f' is Riemann integrable on $[m, M]$, then we have the following representation in terms of the Riemann-Stieltjes integral:

$$\Phi_f(t) = \frac{1}{M - m}\int_m^M K(t, s)\,df'(s), \quad t \in [m, M], \tag{5.25}$$

where the kernel $K : [m, M]^2 \to \mathbb{R}$ is given by

$$K(t, s) := \begin{cases} (M - t)(s - m) & \text{if } m \leq s \leq t \\ (t - m)(M - s) & \text{if } t < s \leq M. \end{cases} \tag{5.26}$$

Indeed, since f' is Riemann integrable on $[m, M]$, it follows that the Riemann-Stieltjes integrals $\int_m^t (s - m)\,df'(s)$ and $\int_t^M (M - s)\,df'(s)$ exist for each $t \in [m, M]$. Now, integrating by parts in the Riemann-Stieltjes integral, we have:

$$\begin{aligned}\int_m^M K(t,s)\,df'(s) &= (M-t)\int_m^t (s-m)\,df'(s) + (t-m)\int_t^M (M-s)\,df'(s)\\ &= (M-t)\left[(s-m)f'(s)\big|_m^t - \int_m^t f'(s)\,ds\right]\\ &+ (t-m)\left[(M-s)f'(s)\big|_t^M - \int_t^M f'(s)\,ds\right]\\ &= (M-t)\left[(t-m)f'(t) - (f(t)-f(m))\right]\\ &+ (t-m)\left[-(M-t)f'(t) + f(M) - f(t)\right]\\ &= (t-m)\left[f(M)-f(t)\right] - (M-t)\left[f(t)-f(m)\right]\\ &= (M-m)\,\Phi_f(t)\end{aligned}$$

for any $t \in [m,M]$, which provides the desired representation (5.25).

Now, utilizing the representation (5.25) and the property (5.3), we have

$$\begin{aligned}(5.27)\qquad &|\Phi_f(t)|\\ &= \frac{1}{M-m}\left|(M-t)\int_m^t (s-m)\,df'(s) + (t-m)\int_t^M (M-s)\,df'(s)\right|\\ &\le \frac{1}{M-m}\left[(M-t)\left|\int_m^t (s-m)\,df'(s)\right| + (t-m)\left|\int_t^M (M-s)\,df'(s)\right|\right]\\ &\le \frac{1}{M-m}\\ &\times \left[(M-t)\bigvee_m^t (f')\sup_{s\in[m,t]}(s-m) + (t-m)\bigvee_t^M (f')\sup_{s\in[t,M]}(M-s)\right]\\ &= \frac{(t-m)(M-t)}{M-m}\left[\bigvee_m^t (f') + \bigvee_t^M (f')\right]\\ &= \frac{(t-m)(M-t)}{M-m}\bigvee_m^M (f') \le \frac{1}{4}(M-m)\bigvee_m^M (f')\end{aligned}$$

for any $t \in [m,M]$.

On making use of the representation (5.2) we deduce the desired result (5.23).

Further, we utilize the fact that for an $L-$Lipschitzian function, $p : [\alpha,\beta] \to \mathbb{C}$ and a Riemann integrable function $v : [\alpha,\beta] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right| \le L\int_\alpha^\beta |p(s)|\,ds.$$

Then, by utilizing (5.27) we have

$$\begin{aligned} (5.28)\qquad & \left|\Phi_f(t)\right| \\ & \leq \frac{1}{M-m}\left[(M-t)\left|\int_m^t (s-m)\,df'(s)\right| + (t-m)\left|\int_t^M (M-s)\,df'(s)\right|\right] \\ & \leq \frac{K}{M-m}\left[(M-t)\int_m^t (s-m)\,ds + (t-m)\int_t^M (M-s)\,ds\right] \\ & = \frac{K}{M-m}\left[\frac{(M-t)(t-m)^2}{2} + \frac{(t-m)(M-t)^2}{2}\right] \\ & = \frac{1}{2}(M-m)(t-m)(M-t)K \leq \frac{1}{8}(M-m)^2 K \end{aligned}$$

for any $t \in [m, M]$.

On making use of the representation (5.2) we deduce the desired result (5.24). ∎

The following inequalities in the operator order are of interest as well:

THEOREM 5.9 (Dragomir, 2010, [**7**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned} (5.29)\qquad & \left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \\ & \leq \frac{(A-m1_H)(M1_H - A)}{M-m}\bigvee_m^M (f') \leq \frac{1}{4}(M-m)\bigvee_m^M (f')\,1_H. \end{aligned}$$

2. If the derivative f' is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$\begin{aligned} (5.30)\qquad & \left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \\ & \leq \frac{1}{2}(M-m)(A-m1_H)(M1_H - A)K \leq \frac{1}{8}(M-m)^2 K1_H. \end{aligned}$$

5.4. Applications for Particular Functions. It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^p$ with $p \neq 0$. On applying Theorem 5.5 we can state the following proposition:

PROPOSITION 5.10. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$. Then for any $x, y \in H$ we have the inequalities*

$$\begin{aligned} (5.31)\qquad & \left|\left\langle \left[\frac{m^p(M1_H - A) + M^p(A - m1_H)}{M-m}\right]x, y\right\rangle - \langle A^p x, y\rangle\right| \\ & \leq \frac{1}{2}(M-m)\Delta_p \|x\|\,\|y\| \end{aligned}$$

where

$$\Delta_p = p \times \begin{cases} M^{p-1} - m^{p-1} & \text{if } p \in (-\infty, 0) \cup [1, \infty) \\ \\ m^{p-1} - M^{p-1} & \text{if } 0 < p < 1. \end{cases}$$

In particular,

$$\left| \left\langle \left[\frac{M(M1_H - A) + m(A - m1_H)}{mM(M-m)} \right] x, y \right\rangle - \left\langle A^{-1}x, y \right\rangle \right| \leq \frac{1}{2} \frac{(M-m)^2 (M+m)}{m^2 M^2} \|x\| \|y\| \tag{5.32}$$

for any $x, y \in H$.

The following inequalities in the operator order also hold:

PROPOSITION 5.11. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$.*

If $p \in (-\infty, 0) \cup [1, \infty)$, then

$$\begin{aligned} 0 &\leq \frac{m^p (M1_H - A) + M^p (A - m1_H)}{M-m} - A^p \\ &\leq p \frac{(M1_H - A)(A - m1_H)}{M-m} \left(M^{p-1} - m^{p-1} \right) \\ &\leq \frac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1} \right) 1_H. \end{aligned} \tag{5.33}$$

If $p \in (0, 1)$, then

$$\begin{aligned} 0 &\leq A^p - \frac{m^p (M1_H - A) + M^p (A - m1_H)}{M-m} \\ &\leq p \frac{(M1_H - A)(A - m1_H)}{M-m} \left(m^{p-1} - M^{p-1} \right) \\ &\leq \frac{1}{4} p (M-m) \left(m^{p-1} - M^{p-1} \right) 1_H. \end{aligned} \tag{5.34}$$

In particular, we have the inequalities

$$\begin{aligned} 0 &\leq \frac{M(M1_H - A) + m(A - m1_H)}{mM(M-m)} - A^{-1} \\ &\leq \frac{(M1_H - A)(A - m1_H)}{M-m} \cdot \frac{M^2 - m^2}{m^2 M^2} \\ &\leq \frac{1}{2} \frac{(M-m)^2 (M+m)}{m^2 M^2} 1_H. \end{aligned} \tag{5.35}$$

The proof follows from (5.21) and the details are omitted.

2. The case of logarithmic function is as follows:

PROPOSITION 5.12. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$. Then for any $x, y \in H$ we have the inequalities*

$$\left| \left\langle \left[\frac{(M1_H - A)\ln m + (A - m1_H)\ln M}{M-m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \leq \frac{1}{4} \frac{(M-m)^2}{mM} \|x\| \|y\| . \tag{5.36}$$

We also have the following inequality in the operator order

$$0 \le \ln A - \frac{(M1_H - A)\ln m + (A - m1_H)\ln M}{M - m} \le \frac{(M1_H - A)(A - m1_H)}{Mm} \le \frac{1}{4}\frac{(M-m)^2}{mM}1_H. \tag{5.37}$$

REMARK 5.1. Similar results can be obtained if ones uses the inequalities from Theorem 5.8 and 5.9. However the details are left to the interested reader.

6. PRODUCT INEQUALITIES

6.1. Some Vector Inequalities. In this section we investigate the quantity

$$|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle|$$

where x, y are vectors in the Hilbert space H and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$, and provide different bounds for some classes of continuous functions $f : [m, M] \to \mathbb{C}$. Applications for some particular cases including the power and logarithmic functions are provided as well.

The following representation in terms of the spectral family is of interest in itself:

LEMMA 6.1 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$ with $f(M) \neq f(m)$, then we have the representation*

$$\begin{aligned}&\frac{1}{[f(M) - f(m)]^2}[f(M)1_H - f(A)][f(A) - f(m)1_H] \\ &= \frac{1}{f(M) - f(m)} \\ &\times \int_{m-0}^{M}\left(E_t - \frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_s df(s)\right)\left(E_t - \frac{1}{2}1_H\right)df(t).\end{aligned} \tag{6.1}$$

PROOF. We observe that,

$$\begin{aligned}&\frac{1}{f(M) - f(m)}\int_{m-0}^{M}\left(E_t - \frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_s df(s)\right) \\ &\times\left(E_t - \frac{1}{2}1_H\right)df(t) \\ &= \frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_t^2 df(t) \\ &- \frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_s df(s)\frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_t df(t) \\ &- \frac{1}{2}\int_{m-0}^{M}E_t df(t) + \frac{1}{2}\int_{m-0}^{M}E_s df(s) \\ &= \frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_t^2 df(t) - \left[\frac{1}{f(M) - f(m)}\int_{m-0}^{M}E_t df(t)\right]^2\end{aligned} \tag{6.2}$$

which is an equality of interest in itself.

Since E_t are projections, we have $E_t^2 = E_t$ for any $t \in [m, M]$ and then we can write that

$$\frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t^2 df(t) - \left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t)\right]^2 \tag{6.3}$$
$$= \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t) - \left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t)\right]^2$$
$$= \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t)\left[1_H - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t)\right].$$

Integrating by parts in the Riemann-Stieltjes integral and utilizing the spectral representation theorem we have

$$\int_{m-0}^{M} E_t df(t) = f(M)1_H - f(A)$$

and

$$1_H - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_t df(t) = \frac{f(A)-f(m)1_H}{f(M)-f(m)},$$

which together with (6.3) and (6.2) produce the desired result (6.1). ∎

The following vector version may be stated as well:

COROLLARY 6.2 (Dragomir, 2010, [**8**]). *With the assumptions of Lemma 6.1 we have the equality*

$$\langle [f(M)1_H - f(A)][f(A)-f(m)1_H]x, y\rangle \tag{6.4}$$
$$= [f(M)-f(m)]$$
$$\times \int_{m-0}^{M} \left\langle \left(E_t - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_s df(s)\right)x, \left(E_t - \frac{1}{2}1_H\right)y\right\rangle df(t),$$

for any $x, y \in [m, M]$.

The following result that provides some bounds for continuous functions of bounded variation may be stated as well:

THEOREM 6.3 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ with $f(M) \neq f(m)$, then we have the inequality*

$$|\langle [f(M)1_H - f(A)][f(A)-f(m)1_H]x, y\rangle| \tag{6.5}$$
$$\leq \frac{1}{2}\|y\| |f(M)-f(m)| \bigvee_m^M (f)$$
$$\times \sup_{t\in[m,M]} \left\| E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_s df(s)\right\| \leq \frac{1}{2}\|x\|\|y\|\left[\bigvee_m^M (f)\right]^2,$$

for any $x, y \in H$.

PROOF. It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, then the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (v)\,, \tag{6.6}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a,b]$.

Utilising this property and the representation (6.4) we have by the Schwarz inequality in Hilbert space H that

$$\begin{aligned}
&|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x,y\rangle| \\
&\le |f(M) - f(m)| \bigvee_m^M (f) \\
&\times \sup_{t\in[m,M]} \left|\left\langle \left(E_t - \frac{1}{f(M)-f(m)}\int_{m-0}^M E_s df(s)\right)x, \left(E_t - \frac{1}{2}1_H\right)y\right\rangle\right| \\
&\le |f(M) - f(m)| \bigvee_m^M (f) \\
&\times \sup_{t\in[m,M]} \left[\left\|E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^M E_s x df(s)\right\| \left\|E_t y - \frac{1}{2}y\right\|\right]
\end{aligned} \tag{6.7}$$

for any $x, y \in [m,M]$.

Since E_t are projections, and in this case we have

$$\begin{aligned}
\left\|E_t y - \frac{1}{2}y\right\|^2 &= \|E_t y\|^2 - \langle E_t y, y\rangle + \frac{1}{4}\|y\|^2 \\
&= \left\langle E_t^2 y, y\right\rangle - \langle E_t y, y\rangle + \frac{1}{4}\|y\|^2 = \frac{1}{4}\|y\|^2\,,
\end{aligned}$$

then from (6.7) we deduce the first part of (6.5).

Now, by the same property (6.6) for vector valued functions p with values in Hilbert spaces, we also have that

$$\begin{aligned}
&\left\|[f(M) - f(m)]E_t x - \int_{m-0}^M E_s x df(s)\right\| \\
&= \left\|\int_{m-0}^M (E_t x - E_s x)\,df(s)\right\| \le \bigvee_m^M (f) \sup_{s\in[m,M]} \|E_t x - E_s x\|
\end{aligned} \tag{6.8}$$

for any $t \in [m,M]$ and $x \in H$.

Since $0 \le E_t \le 1_H$ in the operator order, then $-1_H \le E_t - E_s \le 1$ which gives that $-\|x\|^2 \le \langle (E_t - E_s)x, x\rangle \le \|x\|^2$, i.e., $|\langle (E_t - E_s)x, x\rangle| \le \|x\|^2$ for any $x \in H$, which implies that $\|E_t - E_s\| \le 1$ for any $t, s \in [m,M]$. Therefore $\sup_{s\in[m,M]} \|E_t x - E_s x\| \le \|x\|$ which together with (6.8) prove the last part of (6.5). ∎

The case of Lipschitzian functions is as follows:

THEOREM 6.4 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its* *spectral family. If $f:[m,M]\to\mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on $[m,M]$ and with $f(M)\neq f(m)$, then we have the inequality*

$$\begin{aligned}
(6.9)\qquad & |\langle [f(M)1_H - f(A)][f(A)-f(m)1_H]x,y\rangle| \\
&\le \frac{1}{2}L\|y\|\,|f(M)-f(m)| \\
&\times \int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x df(s)\right\|dt \\
&\le \frac{1}{2}L^2\|y\|\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|dsdt \\
&\le \frac{\sqrt{2}}{2}L^2\|y\|(M-m)\langle Ax-mx, Mx-Ax\rangle^{1/2} \le \frac{\sqrt{2}}{4}L^2\|y\|\|x\|(M-m)^2
\end{aligned}$$

for any $x,y\in H$.

