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SOME NEW MAPPINGS RELATED TO WEIGHTED MEAN INEQUALITIES

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ABSTRACT. In this paper, we define four mappings related to weighted mean inequalities, investigate their properties, and obtain some new refinements of weighted mean inequalities.

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1. INTRODUCTION

In this paper, let $x_i > 0, t_i > 0\,(i = 1, 2, \cdots, n; n \geq 2, n \in \mathbb{N}), 1 \leq k \leq m \leq n, T_n = \sum_{i=1}^{n} t_i, T_0 = 0$. Then

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{1.1}$$

(1.1) is the well-known classical harmonic-geometric-arithmetic mean inequalities with wide applications (see [1]).

For some recent results which generalize, improve and extend these classical inequalities, see [2]-[5]. In [2]-[3], the generalized form of inequalities (1.1) is further introduced: weighted harmonic-geometric-arithmetic mean inequalities

$$T_n \left(\sum_{i=1}^{n} t_i {x_i}^{-1} \right)^{-1} \leq \prod_{i=1}^{n} {x_i}^{t_i/T_n} \leq \sum_{i=1}^{n} \frac{t_i}{T_n} x_i. \tag{1.2}$$

To go further into (1.2), we define four mappings F, G, H and L by

$$F(k, m) = \sum_{i=k}^{m} t_i x_i - (T_m - T_{k-1}) \prod_{i=k}^{m} {x_i}^{t_i/(T_m - T_{k-1})},$$

$$G(k, m) = \sum_{i=k}^{m} \frac{t_i}{x_i} - \frac{T_m - T_{k-1}}{\prod_{i=k}^{m} {x_i}^{t_i/(T_m - T_{k-1})}},$$

$$H(n, k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{C_n^k} \left(\sum_{j=1}^{k} t_{i_j} x_{i_j} - \left(\sum_{j=1}^{k} t_{i_j} \right) \prod_{j=1}^{k} {x_{i_j}}^{t_{i_j} \Big/ \sum_{j=1}^{k} t_{i_j}} \right),$$

$$L(n, k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{C_n^k} \left(\sum_{j=1}^{k} \frac{t_{i_j}}{x_{i_j}} - \frac{\sum_{j=1}^{k} t_{i_j}}{\prod_{j=1}^{k} {x_{i_j}}^{t_{i_j} \Big/ \sum_{j=1}^{k} t_{i_j}}} \right).$$

The aim of this paper is to study monotonicity properties of F, G, H and L, and obtain some new refinements of (1.1) and (1.2).

2. MAIN RESULTS

Theorem 2.1. *Let F be defined as in the first section. For $1 \leq k \leq m \leq n$, we write*

$$\bar{F}(k, m) = \frac{1}{T_n} F(k, m) + \prod_{i=1}^{n} {x_i}^{t_i/T_n}.$$

We have

(1) $F(1, k)$ is monotonically increasing with respect to k, and $F(k, n)$ is monotonically decreasing with respect to k.

(2) For $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$, we have

$$\begin{aligned}\prod_{i=1}^{n} x_i^{t_i/T_n} &= \alpha\bar{F}(1,1) + \beta\bar{F}(n,n) \le \alpha\bar{F}(1,2) + \beta\bar{F}(n-1,n) \\ &\le \cdots \le \alpha\bar{F}(1,n-1) + \beta\bar{F}(2,n) \le \bar{F}(1,n) = \sum_{i=1}^{n}\frac{t_i}{T_n}x_i,\end{aligned} \tag{2.1}$$

(2.1) is the refinements of the right end of (1.2).

Remark 2.1. (1) When $t_i = 1\,(i = 1, 2, \cdots, n)$, (2.1) becomes refinements of the right end of (1.1).

(2) Replace x_i in (2.1) with $x_i^{-1}\,(i = 1, 2, \cdots, n)$, and then take the reciprocal, the refinements of the left end of (1.2) can be obtained.

Theorem 2.2. *Let G be defined as in the first section. For $1 \le k \le m \le n$, we write*

$$\bar{G}(k,m) = \frac{\prod_{i=1}^{n} x_i^{t_i/T_n}}{\sum_{i=1}^{n} t_i x_i^{-1}}\left(G(k,m) + \frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}}\right).$$

We have

(1) $G(1,k)$ is monotonically increasing with respect to k, and $G(k,n)$ is monotonically decreasing with respect to k.

(2) For $\alpha \in [0,1]$ and $\beta = 1-\alpha$, we have

$$\begin{aligned}T_n\left(\sum_{i=1}^{n} t_i x_i^{-1}\right)^{-1} &= \alpha\bar{G}(1,1) + \beta\bar{G}(n,n) \le \alpha\bar{G}(1,2) + \beta\bar{G}(n-1,n) \\ &\le \cdots \le \alpha\bar{G}(1,n-1) + \beta\bar{G}(2,n) \le \bar{G}(1,n) = \prod_{i=1}^{n} x_i^{t_i/T_n},\end{aligned} \tag{2.2}$$

(2.2) is the refinements of the left end of (1.2).

Remark 2.2. When $t_i = 1\,(i = 1, 2, \cdots, n)$, (2.2) becomes refinements of the left end of (1.1).

Theorem 2.3. *Let H be defined as in the first section. For $1 \le k \le n$, we write*

$$\bar{H}(n,k) = \frac{1}{T_n}H(n,k) + \prod_{i=1}^{n} x_i^{t_i/T_n}.$$

We have

(1) $H(n,k)$ is monotonically increasing with respect to k.

(2)

$$\prod_{i=1}^{n} x_i^{t_i/T_n} = \bar{H}(n,1) \le \bar{H}(n,2) \le \cdots \le \bar{H}(n,n) = \sum_{i=1}^{n}\frac{t_i}{T_n}x_i, \tag{2.3}$$

(2.3) is the refinements of the right end of