PROOF. Recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b |p(t)|\,dt. \tag{6.10}$$

Now, on applying this property of the Riemann-Stieltjes integral, then we have from the representation (6.4) that

$$\begin{aligned}
(6.11)\qquad & |\langle [f(M)1_H - f(A)][f(A)-f(m)1_H]x,y\rangle| \\
&\le |f(M)-f(m)| \\
&\times \int_{m-0}^{M}\left|\left\langle\left(E_t - \frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s df(s)\right)x, \left(E_t-\frac{1}{2}1_H\right)y\right\rangle\right|df(t), \\
&\le L|f(M)-f(m)| \\
&\times \int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x df(s)\right\|\left\|E_t y-\frac{1}{2}y\right\|dt \\
&= \frac{1}{2}L\|y\|\,|f(M)-f(m)| \\
&\times \int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x df(s)\right\|dt
\end{aligned}$$

for any $x,y\in H$ and the first inequality in (6.9) is proved.

Further, observe that

$$\begin{aligned}
(6.12)\qquad & |f(M)-f(m)|\int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x df(s)\right\|dt \\
&= \int_{m-0}^{M}\left\|[f(M)-f(m)]E_t x - \int_{m-0}^{M}E_s x df(s)\right\|dt \\
&= \int_{m-0}^{M}\left\|\int_{m-0}^{M}(E_t x - E_s x)\,df(s)\right\|dt
\end{aligned}$$

for any $x\in H$.

If we use the vector valued version of the property (6.10), then we have

$$\int_{m-0}^{M}\left\|\int_{m-0}^{M}(E_t x - E_s x)\,df(s)\right\|dt \le L\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|\,dsdt \tag{6.13}$$

for any $x \in H$ and the second part of (6.9) is proved.

Further on, by applying the double integral version of the Cauchy-Buniakowski-Schwarz inequality we have

$$\begin{aligned}\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|\,dsdt \\ \le (M-m)\left(\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|^2\,dsdt\right)^{1/2}\end{aligned} \tag{6.14}$$

for any $x \in H$.

Now, by utilizing the fact that E_s are projections for each $s \in [m, M]$, then we have

$$\begin{aligned}&\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|^2\,dsdt \\ &= 2\left[(M-m)\int_{m-0}^{M}\|E_t x\|^2\,dt - \left\|\int_{m-0}^{M}E_t x dt\right\|^2\right] \\ &= 2\left[(M-m)\int_{m-0}^{M}\langle E_t x, x\rangle\,dt - \left\|\int_{m-0}^{M}E_t x dt\right\|^2\right]\end{aligned} \tag{6.15}$$

for any $x \in H$.

If we integrate by parts and use the spectral representation theorem, then we get

$$\int_{m-0}^{M}\langle E_t x, x\rangle\,dt = \langle Mx - Ax, x\rangle \text{ and } \int_{m-0}^{M}E_t x dt = Mx - Ax$$

and by (6.15) we then obtain the following equality of interest

$$\int_{m-0}^{M}\int_{m-0}^{M}\|E_t x - E_s x\|^2\,dsdt = 2\,\langle Ax - mx, Mx - Ax\rangle \tag{6.16}$$

for any $x \in H$.

On making use of (6.16) and (6.14) we then deduce the third part of (6.9).

Finally, by utilizing the elementary inequality in inner product spaces

$$\operatorname{Re}\langle a, b\rangle \le \frac{1}{4}\|a+b\|^2,\ a, b \in H, \tag{6.17}$$

we also have that

$$\langle Ax - mx, Mx - Ax\rangle \le \frac{1}{4}(M-m)^2\|x\|^2$$

for any $x \in H$, which proves the last inequality in (6.9). ∎

The case of nondecreasing monotonic functions is as follows:

THEOREM 6.5 (Dragomir, 2010, [**8**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its* *spectral family. If $f:[m,M]\to\mathbb{R}$ is a monotonic nondecreasing function on $[m,M]$, then we have the inequality*

$$\begin{aligned}
&\left|\left\langle\left[f(M)1_H-f(A)\right]\left[f(A)-f(m)1_H\right]x,y\right\rangle\right| \\
&\le\frac{1}{2}\|y\|\left[f(M)-f(m)\right] \\
&\times\int_{m-0}^{M}\left\|E_tx-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sxdf(s)\right\|df(t) \\
&\le\frac{1}{2}\|y\|\left[f(M)-f(m)\right] \\
&\times\left\langle\left[f(M)1_H-f(A)\right]\left[f(A)-f(m)1_H\right]x,x\right\rangle^{1/2} \\
&\le\frac{1}{4}\|y\|\|x\|\left[f(M)-f(m)\right]^2
\end{aligned}\tag{6.18}$$

for any $x,y\in H$.

PROOF. From the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b|p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (6.4) that

$$\begin{aligned}
&\left|\left\langle\left[f(M)1_H-f(A)\right]\left[f(A)-f(m)1_H\right]x,y\right\rangle\right| \\
&\le\left[f(M)-f(m)\right] \\
&\times\int_{m-0}^{M}\left|\left\langle\left(E_t-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sdf(s)\right)x,\left(E_t-\frac{1}{2}1_H\right)y\right\rangle\right|df(t), \\
&\le\left[f(M)-f(m)\right] \\
&\times\int_{m-0}^{M}\left\|E_tx-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sxdf(s)\right\|\left\|E_ty-\frac{1}{2}y\right\|df(t) \\
&=\frac{1}{2}\|y\|\left[f(M)-f(m)\right] \\
&\times\int_{m-0}^{M}\left\|E_tx-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sxdf(s)\right\|df(t)
\end{aligned}\tag{6.19}$$

for any $x,y\in H$, which proves the first inequality in (6.18).

On utilizing the Cauchy-Buniakowski-Schwarz type inequality for the Riemann-Stieltjes integral of monotonic nondecreasing integrators, we have

$$\begin{aligned}
&\int_{m-0}^{M}\left\|E_tx-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sxdf(s)\right\|df(t) \\
&\le\left[\int_{m-0}^{M}df(t)\right]^{1/2} \\
&\times\left[\int_{m-0}^{M}\left\|E_tx-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_sxdf(s)\right\|^2df(t)\right]^{1/2}
\end{aligned}\tag{6.20}$$

for any $x, y \in H$.

Observe that

$$\begin{aligned}
(6.21)\qquad & \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 df(t) \\
& = \int_{m-0}^{M} \left[\|E_t x\|^2 - 2\operatorname{Re}\left\langle E_t x, \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\rangle \right. \\
& \left. + \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 \right] df(t) \\
& = [f(M) - f(m)] \left[\frac{1}{f(M) - f(m)} \int_{m-0}^{M} \|E_t x\|^2 df(t) \right. \\
& \left. - \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 \right]
\end{aligned}$$

and, integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
(6.22)\qquad \int_{m-0}^{M} \|E_t x\|^2 df(t) & = \int_{m-0}^{M} \langle E_t x, E_t x\rangle df(t) \\
& = \int_{m-0}^{M} \langle E_t x, x\rangle df(t) \\
& = f(M)\|x\|^2 - \int_{m-0}^{M} f(t)\, d\langle E_t x, x\rangle \\
& = f(M)\|x\|^2 - \langle f(A)x, x\rangle = \langle [f(M)1_H - f(A)]x, x\rangle
\end{aligned}$$

and

$$(6.23)\qquad \int_{m-0}^{M} E_s x df(s) = f(M)x - f(A)x$$

for any $x \in H$.

On making use of the equalities (6.22) and (6.23) we have

$$\begin{aligned}
(6.24)\qquad & \frac{1}{f(M) - f(m)} \int_{m-0}^{M} \|E_t x\|^2 df(t) \\
& - \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s) \right\|^2 \\
& = \frac{1}{[f(M) - f(m)]^2} \\
& \times \left[[f(M) - f(m)] \langle [f(M)1_H - f(A)]x, x\rangle - \|f(M)x - f(A)x\|^2 \right] \\
& = \frac{\langle f(M)x - f(A)x, f(A)x - f(m)x\rangle}{[f(M) - f(m)]^2}
\end{aligned}$$

for any $x \in H$.

Therefore, we obtain the following equality of interest in itself as well

$$\begin{aligned}(6.25)\qquad &\frac{1}{f(M)-f(m)}\\ &\times\int_{m-0}^{M}\left\|E_t x-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x df(s)\right\|^2 df(t)\\ &=\frac{\langle f(M)x-f(A)x, f(A)x-f(m)x\rangle}{[f(M)-f(m)]^2}\\ &=\frac{\langle [f(M)1_H-f(A)][f(A)-f(m)1_H]x,x\rangle}{[f(M)-f(m)]^2}\end{aligned}$$

for any $x\in H$

On making use of the inequality (6.20) we deduce the second inequality in (6.18).

The last part follows by (6.17) and the details are omitted. ∎

6.2. Applications. We consider the power function $f(t):=t^p$ where $p\in\mathbb{R}\diagdown\{0\}$ and $t>0$. The following power inequalities hold:

PROPOSITION 6.6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0\le m<M$.*

If $p>0$, then for any $x,y\in H$

$$\begin{aligned}(6.26)\qquad &|\langle (M^p1_H-A^p)(A^p-m^p1_H)x,y\rangle|\\ &\le\frac{\sqrt{2}}{2}B_p^2\|y\|(M-m)\langle Ax-mx,Mx-Ax\rangle^{1/2}\\ &\le\frac{\sqrt{2}}{4}B_p^2\|y\|\|x\|(M-m)^2\end{aligned}$$

where

$$B_p=p\times\begin{cases}M^{p-1} & \text{if } p\ge1\\ m^{p-1} & \text{if } 0<p<1, m>0\end{cases}$$

and

$$\begin{aligned}(6.27)\qquad &\left|\left\langle\left(A^{-p}-M^{-p}1_H\right)\left(m^{-p}1_H-A^{-p}\right)x,y\right\rangle\right|\\ &\le\frac{\sqrt{2}}{2}C_p^2\|y\|(M-m)\langle Ax-mx,Mx-Ax\rangle^{1/2}\\ &\le\frac{\sqrt{2}}{4}C_p^2\|y\|\|x\|(M-m)^2,\end{aligned}$$

where

$$C_p=pm^{-p-1}\text{ and } m>0.$$

The proof follows from Theorem 6.4 applied for the power function.

PROPOSITION 6.7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers with $0\le m<M$.*

If $p > 0$, *then for any* $x, y \in H$

$$\begin{aligned} &\left|\left\langle \left(M^p 1_H - A^p\right)\left(A^p - m^p 1_H\right) x, y\right\rangle\right| \\ &\leq \frac{1}{2}\|y\|\left(M^p - m^p\right)\left\langle \left(M^p 1_H - A^p\right)\left(A^p - m^p 1_H\right) x, x\right\rangle^{1/2} \\ &\leq \frac{1}{4}\|y\|\|x\|\left(M^p - m^p\right)^2 \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} &\left|\left\langle \left(A^{-p} - M^{-p} 1_H\right)\left(m^{-p} 1_H - A^{-p}\right) x, y\right\rangle\right| \\ &\leq \frac{1}{2}\|y\|\left(m^{-p} - M^{-p}\right)\left\langle \left(A^{-p} - M^{-p} 1_H\right)\left(m^{-p} 1_H - A^{-p}\right) x, x\right\rangle^{1/2} \\ &\leq \frac{1}{4}\|y\|\|x\|\left(m^{-p} - M^{-p}\right)^2. \end{aligned} \tag{6.29}$$

The proof follows from Theorem 6.5.

Now, consider the logarithmic function $f(t) = \ln t, t > 0$. We have

PROPOSITION 6.8. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers with* $0 < m < M$. *Then we have the inequalities*

$$\begin{aligned} &\left|\left\langle \left[(\ln M) 1_H - \ln A\right]\left[\ln A - (\ln m) 1_H\right] x, y\right\rangle\right| \\ &\leq \frac{\sqrt{2}}{2m^2}\|y\|(M - m)\langle Ax - mx, Mx - Ax\rangle^{1/2} \\ &\leq \frac{\sqrt{2}}{4}\|y\|\|x\|\left(\frac{M}{m} - 1\right)^2 \end{aligned} \tag{6.30}$$

and

$$\begin{aligned} &\left|\left\langle \left[(\ln M) 1_H - \ln A\right]\left[\ln A - (\ln m) 1_H\right] x, y\right\rangle\right| \\ &\leq \frac{1}{2}\|y\|\left\langle \left[(\ln M) 1_H - \ln A\right]\left[\ln A - (\ln m) 1_H\right] x, x\right\rangle^{1/2} \ln\left(\frac{M}{m}\right) \\ &\leq \frac{1}{4}\|y\|\|x\|\left[\ln\left(\frac{M}{m}\right)\right]^2. \end{aligned} \tag{6.31}$$

The proof follows from Theorem 6.4 and 6.5 applied for the logarithmic function.

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CHAPTER 6

Inequalities of Taylor Type

1. Introduction

In approximating n-time differentiable functions around a point, perhaps the classical Taylor's expansion is one of the simplest and most convenient and elegant methods that has been employed in the development of Mathematics for the last three centuries. There is probably no field of Science where Mathematical Modelling is used not to contain in a form or another Taylor's expansion for functions that are differentiable in a certain sense.

In the present chapter, that is intended to be developed to a later stage, we present some error bounds in approximating n-time differentiable functions of selfadjoint operators by the use of operator Taylor's type expansions around a point or two points from its spectrum for which the remainder is known in an integral form.

Some applications for elementary functions including the exponential and logarithmic functions are provided as well.

2. Taylor's Type Inequalities

2.1. Some Identities. In this section, by utilizing the spectral representation theorem of selfadjoint operators in Hilbert spaces, some error bounds in approximating n-time differentiable functions of selfadjoint operators in Hilbert Spaces via a Taylor's type expansion are given. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

The following result provides a Taylor's type representation for a function of selfadjoint operators in Hilbert spaces with integral remainder.

THEOREM 2.1 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k + R_n(f, c, m, M) \tag{2.1}$$

where

$$R_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n d\left(f^{(n)}(t)\right) \right) dE_\lambda. \tag{2.2}$$

PROOF. We utilize the Taylor formula for a function $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n d\left(f^{(n)}(t)\right) \tag{2.3}$$

for any $\lambda, c \in [m, M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m, M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\int_{m-0}^{M} f(\lambda)\, dE_\lambda = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k \, dE_\lambda + \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda$$

which, by the spectral representation theorem, produces the equality (2.1) with the representation of the remainder from (2.2). ∎

The following particular instances are of interest for applications:

COROLLARY 2.2 (Dragomir, 2010, [**5**]). *With the assumptions of the above Theorem 2.1, we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(m) (A - m1_H)^k + L_n(f, c, m, M) \tag{2.4}$$

where

$$L_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_m^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}\left(\frac{m+M}{2}\right) \left(A - \frac{m+M}{2} 1_H\right)^k + M_n(f, c, m, M) \tag{2.5}$$

where

$$M_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_{\frac{m+M}{2}}^\lambda (\lambda - t)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda$$

and

$$f(A) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(M) (M1_H - A)^k + U_n(f, c, m, M) \tag{2.6}$$

where

$$U_n(f, c, m, M) = \frac{(-1)^{n+1}}{n!} \int_{m-0}^{M} \left(\int_\lambda^M (t - \lambda)^n \, d\left(f^{(n)}(t)\right) \right) dE_\lambda, \tag{2.7}$$

respectively.

REMARK 2.1. We remark that, if the n-th derivative of the function f considered above is absolutely continuous on the interval $[m, M]$, then we have the representation (2.1) with the remainder

$$R_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt \right) dE_\lambda. \tag{2.8}$$

Here the integral $\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt$ is considered in the Lebesgue sense. Similar representations hold true when c is taken the particular values m, M or $\frac{m+M}{2}$.

Now, if we consider the exponential function, then for any selfadjoint operator A in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ and with the spectral family $\{E_\lambda\}_\lambda$ we have the representation

$$e^{A-c1_H} = \sum_{k=0}^{n} \frac{1}{k!}(A - c1_H)^k + \frac{1}{n!}\int_{m-0}^{M}\left(\int_c^\lambda (\lambda - t)^n e^{t-c} dt\right) dE_\lambda, \tag{2.9}$$

where c is any real number.

Further, if we consider the logarithmic function, then for any positive definite operator A with $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and with the spectral family $\{E_\lambda\}_\lambda$ we have

$$\begin{aligned} \ln A = (\ln c)\, 1_H &+ \sum_{k=1}^{n} \frac{(-1)^{k-1}(A - c1_H)^k}{kc^k} \\ &+ (-1)^n \int_{m-0}^{M}\left(\int_c^\lambda \frac{(\lambda - t)^n}{t^{n+1}} dt\right) dE_\lambda \end{aligned} \tag{2.10}$$

for any $c > 0$.

2.2. Some Error Bounds.

We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

THEOREM 2.3 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$\begin{aligned} &|\langle R_n(f, c, m, M)x, y\rangle| \\ &= \left|\langle f(A)x, y\rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\left\langle (A - c1_H)^k x, y\right\rangle\right| \\ &\leq \frac{1}{n!}\left[(c - m)^n \bigvee_m^c \left(f^{(n)}\right) \bigvee_m^c \left(\langle E_{(\cdot)}x, y\rangle\right)\right. \\ &\left. + (M - c)^n \bigvee_c^M \left(f^{(n)}\right) \bigvee_c^M \left(\langle E_{(\cdot)}x, y\rangle\right)\right] \\ &\leq \frac{1}{n!}\max\left\{(M - c)^n \bigvee_c^M \left(f^{(n)}\right), (c - m)^n \bigvee_c^M \left(f^{(n)}\right)\right\} \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\leq \frac{1}{n!}\left(\frac{1}{2}(M - m) + \left|c - \frac{m + M}{2}\right|\right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right), \end{aligned} \tag{2.11}$$

for any $x, y \in H$.

PROOF. From the identities (2.1) and (2.2) we have

$$\begin{aligned}(2.12)\qquad &\langle R_n(f,c,m,M)x,y\rangle\\ &=\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle\\ &=\frac{1}{n!}\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\\ &=\frac{1}{n!}\int_{m-0}^{c}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\\ &+\frac{1}{n!}\int_{c}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\end{aligned}$$

for any $x,y\in H.$

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$(2.13)\qquad \left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Taking the modulus in (2.12) and utilizing the inequality (2.13) we have

$$\begin{aligned}(2.14)\qquad &|\langle R_n(f,c,m,M)x,y\rangle|\\ &\le\frac{1}{n!}\left|\int_{m-0}^{c}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\right|\\ &+\frac{1}{n!}\left|\int_{c}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\right|\\ &\le\frac{1}{n!}\max_{\lambda\in[m,c]}\left|\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\bigvee_m^c\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &+\frac{1}{n!}\max_{\lambda\in[c,M]}\left|\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\bigvee_c^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

for any $x,y\in H.$

By the same property (2.13) we have

$$(2.15)\qquad \max_{\lambda\in[m,c]}\left|\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\le(c-m)^n\bigvee_m^c\left(f^{(n)}\right)$$

and

$$(2.16)\qquad \max_{\lambda\in[c,M]}\left|\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\le(M-c)^n\bigvee_c^M\left(f^{(n)}\right).$$

Now, on making use of (2.14)-(2.16) we deduce

$$\begin{aligned}
&\left|\left\langle R_n(f,c,m,M)x,y\right\rangle\right| \\
&\leq \frac{1}{n!}\left[(c-m)^n \bigvee_m^c \left(f^{(n)}\right) \bigvee_m^c \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right. \\
&\left.+(M-c)^n \bigvee_c^M \left(f^{(n)}\right) \bigvee_c^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n \bigvee_m^c \left(f^{(n)}\right),(M-c)^n \bigvee_c^M \left(f^{(n)}\right)\right\} \\
&\times\left[\bigvee_m^c \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\bigvee_c^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n,(M-c)^n\right\} \bigvee_m^M \left(f^{(n)}\right) \bigvee_{m-0}^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&=\frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_{m-0}^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right)
\end{aligned}$$

for any $x,y\in H$ and the proof is complete. ∎

The following particular cases are of interest for applications

COROLLARY 2.4 (Dragomir, 2010, [**5**]). *With the assumption of Theorem 2.3 we have the inequalities*

$$\begin{aligned}
&\left|\left\langle f(A)x,y\right\rangle-\sum_{k=0}^n \frac{1}{k!}f^{(k)}(m)\left\langle (A-m1_H)^k x,y\right\rangle\right| \\
&\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_{m-0}^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right)\|x\|\|y\|,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\left|\left\langle f(A)x,y\right\rangle-\sum_{k=0}^n \frac{(-1)^k}{k!}f^{(k)}(M)\left\langle (M1_H-A)^k x,y\right\rangle\right| \\
&\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_{m-0}^M \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\leq \frac{1}{n!}(M-m)^n \bigvee_m^M \left(f^{(n)}\right)\|x\|\|y\|
\end{aligned} \tag{2.18}$$

and

$$\left|\langle f(A)x,y\rangle - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\right| \tag{2.19}$$
$$\leq \frac{1}{2^n n!}(M-m)^n \max\left\{\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right), \bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\right\}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{2^n n!}(M-m)^n \max\left\{\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right), \bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\right\}\|x\|\|y\|$$

respectively, for any $x, y \in H$.

PROOF. The first part in the inequalities follow from (2.11) by choosing $c = m, c = M$ and $c = \frac{m+M}{2}$ respectively.

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

The following result also holds:

THEOREM 2.5 (Dragomir, 2010, [**5**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$|\langle R_n(f,c,m,M)x,y\rangle| \tag{2.20}$$
$$\leq \frac{1}{(n+1)!}L_n\left[(c-m)^{n+1}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) + (M-c)^{n+1}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\leq \frac{1}{(n+1)!}L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{(n+1)!}L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\|x\|\|y\|$$

for any $x, y \in H$.

PROOF. First of all, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s-t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$\max_{\lambda\in[m,c]}\left|\int_\lambda^c (t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right| \leq \max_{\lambda\in[m,c]}\left[L_n\int_\lambda^c (t-\lambda)^n\,dt\right] \tag{2.21}$$
$$= \frac{L_n}{n+1}(c-m)^{n+1}$$

and

$$\max_{\lambda\in[c,M]}\left|\int_c^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right| \le \max_{\lambda\in[c,M]}\left[L_n\int_c^\lambda(\lambda-t)^n dt\right] = \frac{L_n}{n+1}(M-c)^{n+1}. \tag{2.22}$$

Now, on utilizing the inequality (2.14), then we have from (2.21) and (2.22) that

$$\begin{aligned}
&|\langle R_n(f,c,m,M)x,y\rangle| \\
&\le \frac{1}{(n+1)!}L_n(c-m)^{n+1}\bigvee_m^c\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&+\frac{1}{(n+1)!}L_n(M-c)^{n+1}\bigvee_c^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\le \frac{1}{(n+1)!}L_n\max\left\{(c-m)^{n+1},(M-c)^{n+1}\right\}\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&= \frac{1}{(n+1)!}L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right),
\end{aligned} \tag{2.23}$$

and the proof is complete. ∎

The following particular cases are of interest for applications:

COROLLARY 2.6 (Dragomir, 2010, [**5**]). *With the assumption of Theorem 2.5 we have the inequalities*

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\sum_{k=0}^n\frac{1}{k!}f^{(k)}(m)\left\langle(A-m1_H)^k x,y\right\rangle\right| \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\|x\|\,\|y\|
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\sum_{k=0}^n\frac{(-1)^k}{k!}f^{(k)}(M)\left\langle(M1_H-A)^k x,y\right\rangle\right| \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\bigvee_{m-0}^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\|x\|\,\|y\|
\end{aligned} \tag{2.25}$$

and

$$\left|\langle f(A)x,y\rangle - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\right| \tag{2.26}$$
$$\le \frac{1}{2^{n+1}(n+1)!}(M-m)^{n+1}L_n\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le \frac{1}{2^{n+1}(n+1)!}(M-m)^{n+1}L_n\|x\|\|y\|$$

respectively, for any $x,y\in H$.

The following corollary that provides a perturbed version of Taylor's expansion holds:

COROLLARY 2.7 (Dragomir, 2010, [**5**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m,M]\subset\mathring{I}$ *(the interior of* I) *and let* n *be an integer with* $n\ge 1$. *If* $g:I\to\mathbb{R}$ *is such that the* n*-th derivative* $g^{(n)}$ *is* (l_n,L_n) *−Lipschitzian with the constant* $L_n>l_n>0$ *on the interval* $[m,M]$, *then for any* $c\in[m,M]$ *we have the inequality*

$$\Bigg|\langle g(A)x,y\rangle - g(c)\langle x,y\rangle - \sum_{k=1}^{n}\frac{1}{k!}g^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle \tag{2.27}$$
$$-\frac{l_n+L_n}{2}\Bigg[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{c^{n+1}}{(n+1)!}\langle x,y\rangle$$
$$-\sum_{k=1}^{n}\frac{c^{n-k+1}}{k!(n-k+1)!}\left\langle (A-c1_H)^k x,y\right\rangle\Bigg]\Bigg|$$
$$\le \frac{1}{2(n+1)!}(L_n-l_n)$$
$$\times\left[(c-m)^{n+1}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+(M-c)^{n+1}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\le \frac{1}{2(n+1)!}(L_n-l_n)\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le \frac{1}{2(n+1)!}(L_n-l_n)\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\|x\|\|y\|$$

for any $x,y\in H$.

PROOF. Consider the function $f:I\to\mathbb{R}$ defined by

$$f(t):=g(t)-\frac{1}{(n+1)!}\frac{L_n+l_n}{2}\cdot t^{n+1}.$$

Observe that

$$f^{(k)}(t):=g^{(k)}(t)-\frac{1}{(n-k+1)!}\frac{L_n+l_n}{2}\cdot t^{n-k+1}$$

for any $k=0,...,n$.

Since $g^{(n)}$ is (l_n,L_n) −Lipschitzian it follows that

$$f^{(n)}(t):=g^{(n)}(t)-\frac{L_n+l_n}{2}\cdot t$$

is $\frac{1}{2}(L_n - l_n)$-Lipschitzian and applying Theorem 2.5 for the function f, we deduce after required calculations the desired result (2.11). ∎

REMARK 2.2. In particular, we can state from (2.27) the following inequalities

$$\begin{aligned}(2.28)\quad &\Bigg|\langle g(A)x,y\rangle - g(m)\langle x,y\rangle - \sum_{k=1}^{n}\frac{1}{k!}g^{(k)}(m)\left\langle (A-m1_H)^k x,y\right\rangle \\ &- \frac{l_n+L_n}{2}\left[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{m^{n+1}}{(n+1)!}\langle x,y\rangle \right.\\ &\left.- \sum_{k=1}^{n}\frac{m^{n-k+1}}{k!(n-k+1)!}\left\langle (A-m1_H)^k x,y\right\rangle\right]\Bigg| \\ &\le \frac{1}{2(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{2(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\|y\|\end{aligned}$$

and

$$\begin{aligned}&\Bigg|\langle g(A)x,y\rangle - g(M)\langle x,y\rangle - \sum_{k=1}^{n}\frac{(-1)^k}{k!}g^{(k)}(M)\left\langle (M1_H-A)^k x,y\right\rangle \\ &- \frac{l_n+L_n}{2}\left[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{M^{n+1}}{(n+1)!}\langle x,y\rangle \right.\\ &\left.- \sum_{k=1}^{n}(-1)^k\frac{M^{n-k+1}}{k!(n-k+1)!}\left\langle (M1_H-A)^k x,y\right\rangle\right]\Bigg| \\ &\le \frac{1}{2(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ (2.29)\quad &\le \frac{1}{2(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\|y\|\end{aligned}$$

and

$$\begin{aligned}
(2.30)\qquad & \Bigg|\langle g(A)x,y\rangle - g\left(\frac{m+M}{2}\right)\langle x,y\rangle \\
& - \sum_{k=1}^{n}\frac{1}{k!}g^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle \\
& - \frac{l_n+L_n}{2}\left[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{1}{(n+1)!}\langle x,y\rangle\left(\frac{m+M}{2}\right)^{n+1}\right. \\
& \left.- \sum_{k=1}^{n}\frac{1}{(n-k+1)!k!}\left(\frac{m+M}{2}\right)^{n-k+1}\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\right]\Bigg| \\
& \le \frac{1}{2^{n+2}(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le \frac{1}{2^{n+2}(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\|y\|
\end{aligned}$$

respectively, for any $x, y \in H$.

2.3. Applications. By utilizing Theorem 2.3 and 2.5 for the exponential function, we can state the following result:

PROPOSITION 2.8. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequality*

$$\begin{aligned}
(2.31)\qquad & \left|\left\langle e^A x,y\right\rangle - e^c\sum_{k=0}^{n}\frac{1}{k!}\left\langle (A-c1_H)^k x,y\right\rangle\right| \\
& \le \frac{1}{n!}\left[(c-m)^n(e^c-e^m)\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right. \\
& \left.+(M-c)^n\left(e^M-e^c\right)\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right] \\
& \le \frac{1}{n!}\max\left\{(M-c)^n\left(e^M-e^c\right),(c-m)^n(e^c-e^m)\right\}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le \frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n\left(e^M-e^m\right)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le \frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n\left(e^M-e^m\right)\|x\|\|y\|
\end{aligned}$$

and

$$\left|\left\langle e^{A}x,y\right\rangle - e^{c}\sum_{k=0}^{n}\frac{1}{k!}\left\langle \left(A-c1_{H}\right)^{k}x,y\right\rangle\right| \tag{2.32}$$
$$\leq \frac{1}{(n+1)!}e^{M}\left[(c-m)^{n+1}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) + (M-c)^{n+1}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\leq \frac{1}{(n+1)!}e^{M}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{(n+1)!}e^{M}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\|x\|\,\|y\|$$

for any $x,y\in H$.

REMARK 2.3. We observe that the best inequalities we can get from (2.31) and (2.32) are

$$\left|\left\langle e^{A}x,y\right\rangle - e^{\frac{m+M}{2}}\sum_{k=0}^{n}\frac{1}{k!}\left\langle \left(A-\frac{m+M}{2}1_{H}\right)^{k}x,y\right\rangle\right| \tag{2.33}$$
$$\leq \frac{1}{2^{n}n!}(M-m)^{n}\left(e^{M}-e^{m}\right)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{2^{n}n!}(M-m)^{n}\left(e^{M}-e^{m}\right)\|x\|\,\|y\|$$

and

$$\left|\left\langle e^{A}x,y\right\rangle - e^{\frac{m+M}{2}}\sum_{k=0}^{n}\frac{1}{k!}\left\langle \left(A-\frac{m+M}{2}1_{H}\right)^{k}x,y\right\rangle\right| \tag{2.34}$$
$$\leq \frac{1}{2^{n+1}(n+1)!}e^{M}(M-m)^{n+1}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{2^{n+1}(n+1)!}e^{M}(M-m)^{n+1}\|x\|\,\|y\|$$

for any $x,y\in H$.

The same Theorems 2.3 and 2.5 applied for the logarithmic function produce:

PROPOSITION 2.9. *Let* A *be a positive definite operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]\subset(0,\infty)$ *and* $\{E_{\lambda}\}_{\lambda}$ *be its spectral family, then for any* $c\in[m,M]$

we have the inequalities

$$\left|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle (A - c1_H)^k x, y\right\rangle}{kc^k}\right| \tag{2.35}$$
$$\le \frac{1}{n}\left[\frac{(c-m)^n (c^n - m^n)}{c^n m^n} \bigvee_m^c \left(\langle E_{(\cdot)}x, y\rangle\right) + \frac{(M-c)^n (M^n - c^n)}{M^m c^m} \bigvee_c^M \left(\langle E_{(\cdot)}x, y\rangle\right)\right]$$
$$\le \frac{1}{n}\max\left\{\frac{(c-m)^n (c^n - m^n)}{c^n m^n}, \frac{(M-c)^n (M^n - c^n)}{M^m c^m}\right\} \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$
$$\le \frac{1}{n}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n \frac{(M^n - m^n)}{M^m m^m} \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$
$$\le \frac{1}{n}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n \frac{(M^n - m^n)}{M^m m^m} \|x\| \|y\|$$

and

$$\left|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln c - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle (A - c1_H)^k x, y\right\rangle}{kc^k}\right| \tag{2.36}$$
$$\le \frac{1}{(n+1)m^{n+1}}\left[(c-m)^{n+1} \bigvee_m^c \left(\langle E_{(\cdot)}x, y\rangle\right) + (M-c)^{n+1} \bigvee_c^M \left(\langle E_{(\cdot)}x, y\rangle\right)\right]$$
$$\le \frac{1}{(n+1)m^{n+1}}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1} \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$
$$\le \frac{1}{(n+1)m^{n+1}}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1} \|x\| \|y\|$$

for any $x, y \in H$.

REMARK 2.4. The best inequalities we can get from (2.35) and (2.36) are

$$\left|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln\left(\frac{m+M}{2}\right) - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y\right\rangle}{k\left(\frac{m+M}{2}\right)^k}\right|$$
$$\le \frac{1}{2^n n}(M-m)^n \frac{(M^n - m^n)}{M^m m^m} \bigvee_{m-0}^M \left(\langle E_{(\cdot)}x, y\rangle\right)$$
$$\le \frac{1}{2^n n}(M-m)^n \frac{(M^n - m^n)}{M^m m^m} \|x\| \|y\| \tag{2.37}$$

and

$$\left|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln\left(\frac{m+M}{2}\right) - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y\right\rangle}{k\left(\frac{m+M}{2}\right)^k}\right|$$
$$\leq \frac{1}{2^{n+1}(n+1)}\left(\frac{M}{m}-1\right)^{n+1} \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$$
$$\leq \frac{1}{2^{n+1}(n+1)}\left(\frac{M}{m}-1\right)^{n+1}\|x\|\,\|y\| \tag{2.38}$$

for any $x, y \in H$.

3. PERTURBED VERSION

3.1. Some Identities. The following result provides a perturbed Taylor's type representation for a function of selfadjoint operators in Hilbert spaces.

THEOREM 3.1 (Dragomir, 2010, [**4**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the equalities*

$$\begin{aligned} f(A) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k \\ &\quad + \left[f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(M-c)^k\right] 1_H \\ &\quad + V_n(f, c, m, M) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &V_n(f, c, m, M) \\ :&= \frac{(-1)^n}{(n-1)!}\int_{m-0}^{M}\left(\int_c^\lambda (t-\lambda)^{n-1} d\left(f^{(n)}(t)\right)\right) E_\lambda d\lambda. \end{aligned} \tag{3.2}$$

PROOF. We utilize the Taylor's formula for functions $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!}\int_c^\lambda (\lambda - t)^n d\left(f^{(n)}(t)\right) \tag{3.3}$$

for any $\lambda, c \in [m, M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m, M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\begin{aligned} \int_{m-0}^{M} f(\lambda)\, dE_\lambda &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\int_{m-0}^{M} (\lambda - c)^k dE_\lambda \\ &\quad + \frac{1}{n!}\int_{m-0}^{M}\left(\int_c^\lambda (\lambda - t)^n d\left(f^{(n)}(t)\right)\right) dE_\lambda \end{aligned}$$

which, by the spectral representation theorem, produces the equality

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(A-c1_H)^k+\frac{1}{n!}\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda \tag{3.4}$$

that is of interest in itself as well.

Now, integrating by parts in the Riemann-Stieltjes integral and using the Leibnitz formula for integrals with parameters, we have

$$\begin{aligned}&\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda \\ &=E_\lambda\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)\Bigg|_{m-0}^{M} \\ &-\int_{m-0}^{M}E_\lambda d\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right) \\ &=\left(\int_c^{M}(M-t)^n d\left(f^{(n)}(t)\right)\right)1_H \\ &-n\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^{n-1} d\left(f^{(n)}(t)\right)\right)E_\lambda d\lambda\end{aligned} \tag{3.5}$$

and, since by the Taylor's formula (3.3) we have

$$\frac{1}{n!}\int_c^{M}(M-t)^n d\left(f^{(n)}(t)\right)=f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(M-c)^k, \tag{3.6}$$

then, by (3.4) and (3.6), we deduce the equality (3.1) with the integral representation for the remainder provided by (3.2). ∎

The following particular instances are of interest for applications:

COROLLARY 3.2 (Dragomir, 2010, [**4**]). *With the assumptions of the above Theorem 3.1, we have the equalities*

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)(A-m1_H)^k+\left[f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)(M-m)^k\right]1_H+T_n(f,c,m,M) \tag{3.7}$$

where

$$T_n(f,m,M):=-\frac{1}{(n-1)!}\int_{m-0}^{M}\left(\int_m^{\lambda}(\lambda-t)^{n-1} d\left(f^{(n)}(t)\right)\right)E_\lambda d\lambda \tag{3.8}$$

and

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left(A-\frac{m+M}{2}1_H\right)^k + \left[f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left(\frac{M-m}{2}\right)^k\right]1_H + W_n(f,c,m,M) \tag{3.9}$$

where

$$W_n(f,m,M) := \frac{(-1)^n}{(n-1)!}\int_{m-0}^{M}\left(\int_{\frac{m+M}{2}}^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)E_\lambda d\lambda \tag{3.10}$$

and

$$f(A)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(M)(M1_H-A)^k+Y_n(f,c,m,M) \tag{3.11}$$

where

$$Y_n(f,m,M) := \frac{(-1)^{n+1}}{(n-1)!}\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)E_\lambda d\lambda, \tag{3.12}$$

respectively.

REMARK 3.1. In order to give some examples we use the simplest representation, namely (3.11) for the exponential and the logarithmic functions.

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then we have the representation

$$e^A=e^M\sum_{k=0}^{n}\frac{(-1)^k}{k!}(M1_H-A)^k + \frac{(-1)^{n+1}}{(n-1)!}\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}e^t dt\right)E_\lambda d\lambda. \tag{3.13}$$

In the case when A is positive definite, i.e., $m>0$, then we have the representation

$$\ln A=(\ln M)1_H-\sum_{k=1}^{n}\frac{(M1_H-A)^k}{kM^k} - n\int_{m-0}^{M}\left(\int_{\lambda}^{M}\frac{(t-\lambda)^{n-1}}{t^{n+1}}dt\right)E_\lambda d\lambda. \tag{3.14}$$

3.2. Error Bounds for $f^{(n)}$ of Bounded Variation. We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

THEOREM 3.3 (Dragomir, 2010, [**4**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ (the interior of I) and let n be an*

integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequalities

$$
\begin{aligned}
&|\langle V_n(f, c, m, M) x, y\rangle| \\
&\leq \frac{1}{(n-1)!} \int_{m-0}^{c} (c-\lambda)^{n-1} \bigvee_{\lambda}^{c} \left(f^{(n)}\right) |\langle E_\lambda x, y\rangle| \, d\lambda \\
&+ \frac{1}{(n-1)!} \int_{c}^{M} (\lambda - c)^{n-1} \bigvee_{c}^{\lambda} \left(f^{(n)}\right) |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{(n-1)!} \bigvee_{m}^{c} \left(f^{(n)}\right) \int_{m-0}^{c} (c-\lambda)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&+ \frac{1}{(n-1)!} \bigvee_{c}^{M} \left(f^{(n)}\right) \int_{c}^{M} (\lambda - c)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{(n-1)!} \max\left\{ \bigvee_{m}^{c} \left(f^{(n)}\right), \bigvee_{c}^{M} \left(f^{(n)}\right) \right\} \int_{m-0}^{M} |\lambda - c|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{n!} \max\left\{ \bigvee_{m}^{c} \left(f^{(n)}\right), \bigvee_{c}^{M} \left(f^{(n)}\right) \right\} B_n(c, m, M, x, y),
\end{aligned}
\tag{3.15}
$$

for any $x, y \in H$, where

$$
\begin{aligned}
&B_n(c, m, M, x, y) \\
&:= \begin{cases}
[(M-c)^n + (c-m)^n] \|x\| \|y\| ; \\[2ex]
C_n(c, m, M, x, y); \\[2ex]
n \left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^{n-1} \\
\times \left[\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle\right]^{1/2}
\end{cases}
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
&C_n(c, m, M, x, y) \\
&:= \left[\langle [(M-c)^n 1_H - sgn(A - c1_H) |A - c1_H|^n] x, x\rangle\right]^{1/2} \\
&\times \left[\langle [(M-c)^n 1_H - sgn(A - c1_H) |A - c1_H|^n] y, y\rangle\right]^{1/2}.
\end{aligned}
\tag{3.17}
$$

Here the operator function $sgn(A - c1_H) |A - c1_H|^n$ is generated by the continuous function $sgn(\cdot - c) |\cdot - c|^n$ defined on the interval $[m, M]$.

PROOF. From the identities (3.1) and (3.2) we have

$$\begin{aligned}
&|\langle V_n(f,c,m,M)x,y\rangle| \\
&= \left|\frac{1}{(n-1)!}\int_{m-0}^{M}\left(\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\langle E_\lambda x,y\rangle\, d\lambda\right| \\
&\le \frac{1}{(n-1)!}\left|\int_{m-0}^{c}\left(\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\langle E_\lambda x,y\rangle\, d\lambda\right| \\
&+ \frac{1}{(n-1)!}\left|\int_{c}^{M}\left(\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\langle E_\lambda x,y\rangle\, d\lambda\right| \\
&\le \frac{1}{(n-1)!}\int_{m-0}^{c}\left|\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| |\langle E_\lambda x,y\rangle|\, d\lambda \\
&+ \frac{1}{(n-1)!}\int_{c}^{M}\left|\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| |\langle E_\lambda x,y\rangle|\, d\lambda
\end{aligned} \tag{3.18}$$

for any $x,y\in H.$

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{3.19}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

By the same property (3.19) we have

$$\left|\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| \le (c-\lambda)^{n-1}\bigvee_\lambda^c\left(f^{(n)}\right) \tag{3.20}$$

for $\lambda\in[m,c]$ and

$$\left|\int_c^\lambda (t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| \le (\lambda-c)^{n-1}\bigvee_c^\lambda\left(f^{(n)}\right) \tag{3.21}$$

for $\lambda\in[c,M]$.

Now, on making use of (3.18) and (3.20)-(3.21) we deduce

$$\begin{aligned}
&|\langle V_n(f,c,m,M)x,y\rangle| \\
&\le \frac{1}{(n-1)!}\int_{m-0}^{c}(c-\lambda)^{n-1}\bigvee_\lambda^c\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\, d\lambda \\
&+ \frac{1}{(n-1)!}\int_{c}^{M}(\lambda-c)^{n-1}\bigvee_c^\lambda\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\, d\lambda
\end{aligned}$$

for any $x,y\in H$ which proves the first part of (3.15).

The second and the third inequalities follow by the properties of the integral.

For the last part we observe that

$$\int_{m-0}^{M} |\lambda - c|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \le \max_{\lambda\in[m,M]} |\langle E_\lambda x, y\rangle| \int_{m}^{M} |\lambda - c|^{n-1} \, d\lambda$$
$$\le \frac{1}{n} \|x\| \|y\| [(M-c)^n + (c-m)^n]$$

for any $x, y \in H$,and the proof for the first branch of $B(c, m, M, x, y)$ is complete.

Now, to prove the inequality for the second branch of $B(c, m, M, x, y)$ we use the fact that if P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality that provides a generalization of the Schwarz inequality in H can be stated

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle \tag{3.22}$$

for any $x, y \in H$.

If we use (3.22) and the Cauchy-Buniakowski-Schwarz weighted integral inequality we can write that

$$\begin{aligned}
&\int_{m-0}^{M} |\lambda - c|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\le \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \, d\lambda \\
&\le \left(\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda\right)^{1/2} \left(\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda y, y\rangle \, d\lambda\right)^{1/2}
\end{aligned} \tag{3.23}$$

for any $x, y \in H$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$
\begin{aligned}
&\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_{\lambda}x,x\rangle\,d\lambda \\
&=\int_{m-0}^{c}(c-\lambda)^{n-1}\langle E_{\lambda}x,x\rangle\,d\lambda+\int_{c}^{M}(\lambda-c)^{n-1}\langle E_{\lambda}x,x\rangle\,d\lambda \\
&=\frac{1}{n}\left[-\int_{m-0}^{c}\langle E_{\lambda}x,x\rangle\,d(c-\lambda)^{n}+\int_{c}^{M}\langle E_{\lambda}x,x\rangle\,d(\lambda-c)^{n}\right] \\
&=\frac{1}{n}\left[-(c-\lambda)^{n}\langle E_{\lambda}x,x\rangle\Big|_{m-0}^{c}+\int_{m-0}^{c}(c-\lambda)^{n}\,d\langle E_{\lambda}x,x\rangle\right] \\
&+\frac{1}{n}\left[\langle E_{\lambda}x,x\rangle(\lambda-c)^{n}\Big|_{c}^{M}-\int_{c}^{M}(\lambda-c)^{n}\,d\langle E_{\lambda}x,x\rangle\right] \\
&=\frac{1}{n}\int_{m-0}^{c}(c-\lambda)^{n}\,d\langle E_{\lambda}x,x\rangle \\
&+\frac{1}{n}\left[\|x\|^{2}(M-c)^{n}-\int_{c}^{M}(\lambda-c)^{n}\,d\langle E_{\lambda}x,x\rangle\right] \\
&=\frac{1}{n}\|x\|^{2}(M-c)^{n} \\
&+\frac{1}{n}\left[\int_{m-0}^{c}(c-\lambda)^{n}\,d\langle E_{\lambda}x,x\rangle-\int_{c}^{M}(\lambda-c)^{n}\,d\langle E_{\lambda}x,x\rangle\right] \\
&=\frac{1}{n}\left[\|x\|^{2}(M-c)^{n}-\int_{m-0}^{M}sgn(\lambda-c)\,|\lambda-c|^{n}\,d\langle E_{\lambda}x,x\rangle\right] \\
&=\frac{1}{n}\left[\langle[(M-c)^{n}1_{H}-sgn(A-c1_{H})\,|A-c1_{H}|^{n}]\,x,x\rangle\right]
\end{aligned}
\tag{3.24}
$$

for any $x\in H$, and a similar relation for y, namely

$$
\begin{aligned}
&\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_{\lambda}y,y\rangle\,d\lambda \\
&=\frac{1}{n}\left[\langle[(M-c)^{n}1_{H}-sgn(A-c1_{H})\,|A-c1_{H}|^{n}]\,y,y\rangle\right]
\end{aligned}
\tag{3.25}
$$

for any $y\in H$.

The inequality (3.23) and the equalities (3.24) and (3.25) produce the second bound in (3.16).

Finally, observe also that

$$
\begin{aligned}
(3.26)\quad & \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&= \int_{m-0}^{c} (c-\lambda)^{n-1} \langle E_\lambda x, x\rangle \, d\lambda + \int_{c}^{M} (\lambda - c)^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&\leq (c-m)^{n-1} \int_{m-0}^{c} \langle E_\lambda x, x\rangle \, d\lambda + (M-c)^{n-1} \int_{c}^{M} \langle E_\lambda x, x\rangle \, d\lambda \\
&\leq \max\left\{(c-m)^{n-1}, (M-c)^{n-1}\right\} \int_{m-0}^{M} \langle E_\lambda x, x\rangle \, d\lambda \\
&= \left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^{n-1} \\
&\times \left[\langle E_\lambda x, x\rangle \lambda\big|_{m-0}^{M} - \int_{m-0}^{M} \lambda d \langle E_\lambda x, x\rangle\right] \\
&= \left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^{n-1} \langle (M 1_H - A) x, x\rangle
\end{aligned}
$$

for any $x \in H$ and similarly,

$$
\begin{aligned}
(3.27)\quad & \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&\leq \left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^{n-1} \langle (M 1_H - A) y, y\rangle
\end{aligned}
$$

for any $y \in H$.

On making use of (3.23), (3.26) and (3.27) we deduce the last bound provided in (3.16). ∎

The following particular cases are of interest for applications

COROLLARY 3.4 (Dragomir, 2010, [**4**]). *With the assumption of Theorem 3.3 we have the inequalities*

$$
\begin{aligned}
(3.28)\quad & |\langle T_n(f, m, M) x, y\rangle| \\
&\leq \frac{1}{(n-1)!} \int_{m-0}^{M} (\lambda - m)^{n-1} \bigvee_{m}^{\lambda} \left(f^{(n)}\right) |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)}\right) \int_{m-0}^{M} (\lambda - m)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)}\right) B_n(m, M, x, y),
\end{aligned}
$$

for any $x, y \in H$, *where*

(3.29) $$B_n(m, M, x, y)$$

$$:= \begin{cases} (M-m)^n \|x\| \|y\|; \\ C_n(m, M, x, y); \\ n (M-m)^{n-1} [\langle (M 1_H - A) x, x \rangle \langle (M 1_H - A) y, y \rangle]^{1/2} \end{cases}$$

and

(3.30) $$\begin{aligned} & C_n(m, M, x, y) \\ & := [\langle [(M-m)^n 1_H - (A - m 1_H)^n] x, x \rangle]^{1/2} \\ & \times [\langle [(M-m)^n 1_H - (A - m 1_H)^n] y, y \rangle]^{1/2}. \end{aligned}$$

The proof follows from Theorem 3.3 by choosing $c = m$ and performing the corresponding calculations.

COROLLARY 3.5 (Dragomir, 2010, [**4**]). *With the assumption of Theorem 3.3 we have the inequalities*

(3.31) $$\begin{aligned} & |\langle Y_n(f, m, M) x, y \rangle| \\ & \le \frac{1}{(n-1)!} \int_{m-0}^{M} (M-\lambda)^{n-1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right) |\langle E_\lambda x, y \rangle| \, d\lambda \\ & \le \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)}\right) \int_{m-0}^{M} (M-\lambda)^{n-1} |\langle E_\lambda x, y \rangle| \, d\lambda \\ & \le \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)}\right) \tilde{B}_n(m, M, x, y), \end{aligned}$$

for any $x, y \in H$, *where*

(3.32) $$\tilde{B}_n(m, M, x, y)$$

$$:= \begin{cases} (M-m)^n \|x\| \|y\|; \\ \tilde{C}_n(m, M, x, y); \\ n (M-m)^{n-1} [\langle (M 1_H - A) x, x \rangle \langle (M 1_H - A) y, y \rangle]^{1/2} \end{cases}$$

and

(3.33) $$\begin{aligned} & \tilde{C}_n(m, M, x, y) \\ & := [\langle (M 1_H - A)^n x, x \rangle]^{1/2} [\langle (M 1_H - A)^n y, y \rangle]^{1/2}. \end{aligned}$$

The proof follows from Theorem 3.3 by choosing $c = M$ and performing the corresponding calculations.

The best bound we can get is incorporated in

COROLLARY 3.6 (Dragomir, 2010, [**4**]). *With the assumption of Theorem 3.3 we have the inequalities*

$$(3.34)\quad \begin{aligned} &\left|\left\langle W_n(f,m,M)x,y\right\rangle\right| \\ &\leq \frac{1}{(n-1)!}\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-\lambda\right)^{n-1}\bigvee_{\lambda}^{\frac{m+M}{2}}\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\ &+\frac{1}{(n-1)!}\int_{\frac{m+M}{2}}^{M}\left(\lambda-\frac{m+M}{2}\right)^{n-1}\bigvee_{\frac{m+M}{2}}^{\lambda}\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\ &\leq \frac{1}{(n-1)!}\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-\lambda\right)^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\ &+\frac{1}{(n-1)!}\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\int_{\frac{m+M}{2}}^{M}\left(\lambda-\frac{m+M}{2}\right)^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\ &\leq \frac{1}{(n-1)!}\max\left\{\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right),\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\right\} \\ &\times\int_{m-0}^{M}\left|\lambda-\frac{m+M}{2}\right|^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\ &\leq \frac{1}{n!}\max\left\{\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right),\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\right\}\breve{B}_n(m,M,x,y), \end{aligned}$$

for any $x,y\in H$, *where*

$$(3.35)\quad \begin{aligned} &\breve{B}_n(m,M,x,y) \\ :&=\begin{cases} \frac{(M-m)^n}{2^{n-1}}\left\|x\right\|\left\|y\right\|; \\ \\ \breve{C}(m,M,x,y) \\ \\ \frac{n}{2^{n-1}}\left(M-m\right)^{n-1}\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(M1_H-A\right)y,y\right\rangle\right]^{1/2} \end{cases} \end{aligned}$$

and

$$(3.36)\quad \begin{aligned} &\breve{C}_n(m,M,x,y) \\ &:=\left[\left\langle\left[\frac{(M-m)^n}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^n\right]x,x\right\rangle\right]^{1/2} \\ &\times\left[\left\langle\left[\frac{(M-m)^n}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^n\right]y,y\right\rangle\right]^{1/2}. \end{aligned}$$

3.3. Error Bounds for $f^{(n)}$ Lipschitzian. The case when the n-th derivative is Lipschitzian is incorporated in the following result:

THEOREM 3.7 (Dragomir, 2010, [**4**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m,M]\subset\mathring{I}$ *(the interior of* I*) and let* n *be an*

integer with $n \geq 1$*. If* $f : I \to \mathbb{C}$ *is such that the* n*-th derivative* $f^{(n)}$ *is Lipschitzian with the constant* $L_n > 0$ *on the interval* $[m, M]$*, then for any* $c \in [m, M]$ *we have the inequalities*

$$\begin{aligned} &|\langle V_n(f, c, m, M)x, y\rangle| \\ &\leq \frac{1}{n!}L_n \int_{m-0}^{M} |\lambda - c|^n |\langle E_\lambda x, y\rangle| d\lambda \\ &\leq \frac{1}{(n+1)!}L_n \\ &\times \begin{cases} \left[(M-c)^{n+1} + (c-m)^{n+1}\right] \|x\| \|y\|; \\ \\ \left[\left\langle \left[(M-c)^{n+1} 1_H - sgn(A - c1_H)|A - c1_H|^{n+1}\right] x, x\right\rangle\right]^{1/2} \\ \times \left[\left\langle \left[(M-c)^{n+1} 1_H - sgn(A - c1_H)|A - c1_H|^{n+1}\right] y, y\right\rangle\right]^{1/2}; \\ \\ (n+1)\left[\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right]^n \\ \times \left[\langle (M1_H - A)x, x\rangle \langle (M1_H - A)y, y\rangle\right]^{1/2}; \end{cases} \end{aligned} \tag{3.37}$$

for any $x, y \in H$.

PROOF. From the inequality (3.18) in the proof of Theorem 3.3 we have

$$\begin{aligned} &|\langle V_n(f, c, m, M)x, y\rangle| \\ &\leq \frac{1}{(n-1)!}\int_{m-0}^{c} \left|\int_{\lambda}^{c} (t-\lambda)^{n-1} d\left(f^{(n)}(t)\right)\right| |\langle E_\lambda x, y\rangle| d\lambda \\ &+ \frac{1}{(n-1)!}\int_{c}^{M} \left|\int_{c}^{\lambda} (t-\lambda)^{n-1} d\left(f^{(n)}(t)\right)\right| |\langle E_\lambda x, y\rangle| d\lambda \end{aligned} \tag{3.38}$$

for any $x, y \in H$.

Further, we utilize the fact that for an $L-$Lipschitzian function, $p : [\alpha, \beta] \to \mathbb{C}$ and a Riemann integrable function $v : [\alpha, \beta] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\, dv(s)\right| \leq L \int_\alpha^\beta |p(s)|\, ds.$$

On making use of this property we have for $\lambda \in [m, c]$ that

$$\left|\int_\lambda^c (t-\lambda)^{n-1} d\left(f^{(n)}(t)\right)\right| \leq L_n \int_\lambda^c (t-\lambda)^{n-1} dt = \frac{1}{n}L_n (c-\lambda)^n$$

and for $\lambda \in [c, M]$ that

$$\left|\int_c^\lambda (t-\lambda)^{n-1} d\left(f^{(n)}(t)\right)\right| \leq L_n \int_c^\lambda (\lambda - t)^{n-1} dt = \frac{1}{n}L_n (\lambda - c)^n$$

which, by (3.38) produces the inequality

$$\begin{aligned} &|\langle V_n(f, c, m, M)x, y\rangle| \\ &\leq \frac{1}{n!}L_n \int_{m-0}^{c} (c-\lambda)^n |\langle E_\lambda x, y\rangle| d\lambda + \frac{1}{n!}L_n \int_{c}^{M} (\lambda - c)^n |\langle E_\lambda x, y\rangle| d\lambda \\ &= \frac{1}{n!}L_n \int_{m-0}^{M} |\lambda - c|^n |\langle E_\lambda x, y\rangle| d\lambda, \end{aligned} \tag{3.39}$$

for any $x, y \in H$, and the first part of (3.37) is proved.

Finally, we observe that the bounds for the integral $\int_{m-0}^{M} |\lambda - c|^n |\langle E_\lambda x, y\rangle| \, d\lambda$ can be obtained in a similar manner to the ones from the proof of Theorem 3.3 and the details are omitted. ∎

The following result contains error bounds for the particular expansions considered in Corollary 3.2:

COROLLARY 3.8 (Dragomir, 2010, [**4**]). *With the assumptions in Theorem 3.7 we have the inequalities*

$$
\begin{aligned}
&|\langle T_n(f,m,M)x,y\rangle| \\
&\le \frac{1}{n!}L_n \int_{m-0}^{M} (\lambda - m)^n |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\le \frac{1}{(n+1)!}L_n \\
&\times \begin{cases} (M-m)^{n+1}\|x\|\|y\|; \\ \\ \left[\left\langle\left[(M-m)^{n+1}1_H - (A - m1_H)^{n+1}\right]x,x\right\rangle\right]^{1/2} \\ \times \left[\left\langle\left[(M-m)^{n+1}1_H - (A - m1_H)^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\ \\ (n+1)(M-m)^n\left[\langle (M1_H - A)x,x\rangle \langle (M1_H - A)y,y\rangle\right]^{1/2}; \end{cases}
\end{aligned} \tag{3.40}
$$

and

$$
\begin{aligned}
&|\langle Y_n(f,m,M)x,y\rangle| \\
&\le \frac{1}{n!}L_n \int_{m-0}^{M} (M - \lambda)^n |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\le \frac{1}{(n+1)!}L_n \\
&\times \begin{cases} (M-m)^{n+1}\|x\|\|y\|; \\ \\ \left[\left\langle\left[(M1_H - A)^{n+1}\right]x,x\right\rangle\right]^{1/2}\left[\left\langle\left[(M1_H - A)^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\ \\ (n+1)\left[(M-m)\right]^n\left[\langle (M1_H - A)x,x\rangle \langle (M1_H - A)y,y\rangle\right]^{1/2}; \end{cases}
\end{aligned} \tag{3.41}
$$

and

$$
\begin{aligned}
&|\langle W_n(f,m,M)x,y\rangle| \\
&\le \frac{1}{n!}L_n \int_{m-0}^{M} \left|\lambda - \frac{m+M}{2}\right|^n |\langle E_\lambda x, y\rangle| \, d\lambda \le \frac{1}{(n+1)!}L_n \\
&\times \begin{cases} \frac{(M-m)^{n+1}}{2^n}\|x\|\|y\|; \\ \\ \left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H - sgn\left(A - \frac{m+M}{2}1_H\right)\left|A - \frac{m+M}{2}1_H\right|^{n+1}\right]x,x\right\rangle\right]^{1/2} \\ \times \left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H - sgn\left(A - \frac{m+M}{2}1_H\right)\left|A - \frac{m+M}{2}1_H\right|^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\ \\ \frac{n+1}{2^n}(M-m)^n\left[\langle (M1_H - A)x,x\rangle \langle (M1_H - A)y,y\rangle\right]^{1/2}; \end{cases}
\end{aligned} \tag{3.42}
$$

for any $x, y \in H$, *respectively.*

3.4. Applications. In order to obtain various vectorial operator inequalities one can use the above results for particular elementary functions. We restrict ourself to only two examples of functions, namely the exponential and the logarithmic functions.

If we apply Corollary 3.5 for the exponential function, we can state the following result:

PROPOSITION 3.9. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. Then we have*

$$\left| \left\langle e^A x, y \right\rangle - e^M \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left\langle (M1_H - A)^k x, y \right\rangle \right| \tag{3.43}$$
$$\begin{aligned}
&\leq \frac{1}{(n-1)!} \int_{m-0}^{M} (M-\lambda)^{n-1} \left(e^M - e^\lambda\right) |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{(n-1)!} \left(e^M - e^m\right) \int_{m-0}^{M} (M-\lambda)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{n!} \left(e^M - e^m\right) \\
&\times \begin{cases} (M-m)^n \|x\| \|y\| ; \\ [\langle (M1_H - A)^n x, x\rangle]^{1/2} [\langle (M1_H - A)^n y, y\rangle]^{1/2} \\ n (M-m)^{n-1} [\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle]^{1/2} \end{cases}
\end{aligned}$$

for any $x, y \in H$.

If we use Corollary 3.8 then we can provide other bounds as follows:

PROPOSITION 3.10. *With the assumptions of Proposition 3.9 we have*

$$\left| \left\langle e^A x, y \right\rangle - e^M \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left\langle (M1_H - A)^k x, y \right\rangle \right| \tag{3.44}$$
$$\begin{aligned}
&\leq \frac{1}{n!} e^M \int_{m-0}^{M} (M-\lambda)^n |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\leq \frac{1}{(n+1)!} e^M \\
&\times \begin{cases} (M-m)^{n+1} \|x\| \|y\| ; \\ \left[\left\langle \left[(M1_H - A)^{n+1}\right] x, x\right\rangle\right]^{1/2} \left[\left\langle \left[(M1_H - A)^{n+1}\right] y, y\right\rangle\right]^{1/2} ; \\ (n+1) [(M-m)]^n [\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle]^{1/2} ; \end{cases}
\end{aligned}$$

Finally, the Corollaries 3.5 and 3.8 produce the following results for the logarithmic function:

PROPOSITION 3.11. *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ be its spectral family, then*

$$\left| \langle \ln Ax, y\rangle - \langle x, y\rangle \ln M + \sum_{k=1}^{n} \frac{\left\langle (M1_H - A)^k x, y\right\rangle}{kM^k} \right| \tag{3.45}$$
$$\leq \int_{m-0}^{M} (M-\lambda)^{n-1} \frac{M^n - \lambda^n}{M^n \lambda^n} |\langle E_\lambda x, y\rangle| \, d\lambda$$
$$\leq \frac{M^n - m^n}{M^n m^n} \int_{m-0}^{M} (M-\lambda)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda$$
$$\leq \frac{M^n - m^n}{nM^n m^n} \times \begin{cases} (M-m)^n \|x\| \|y\|; \\ [\langle (M1_H - A)^n x, x\rangle]^{1/2} [\langle (M1_H - A)^n y, y\rangle]^{1/2} \\ n(M-m)^{n-1} [\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle]^{1/2} \end{cases}$$

and

$$\left| \langle \ln Ax, y\rangle - \langle x, y\rangle \ln M + \sum_{k=1}^{n} \frac{\left\langle (M1_H - A)^k x, y\right\rangle}{kM^k} \right| \tag{3.46}$$
$$\leq \frac{1}{m^{n+1}} \int_{m-0}^{M} (M-\lambda)^n |\langle E_\lambda x, y\rangle| \, d\lambda$$
$$\leq \frac{1}{(n+1) m^{n+1}} \times \begin{cases} (M-m)^{n+1} \|x\| \|y\|; \\ \left[\left\langle (M1_H - A)^{n+1} x, x\right\rangle\right]^{1/2} \left[\left\langle (M1_H - A)^{n+1} y, y\right\rangle\right]^{1/2}; \\ (n+1) [(M-m)]^n [\langle (M1_H - A) x, x\rangle \langle (M1_H - A) y, y\rangle]^{1/2}; \end{cases}$$

4. TWO POINTS TAYLOR'S TYPE INEQUALITIES

4.1. Representation Results. We start with the following identity that has been obtained in [**2**]. For the sake of completeness we give here a short proof as well.

LEMMA 4.1 (Dragomir, 2010, [**2**]). *Let I be a closed subinterval on $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let n be a nonnegative integer. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is of* *bounded variation on the interval* $[a,b]$ *, then, for any* $x \in [a,b]$ *we have the representation*

$$f(x) = \frac{1}{b-a}\left[(b-x) f(a) + (x-a) f(b)\right] + \frac{(b-x)(x-a)}{b-a} \times \sum_{k=1}^{n} \frac{1}{k!}\left\{(x-a)^{k-1} f^{(k)}(a) + (-1)^k (b-x)^{k-1} f^{(k)}(b)\right\} + \frac{1}{b-a}\int_a^b S_n(x,t)\, d\left(f^{(n)}(t)\right), \tag{4.1}$$

where the kernel $S_n : [a,b]^2 \to \mathbb{R}$ *is given by*

$$S_n(x,t) = \frac{1}{n!} \times \begin{cases} (x-t)^n (b-x) & \text{if } a \le t \le x; \\ (-1)^{n+1} (t-x)^n (x-a) & \text{if } x < t \le b \end{cases} \tag{4.2}$$

and the integral in the remainder is taken in the Riemann-Stieltjes sense.

PROOF. We utilize the following Taylor's representation formula for functions $f : I \to \mathbb{R}$ such that the n-th derivatives $f^{(n)}$ are of locally bounded variation on the interval I,

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-c)^k f^{(k)}(c) + \frac{1}{n!}\int_c^x (x-t)^n\, d\left(f^{(n)}(t)\right), \tag{4.3}$$

where x and c are in I and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $c = a$ and then $c = b$ in (4.3) we can write that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-a)^k f^{(k)}(a) + \frac{1}{n!}\int_a^x (x-t)^n\, d\left(f^{(n)}(t)\right), \tag{4.4}$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}(b-x)^k f^{(k)}(b) + \frac{(-1)^{n+1}}{n!}\int_x^b (t-x)^n\, d\left(f^{(n)}(t)\right), \tag{4.5}$$

for any $x \in [a,b]$.

Now, by multiplying (4.4) with $(b-x)$ and (4.5) with $(x-a)$ we get

$$(b-x) f(x) = (b-x) f(a) + (b-x)(x-a)\sum_{k=1}^{n} \frac{1}{k!}(x-a)^{k-1} f^{(k)}(a) + \frac{1}{n!}(b-x)\int_a^x (x-t)^n\, d\left(f^{(n)}(t)\right) \tag{4.6}$$

and

$$(x-a) f(x) = (x-a) f(b) + (b-x)(x-a) \sum_{k=1}^{n} \frac{(-1)^k}{k!} (b-x)^{k-1} f^{(k)}(b) + \frac{(-1)^{n+1}}{n!} (x-a) \int_x^b (t-x)^n d\left(f^{(n)}(t)\right) \tag{4.7}$$

respectively.

Finally, by adding the equalities (4.6) and (4.7) and dividing the sum with $(b-a)$, we obtain the desired representation (4.2). ∎

REMARK 4.1. The case $n = 0$ provides the representation

$$f(x) = \frac{1}{b-a} [(b-x) f(a) + (x-a) f(b)] + \frac{1}{b-a} \int_a^b S(x,t)\, d(f(t)) \tag{4.8}$$

for any $x \in [a, b]$, where

$$S(x,t) = \begin{cases} b-x & \text{if } a \le t \le x, \\ a-x & \text{if } x < t \le b, \end{cases}$$

and f is of bounded variation on $[a, b]$. This result was obtained by a different approach in [**1**].

The case $n = 1$ provides the representation

$$f(x) = \frac{1}{b-a} [(b-x) f(a) + (x-a) f(b)] + \frac{1}{b-a} \int_a^b Q(x,t)\, d(f'(t)), \tag{4.9}$$

where

$$Q(x,t) = \begin{cases} (a-t)(b-x) & \text{if } a \le t \le x, \\ (t-b)(x-a) & \text{if } x \le t \le b. \end{cases}$$

Notice that the representation (4.9) was obtained by a different approach in [**1**].

THEOREM 4.2 (Dragomir, 2010, [**3**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the representation*

$$\begin{aligned} f(A) = {} & \frac{1}{M-m} [f(m)(M1_H - A) + f(M)(A - m1_H)] \\ & + \frac{(M1_H - A)(A - m1_H)}{M-m} \\ & \times \sum_{k=1}^{n} \frac{1}{k!} \left\{ f^{(k)}(m)(A - m1_H)^{k-1} + (-1)^k f^{(k)}(M)(M1_H - A)^{k-1} \right\} \\ & + T_n(f, m, M), \end{aligned} \tag{4.10}$$

where the remainder $T_n(f,m,M)$ *is given by*

$$T_n(f,m,M) := \frac{1}{(M-m)\,n!}\int_{m-0}^{M} K_n(m,M,f;\lambda)\,dE_\lambda \tag{4.11}$$

and the kernel $K_n(m,M,f;\cdot)$ *has the representation*

$$\begin{aligned}K_n(m,M,f;\lambda) &:= (M-\lambda)\left(\int_m^\lambda (\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right)\\ &\quad+(-1)^{n+1}(\lambda-m)\left(\int_\lambda^M (t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right)\end{aligned} \tag{4.12}$$

for $\lambda\in[m,M]$.

PROOF. Utilising Lemma 4.1 we have the representation

$$\begin{aligned}f(\lambda) &= \frac{1}{M-m}\left[(M-\lambda)f(m)+(\lambda-m)f(M)\right]\\ &\quad+\frac{(M-\lambda)(\lambda-m)}{M-m}\\ &\quad\times\sum_{k=1}^{n}\frac{1}{k!}\left\{(\lambda-m)^{k-1}f^{(k)}(m)+(-1)^k(M-\lambda)^{k-1}f^{(k)}(M)\right\}\\ &\quad+\frac{1}{(M-m)\,n!}\left[(M-\lambda)\int_m^\lambda(\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right.\\ &\quad\left.+(-1)^{n+1}(\lambda-m)\int_\lambda^M(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right],\end{aligned} \tag{4.13}$$

for any $\lambda\in[m,M]$.

If we integrate (4.13) in the Riemann-Stieltjes sense on the interval $[m,M]$ with the integrator E_λ, then we get

$$\begin{aligned}&\int_{m-0}^{M} f(\lambda)\,dE_\lambda\\ &=\frac{1}{M-m}\int_{m-0}^{M}\left[(M-\lambda)f(m)+(\lambda-m)f(M)\right]dE_\lambda\\ &\quad+\int_{m-0}^{M}\frac{(M-\lambda)(\lambda-m)}{M-m}\sum_{k=1}^{n}\frac{1}{k!}\left\{(\lambda-m)^{k-1}f^{(k)}(m)\right.\\ &\quad\left.+(-1)^k(M-\lambda)^{k-1}f^{(k)}(M)\right\}dE_\lambda+\frac{1}{(M-m)\,n!}\\ &\quad\times\left[\int_{m-0}^{M}(M-\lambda)\left(\int_m^\lambda(\lambda-t)^n\,d\left(f^{(n)}(t)\right)\right)dE_\lambda\right.\\ &\quad\left.+(-1)^{n+1}\int_{m-0}^{M}(\lambda-m)\left(\int_\lambda^M(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right)dE_\lambda\right].\end{aligned} \tag{4.14}$$

Now, on making use of the spectral representation theorem we deduce from (4.14) the equality (4.1) with the remainder representation (4.2). ∎

REMARK 4.2. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its *spectral family.* In the case when

the function f is continuous and of bounded variation on $[m, M]$, then we get the representation

$$f(A) = \frac{1}{M-m}\left[f(m)(M1_H - A) + f(M)(A - m1_H)\right] + \frac{1}{(M-m)}\left[\int_{m-0}^{M}(M-\lambda)\left[f(\lambda) - f(m)\right]dE_\lambda - \int_{m-0}^{M}(\lambda - m)\left[f(M) - f(\lambda)\right]dE_\lambda\right]. \tag{4.15}$$

Also, if the derivative f' is of bounded variation, then we have the representation

$$f(A) = \frac{1}{M-m}\left[f(m)(M1_H - A) + f(M)(A - m1_H)\right] + \frac{1}{(M-m)}\left[\int_{m-0}^{M}(M-\lambda)\left(\int_{m}^{\lambda}(\lambda - t)\,d(f'(t))\right)dE_\lambda + \int_{m-0}^{M}(\lambda - m)\left(\int_{\lambda}^{M}(t-\lambda)\,d(f'(t))\right)dE_\lambda\right]. \tag{4.16}$$

EXAMPLE 4.1. *a. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family. If we consider the exponential function, then we get from (4.10) and (4.11) that*

$$e^A = \frac{1}{M-m}\left[e^m(M1_H - A) + e^M(A - m1_H)\right] + \frac{(M1_H - A)(A - m1_H)}{M-m} \times \sum_{k=1}^{n}\frac{1}{k!}\left\{e^m(A - m1_H)^{k-1} + (-1)^k e^M (M1_H - A)^{k-1}\right\} + \frac{1}{(M-m)\,n!} \times \left[\int_{m-0}^{M}(M-\lambda)\left(\int_{m}^{\lambda}(\lambda - t)^n e^t dt\right)dE_\lambda + (-1)^{n+1}\int_{m-0}^{M}(\lambda - m)\left(\int_{\lambda}^{M}(t-\lambda)^n e^t dt\right)dE_\lambda\right]. \tag{4.17}$$

b. If A is a positive definite selfadjoint operator with the spectrum $Sp(A) \subseteq [m, M] \subset (0, \infty)$ and $\{E_\lambda\}_\lambda$ is its spectral family, then we have the representation

$$\ln A = \frac{1}{M-m}\left[(M1_H - A)\ln m + (A - m1_H)\ln M\right] + \frac{(M1_H - A)(A - m1_H)}{M-m} \times \sum_{k=1}^{n}\frac{1}{k}\left\{(-1)^{k-1}\frac{(A - m1_H)^{k-1}}{m^k} - \frac{(M1_H - A)^{k-1}}{M^k}\right\} + \frac{1}{(M-m)}\left[(-1)^n\int_{m-0}^{M}(M-\lambda)\left(\int_{m}^{\lambda}\frac{(\lambda - t)^n}{t^{n+1}}dt\right)dE_\lambda - \int_{m-0}^{M}(\lambda - m)\left(\int_{\lambda}^{M}\frac{(t-\lambda)^n}{t^{n+1}}dt\right)dE_\lambda\right]. \tag{4.18}$$

The case of functions for which the n-th derivative $f^{(n)}$ is absolutely continuous is of interest for applications. In this case the remainder can be represented as follows:

THEOREM 4.3 (Dragomir, 2010, [**3**]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m, M]$, then we have the representation (4.10) where the remainder $T_n(f, m, M)$ is given by*

$$T_n(f, m, M) := \frac{1}{(M-m)\,n!} \int_{m-0}^{M} W_n(m, M, f; \lambda)\, E_\lambda d\lambda \tag{4.19}$$

and the kernel $W_n(m, M, f; \cdot)$ has the representation

$$\begin{aligned} W_n(m, M, f; \lambda) &:= (-1)^n \int_m^\lambda (\lambda - t)^{n-1} [nM + t - (n+1)\lambda] f^{(n+1)}(t)\, dt \\ &- \int_\lambda^M (t - \lambda)^{n-1} [t + nm - (n+1)\lambda] f^{(n+1)}(t)\, dt \end{aligned} \tag{4.20}$$

for $\lambda \in [m, M]$.

PROOF. Observe that, by Leibnitz's rule for differentiation under the integral sign, we have

$$\begin{aligned} &\frac{d}{d\lambda}\left[(M-\lambda)\left(\int_m^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt\right)\right] \\ &= -\int_m^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt + (M-\lambda)\frac{d}{d\lambda}\left(\int_m^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt\right) \\ &= -\int_m^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt + n(M-\lambda)\int_m^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t)\, dt \\ &= \int_m^\lambda (\lambda - t)^{n-1} [nM + t - (n+1)\lambda] f^{(n+1)}(t)\, dt \end{aligned} \tag{4.21}$$

for any $\lambda \in [m, M]$.

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} &\int_{m-0}^{M} (M-\lambda)\left(\int_m^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt\right) dE_\lambda \\ &= (M-\lambda)\left(\int_m^\lambda (\lambda - t)^n d\left(f^{(n)}(t)\right)\right) E_\lambda \Big|_{m-0}^{M} \\ &- \int_{m-0}^{M}\left(\int_m^\lambda (\lambda - t)^{n-1} [nM + t - (n+1)\lambda] f^{(n+1)}(t)\, dt\right) E_\lambda d\lambda \\ &= -\int_{m-0}^{M}\left(\int_m^\lambda (\lambda - t)^{n-1} [nM + t - (n+1)\lambda] f^{(n+1)}(t)\, dt\right) E_\lambda d\lambda. \end{aligned} \tag{4.22}$$

By Leibnitz's rule we also have

$$\frac{d}{d\lambda}\left[(\lambda-m)\left(\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt\right)\right] \tag{4.23}$$
$$=\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt+(\lambda-m)\frac{d}{d\lambda}\left(\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt\right)$$
$$=\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt-n(\lambda-m)\int_{\lambda}^{M}(t-\lambda)^{n-1}f^{(n+1)}(t)\,dt$$
$$=\int_{\lambda}^{M}(t-\lambda)^{n-1}[t+nm-(n+1)\lambda]f^{(n+1)}(t)\,dt$$

for any $\lambda\in[m,M]$.

Utilising the integration by parts and (4.24) we get

$$\int_{m-0}^{M}(\lambda-m)\left(\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt\right)dE_{\lambda} \tag{4.24}$$
$$=(\lambda-m)\left(\int_{\lambda}^{M}(t-\lambda)^{n}f^{(n+1)}(t)\,dt\right)E_{\lambda}\Big|_{m-0}^{M}$$
$$-\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}[t+nm-(n+1)\lambda]f^{(n+1)}(t)\,dt\right)E_{\lambda}d\lambda$$
$$=-\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}[t+nm-(n+1)\lambda]f^{(n+1)}(t)\,dt\right)E_{\lambda}d\lambda.$$

Finally, on utilizing the representation (4.11) for the remainder $T_n(f,m,M)$ and the equalities (4.22) and (4.24) we deduce (4.19). The details are omitted. ∎

REMARK 4.3. The case $n=1$ provides the following equality

$$f(A)=\frac{1}{M-m}[f(m)(M1_H-A)+f(M)(A-m1_H)] \tag{4.25}$$
$$+\frac{1}{(M-m)}\int_{m-0}^{M}W_1(m,M,f;\lambda)E_{\lambda}d\lambda,$$

where

$$W_1(m,M,f;\lambda) \tag{4.26}$$
$$:=\int_{m}^{\lambda}(2\lambda-M-t)f''(t)\,dt+\int_{\lambda}^{M}(2\lambda-t-m)f''(t)\,dt$$

for $\lambda\in[m,M]$.

4.2. Error Bounds for $f^{(n)}$ of Bonded Variation. The following result that provides bounds for the absolute value of the kernel $K_n(m,M,f;\cdot)$ holds:

LEMMA 4.4 (Dragomir, 2010, [**3**]). *Let I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$, let n be an integer with $n\geq 1$ and assume that $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ exists on the interval $[m,M]$.*

1. If $f^{(n)}$ is of bounded variation on $[m,M]$, then

$$\begin{aligned}
(4.27)\qquad & \left|K_n(m,M,f;\lambda)\right| \\
&\le (M-\lambda)(\lambda-m)^n \bigvee_m^\lambda \left(f^{(n)}\right) + (\lambda-m)(M-\lambda)^n \bigvee_\lambda^M \left(f^{(n)}\right) \\
&\le \frac{1}{4}(M-m)^2\left[(\lambda-m)^{n-1}\bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n-1}\bigvee_\lambda^M \left(f^{(n)}\right)\right] \\
&\le \frac{1}{4}(M-m)^2 J_n(m,M;\lambda)
\end{aligned}$$

where

$$(4.28)\qquad J_n(m,M;\lambda) := \begin{cases} \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\bigvee_m^M \left(f^{(n)}\right); \\ \\ \left[(\lambda-m)^{p(n-1)}+(M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(\bigvee_m^\lambda \left(f^{(n)}\right)\right)^q + \left(\bigvee_\lambda^M \left(f^{(n)}\right)\right)^q\right]^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_m^M \left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_m^\lambda \left(f^{(n)}\right)-\bigvee_\lambda^M \left(f^{(n)}\right)\right|\right] \\ \times\left[(\lambda-m)^{n-1}+(M-\lambda)^{n-1}\right] \end{cases}$$

and $\lambda\in[m,M]$.

2. If $\lambda\in(m,M)$ and $f^{(n)}$ is $L_{n,1,\lambda}$-Lipschitzian on $[m,\lambda]$ and $L_{n,2,\lambda}$-Lipschitzian on $[\lambda,M]$, then

$$\begin{aligned}
(4.29)\qquad & \left|K_n(m,M,f;\lambda)\right| \\
&\le \frac{1}{n+1}\left[L_{n,1,\lambda}(M-\lambda)(\lambda-m)^{n+1}+L_{n,2,\lambda}(\lambda-m)(M-\lambda)^{n+1}\right] \\
&\le \frac{1}{4(n+1)}\left[L_{n,1,\lambda}(\lambda-m)^n+L_{n,2,\lambda}(M-\lambda)^n\right] \\
&\le \frac{1}{4(n+1)} \\
&\quad\times\begin{cases} \left[(\lambda-m)^n+(M-\lambda)^n\right]\max\left\{L_{n,1,\lambda},L_{n,2,\lambda}\right\} \\ \\ \left[(\lambda-m)^{pn}+(M-\lambda)^{pn}\right]^{1/p}\left(L_{n,1,\lambda}^q+L_{n,2,\lambda}^q\right)^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^n\left(L_{n,1,\lambda}+L_{n,2,\lambda}\right) \end{cases}
\end{aligned}$$

and $\lambda\in[m,M]$.

In particular, if $f^{(n)}$ is L_n-Lipschitzian on $[m, M]$, then

$$|K_n(m, M, f; \lambda)| \le \frac{L_n}{n+1}\left[(M-\lambda)(\lambda-m)^{n+1} + (\lambda-m)(M-\lambda)^{n+1}\right] \le \frac{L_n(M-m)^2}{4(n+1)}\left[(\lambda-m)^n + (M-\lambda)^n\right] \tag{4.30}$$

and $\lambda \in [m, M]$.

3. If the function $f^{(n)}$ is monotonic nondecreasing on $[m, M]$, then

$$\begin{aligned} &|K_n(m, M, f; \lambda)| \\ &\le (M-\lambda)\left[n\int_m^\lambda (\lambda-t)^{n-1} f^{(n)}(t)\,dt - (\lambda-m)^n f^{(n)}(m)\right] \\ &+ (\lambda-m)\left[(M-\lambda)^n f^{(n)}(M) - n\int_\lambda^M (t-\lambda)^{n-1} f^{(n)}(t)\,dt\right] \\ &\le (M-\lambda)(\lambda-m) \\ &\times \left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M-\lambda)^{n-1}\left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right] \\ &\le \frac{1}{4}(M-m)^2 \\ &\times \left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda) - f^{(n)}(m)\right] + (M-\lambda)^{n-1}\left[f^{(n)}(M) - f^{(n)}(\lambda)\right]\right] \\ &\le \frac{1}{4}(M-m)^2 T_n(m, M; \lambda) \end{aligned} \tag{4.31}$$

where

$$T_n(m, M; \lambda) := \begin{cases} \left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n-1}\left[f^{(n)}(M) - f^{(n)}(m)\right]; \\ \\ \left[(\lambda-m)^{p(n-1)} + (M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(f^{(n)}(M) - f^{(n)}(\lambda)\right)^q + \left(f^{(n)}(\lambda) - f^{(n)}(m)\right)^q\right]^{1/q} \\ \textit{if } p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\left[f^{(n)}(M) - f^{(n)}(m)\right] + \left|f^{(n)}(\lambda) - \frac{f^{(n)}(M)+f^{(n)}(m)}{2}\right|\right] \\ \times\left[(\lambda-m)^{n-1} + (M-\lambda)^{n-1}\right]. \end{cases} \tag{4.32}$$

PROOF. 1. It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v), \tag{4.33}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising the representation (4.12) and the property (4.33) we have successively

$$
\begin{aligned}
(4.34)\qquad & \left|K_n(m,M,f;\lambda)\right| \\
&\le (M-\lambda)\left|\int_m^\lambda (\lambda-t)^n\, d\left(f^{(n)}(t)\right)\right| + (\lambda-m)\left|\int_\lambda^M (t-\lambda)^n\, d\left(f^{(n)}(t)\right)\right| \\
&\le (M-\lambda)(\lambda-m)^n \bigvee_m^\lambda \left(f^{(n)}\right) + (\lambda-m)(M-\lambda)^n \bigvee_\lambda^M \left(f^{(n)}\right) \\
&= (M-\lambda)(\lambda-m)\left[(\lambda-m)^{n-1}\bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n-1}\bigvee_\lambda^M \left(f^{(n)}\right)\right] \\
&\le \frac{1}{4}(M-m)^2\left[(\lambda-m)^{n-1}\bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n-1}\bigvee_\lambda^M \left(f^{(n)}\right)\right] \\
&\le \frac{1}{4}(M-m)^2 I_n(m,M;\lambda)
\end{aligned}
$$

for any $\lambda \in [m,M]$.

By Hölder's inequality we also have

$$
(4.35)\qquad I_n(m,M;\lambda) \le \begin{cases}
\left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n-1} \bigvee_m^M \left(f^{(n)}\right); \\[2ex]
\left[(\lambda-m)^{p(n-1)} + (M-\lambda)^{p(n-1)}\right]^{1/p} \\
\times \left[\left(\bigvee_m^\lambda \left(f^{(n)}\right)\right)^q + \left(\bigvee_\lambda^M \left(f^{(n)}\right)\right)^q\right]^{1/q} \\
\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\bigvee_m^M \left(f^{(n)}\right) + \frac{1}{2}\left|\bigvee_m^\lambda \left(f^{(n)}\right) - \bigvee_\lambda^M \left(f^{(n)}\right)\right|\right] \\
\times \left[(\lambda-m)^{n-1} + (M-\lambda)^{n-1}\right].
\end{cases}
$$

for any $\lambda \in [m,M]$.

On making use of (4.34) and (4.35) we deduce (4.27).

2. We recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$
|f(s)-f(t)| \le L|s-t| \text{ for any } t,s\in[a,b],
$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$
\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt.
$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$
\begin{aligned}
(4.36)\qquad & |K_n(m,M,f;\lambda)| \\
& \leq (M-\lambda)\left|\int_m^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right| + (\lambda-m)\left|\int_{\lambda}^{M}(t-\lambda)^n d\left(f^{(n)}(t)\right)\right| \\
& \leq \frac{1}{n+1}\left[L_{n,1,\lambda}(M-\lambda)(\lambda-m)^{n+1} + L_{n,2,\lambda}(\lambda-m)(M-\lambda)^{n+1}\right] \\
& = \frac{(M-\lambda)(\lambda-m)}{n+1}\left[L_{n,1,\lambda}(\lambda-m)^n + L_{n,2,\lambda}(M-\lambda)^n\right] \\
& \leq \frac{(M-m)^2}{4(n+1)}\left[L_{n,1,\lambda}(\lambda-m)^n + L_{n,2,\lambda}(M-\lambda)^n\right] \\
& \leq \frac{(M-m)^2}{4(n+1)} \\
& \times \begin{cases} \left[(\lambda-m)^n + (M-\lambda)^n\right]\max\left\{L_{n,1,\lambda}, L_{n,2,\lambda}\right\} \\ \\ \left[(\lambda-m)^{pn} + (M-\lambda)^{pn}\right]^{1/p}\left(L_{n,1,\lambda}^q + L_{n,2,\lambda}^q\right)^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^n \left(L_{n,1,\lambda} + L_{n,2,\lambda}\right) \end{cases}
\end{aligned}
$$

which prove the desired result (4.30).

3. From the theory of Riemann-Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$
(4.37)\qquad \left|\int_a^b p(t)\,dv(t)\right| \leq \int_a^b |p(t)|\,dv(t) \leq \max_{t\in[a,b]}|p(t)|\left[v(b)-v(a)\right].
$$

By utilizing this property, we have

$$
\begin{aligned}
(4.38)\qquad & |K_n(m,M,f;\lambda)| \\
& \leq (M-\lambda)\left|\int_m^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right| + (\lambda-m)\left|\int_{\lambda}^{M}(t-\lambda)^n d\left(f^{(n)}(t)\right)\right| \\
& \leq (M-\lambda)\int_m^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right) + (\lambda-m)\int_{\lambda}^{M}(t-\lambda)^n d\left(f^{(n)}(t)\right) \\
& = H_n(m,M;\lambda)
\end{aligned}
$$

By the second part of (4.37) we also have that

$$\begin{aligned}
(4.39)\qquad & H_n(m,M;\lambda) \\
& \leq (M-\lambda)(\lambda-m)^n\left[f^{(n)}(\lambda)-f^{(n)}(m)\right] \\
& + (\lambda-m)(M-\lambda)^n\left[f^{(n)}(M)-f^{(n)}(\lambda)\right] \\
& = (M-\lambda)(\lambda-m) \\
& \times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]+(M-\lambda)^{n-1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\right] \\
& \leq \frac{1}{4}(M-m)^2 \\
& \times\left[(\lambda-m)^{n-1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]+(M-\lambda)^{n-1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\right] \\
& = \frac{1}{4}(M-m)^2 L_n(m,M;\lambda)
\end{aligned}$$

with

$$(4.40)\qquad L_n(m,M;\lambda) \leq \begin{cases} \left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\left[f^{(n)}(M)-f^{(n)}(m)\right]; \\ \\ \left[(\lambda-m)^{p(n-1)}+(M-\lambda)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(f^{(n)}(M)-f^{(n)}(\lambda)\right)^q+\left(f^{(n)}(\lambda)-f^{(n)}(m)\right)^q\right]^{1/q} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\left[f^{(n)}(M)-f^{(n)}(m)\right]+\left|f^{(n)}(\lambda)-\frac{f^{(n)}(M)+f^{(n)}(m)}{2}\right|\right] \\ \times\left[(\lambda-m)^{n-1}+(M-\lambda)^{n-1}\right]. \end{cases}$$

Integrating by parts we have

$$\begin{aligned}
(4.41)\qquad & H_n(m,M;\lambda) \\
& = (M-\lambda)\int_m^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)+(\lambda-m)\int_\lambda^M(t-\lambda)^n d\left(f^{(n)}(t)\right) \\
& = (M-\lambda)\left[(\lambda-t)^n f^{(n)}(t)\Big|_m^\lambda+n\int_m^\lambda(\lambda-t)^{n-1}f^{(n)}(t)\,dt\right] \\
& + (\lambda-m)\left[(t-\lambda)^n f^{(n)}(t)\Big|_\lambda^M-n\int_\lambda^M(t-\lambda)^{n-1}f^{(n)}(t)\,dt\right] \\
& = (M-\lambda)\left[n\int_m^\lambda(\lambda-t)^{n-1}f^{(n)}(t)\,dt-(\lambda-m)^n f^{(n)}(m)\right] \\
& + (\lambda-m)\left[(M-\lambda)^n f^{(n)}(M)-n\int_\lambda^M(t-\lambda)^{n-1}f^{(n)}(t)\,dt\right].
\end{aligned}$$

On making use of (4.38)-(4.41) we deduce the desired result (4.31). ∎

On making use of the bounds for the kernel $K_n(m,M,f;\cdot)$ provided above, we can establish the following error estimates for the remainder $T_n(f,m,M)$ in the representation formula (4.10).

THEOREM 4.5 (Dragomir, 2010, [**3**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$, $\{E_\lambda\}_\lambda$ *be its spectral*

family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset \mathring{I}$ and let n be an integer with $n\geq 1$. If $f:I\rightarrow \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then we have the representation

$$\begin{aligned} \langle f(A)x,y\rangle &= \frac{1}{M-m}\left[f(m)\langle (M1_H-A)x,y\rangle + f(M)\langle (A-m1_H)x,y\rangle\right] \\ &+ \frac{1}{M-m} \\ &\times \left\{\sum_{k=1}^{n}\frac{1}{k!}f^{(k)}(m)\left\langle (M1_H-A)(A-m1_H)^k x,y\right\rangle \right. \\ &\left. +\sum_{k=1}^{n}\frac{1}{k!}(-1)^k f^{(k)}(M)\left\langle (A-m1_H)(M1_H-A)^k x,y\right\rangle\right\} \\ &+ T_n(f,m,M;x,y), \end{aligned} \tag{4.42}$$

where the remainder $T_n(f,m,M;x,y)$ is given by

$$T_n(f,m,M;x,y) := \frac{1}{(M-m)\,n!}\int_{m-0}^{M} K_n(m,M,f;\lambda)\,d\langle E_\lambda x,y\rangle \tag{4.43}$$

and the kernel $K_n(m,M,f;\cdot)$ has the representation (4.12).

Moreover, we have the error estimate

$$\begin{aligned} &|T_n(f,m,M;x,y)| \\ &\leq \frac{1}{4n!}(M-m)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\times \max_{\lambda\in[m,M]}\left[(\lambda-m)^{n-1}\bigvee_{m}^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\ &\leq \frac{1}{4n!}(M-m)^n\bigvee_{m}^{M}\left(f^{(n)}\right)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\leq \frac{1}{4n!}(M-m)^n\bigvee_{m}^{M}\left(f^{(n)}\right)\|x\|\,\|y\| \end{aligned} \tag{4.44}$$

for any $x,y\in H$.

PROOF. The identity (4.42) with the remainder representation (4.43) follows from (4.10) and (4.11).

Now, on utilizing the property (4.33) for the Riemann-Stieltjes integral we deduce from (4.43) that

$$\begin{aligned} &|T_n(f,m,M;x,y)| \\ &\leq \frac{1}{(M-m)\,n!}\max_{\lambda\in[m,M]}|K_n(m,M,f;\lambda)|\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \end{aligned} \tag{4.45}$$

for any $x,y\in H$.

Further, by (4.27) and (4.28) we have the bounds

$$\begin{aligned} &|K_n(m,M,f;\lambda)| \\ &\le \frac{1}{4}(M-m)^2\left[(\lambda-m)^{n-1}\bigvee_m^{\lambda}\left(f^{(n)}\right)+(M-\lambda)^{n-1}\bigvee_{\lambda}^{M}\left(f^{(n)}\right)\right] \\ &\le \frac{1}{4}(M-m)^2\left[\frac{1}{2}(M-m)+\left|\lambda-\frac{m+M}{2}\right|\right]^{n-1}\bigvee_m^M\left(f^{(n)}\right); \end{aligned} \tag{4.46}$$

for any $\lambda \in [m,M]$.

Taking the maximum over $\lambda \in [m,M]$ in (4.46) we deduce the first and the second inequalities in (4.44).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. ∎

COROLLARY 4.6 (Dragomir, 2010, [**3**]). *With the assumptions from Theorem 4.5 and if $f^{(n)}$ is L_n-Lipschitzian on $[m,M]$, then*

$$\begin{aligned} &|T_n(f,m,M;x,y)| \\ &\le \frac{1}{(n+1)!\,(M-m)}L_n\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\times \max_{\lambda\in[m,M]}\left[(M-\lambda)(\lambda-m)^{n+1}+(\lambda-m)(M-\lambda)^{n+1}\right] \\ &\le \frac{1}{4(n+1)!}(M-m)^{n+1}L_n\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{4(n+1)!}(M-m)^{n+1}L_n\|x\|\|y\| \end{aligned} \tag{4.47}$$

for any $x,y \in H$.

4.3. Error Bounds for $f^{(n)}$ Absolutely Continuous. The following result that provides bounds for the absolute value of the kernel $W_n(m,M,f;\cdot)$ holds:

LEMMA 4.7 (Dragomir, 2010, [**3**]). *Let I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$, let n be an integer with $n \ge 1$ and assume that $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m,M]$. Then we have the bound*

$$|W_n(m,M,f;\lambda)| \le \sum_{i=1}^{4} B_n^{(i)}(m,M,f;\lambda) \tag{4.48}$$

where

$$B_n^{(1)}(m,M,f;\lambda) := n(M-\lambda)\int_m^\lambda (\lambda-t)^{n-1}\left|f^{(n+1)}(t)\right|dt \le n(M-\lambda) \times \begin{cases} \frac{1}{n}(\lambda-m)^n \left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} \text{ if } f^{(n+1)} \in L_\infty[m,\lambda]; \\ \frac{1}{[(n-1)p_1+1]^{1/p_1}}(\lambda-m)^{n-1+1/p_1}\left\|f^{(n+1)}\right\|_{[m,\lambda],q_1} \\ \text{if } f^{(n+1)} \in L_{q_1}[m,\lambda], p_1>1, \frac{1}{p_1}+\frac{1}{q_1}=1; \\ (\lambda-m)^{n-1}\left\|f^{(n+1)}\right\|_{[m,\lambda],1}; \end{cases} \tag{4.49}$$

$$B_n^{(2)}(m,M,f;\lambda) := \int_m^\lambda (\lambda-t)^{n}\left|f^{(n+1)}(t)\right|dt \le \begin{cases} \frac{1}{n+1}(\lambda-m)^{n+1} \left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} \text{ if } f^{(n+1)} \in L_\infty[m,\lambda]; \\ \frac{1}{(np_2+1)^{1/p_2}}(\lambda-m)^{n+1/p_2}\left\|f^{(n+1)}\right\|_{[m,\lambda],q_2} \text{ if } f^{(n+1)} \in L_{q_2}[m,\lambda], \\ p_2>1, \frac{1}{p_2}+\frac{1}{q_2}=1; \\ (\lambda-m)^{n}\left\|f^{(n+1)}\right\|_{[m,\lambda],1} \end{cases} \tag{4.50}$$

$$B_n^{(3)}(m,M,f;\lambda) := \int_\lambda^M (t-\lambda)^{n}\left|f^{(n+1)}(t)\right|dt \le \begin{cases} \frac{1}{n+1}(M-\lambda)^{n+1} \left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} \text{ if } f^{(n+1)} \in L_\infty[\lambda,M]; \\ \frac{1}{(np_3+1)^{1/p_3}}(M-\lambda)^{n+1/p_3}\left\|f^{(n+1)}\right\|_{[\lambda,M],q_3} \text{ if } f^{(n+1)} \in L_{q_3}[\lambda,M], \\ p_3>1, \frac{1}{p_3}+\frac{1}{q_3}=1; \\ (M-\lambda)^{n}\left\|f^{(n+1)}\right\|_{[\lambda,M],1} \end{cases} \tag{4.51}$$

and

$$B_n^{(4)}(m,M,f;\lambda) := n(\lambda-m)\int_\lambda^M (t-\lambda)^{n-1}\left|f^{(n+1)}(t)\right|dt \le n(\lambda-m) \times \begin{cases} \frac{1}{n}(M-\lambda)^n \left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} \text{ if } f^{(n+1)} \in L_\infty[\lambda,M]; \\ \frac{1}{[(n-1)p_4+1]^{1/p_4}}(M-\lambda)^{n-1+1/p_4}\left\|f^{(n+1)}\right\|_{[\lambda,M],q_4} \\ \text{if } f^{(n+1)} \in L_{q_1}[\lambda,M], p_4>1, \frac{1}{p4}+\frac{1}{q_4}=1; \\ (M-\lambda)^{n-1}\left\|f^{(n+1)}\right\|_{[m,\lambda],1}; \end{cases} \tag{4.52}$$

for any $\lambda \in [m, M]$, *where the Lebesgue norms* $\|\cdot\|_{[a,b],p}$ *are defined by*

$$\|g\|_{[a,b],p} := \begin{cases} \left(\int_a^b |g(t)|^p\, dt\right)^{1/p} & \text{if } g \in L_p[a,b]\,, p \geq 1 \\ \\ ess\sup_{t\in[a,b]} |g(t)| & \text{if } g \in L_\infty[a,b]\,. \end{cases}$$

PROOF. From (4.20) we have

$$\begin{aligned} &|W_n(m,M,f;\lambda)| \\ &\leq \left|\int_m^\lambda (\lambda-t)^{n-1}[nM+t-(n+1)\lambda] f^{(n+1)}(t)\,dt\right| \\ &+\left|\int_\lambda^M (t-\lambda)^{n-1}[t+nm-(n+1)\lambda] f^{(n+1)}(t)\,dt\right| \\ &\leq \int_m^\lambda (\lambda-t)^{n-1}|nM+t-(n+1)\lambda|\left|f^{(n+1)}(t)\right|dt \\ &+\int_\lambda^M (t-\lambda)^{n-1}|t+nm-(n+1)\lambda|\left|f^{(n+1)}(t)\right|dt \\ &\leq \int_m^\lambda (\lambda-t)^{n-1}[n(M-\lambda)+(\lambda-t)]\left|f^{(n+1)}(t)\right|dt \\ &+\int_\lambda^M (t-\lambda)^{n-1}[(t-\lambda)+n(\lambda-m)]\left|f^{(n+1)}(t)\right|dt \\ &=\sum_{i=1}^4 B_n^{(i)}(m,M,f;\lambda) \end{aligned} \tag{4.53}$$

for any $\lambda \in [m, M]$, which proves (4.48).

The other bounds follows by Hölder's integral inequality and the details are omitted. ∎

REMARK 4.4. It is obvious that the inequalities (4.48)-(4.52) can produce 12 different bounds for $|W_n(m,M,f;\lambda)|$. However, we mention here only the case when $f^{(n+1)} \in L_\infty[\lambda, M]$, namely

$$\begin{aligned} &|W_n(m,M,f;\lambda)| \\ &\leq (M-\lambda)(\lambda-m)^n\left\|f^{(n+1)}\right\|_{[m,\lambda],\infty}+\frac{1}{n+1}(\lambda-m)^{n+1}\left\|f^{(n+1)}\right\|_{[m,\lambda],\infty} \\ &+\frac{1}{n+1}(M-\lambda)^{n+1}\left\|f^{(n+1)}\right\|_{[\lambda,M],\infty}+(\lambda-m)(M-\lambda)^n\left\|f^{(n+1)}\right\|_{[\lambda,M],\infty} \\ &\leq \Big[(M-\lambda)(\lambda-m)^n+(\lambda-m)(M-\lambda)^n \\ &+\frac{1}{n+1}(\lambda-m)^{n+1}+\frac{1}{n+1}(M-\lambda)^{n+1}\Big]\left\|f^{(n+1)}\right\|_{[m,M],\infty} \end{aligned} \tag{4.54}$$

for any $\lambda \in [m, M]$.

Finally, we can state the following result as well:

THEOREM 4.8 (Dragomir, 2010, [**3**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m, M] \subset \mathring{I}$ *and let* n *be an integer with* $n \geq 1$. *If*

$f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is absolutely continuous on the interval $[m, M]$, then we have the representation (4.42) where the remainder $T_n(f, m, M; x, y)$ is given by

$$T_n(f, m, M; x, y) := \frac{1}{(M-m)\, n!} \int_{m-0}^{M} W_n(m, M, f; \lambda) \langle E_\lambda x, y \rangle \, d\lambda \tag{4.55}$$

and the kernel $W_n(m, M, f; \cdot)$ has the representation (4.20).

We also have the error bounds

$$\begin{aligned} &|T_n(f, m, M; x, y)| \\ &\leq \frac{1}{(M-m)\, n!} \int_{m-0}^{M} |W_n(m, M, f; \lambda)| \, |\langle E_\lambda x, y \rangle| \, d\lambda \\ &\leq \frac{1}{(M-m)\, n!} \int_{m-0}^{M} |W_n(m, M, f; \lambda)| \langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \, d\lambda \\ &\leq \frac{1}{(M-m)\, n!} \|x\| \, \|y\| \int_{m}^{M} |W_n(m, M, f; \lambda)| \, d\lambda \end{aligned} \tag{4.56}$$

for any $x, y \in H$.

REMARK 4.5. On making use of Lemma 4.7 one can produce further bounds. However, the details are left to the interested reader.

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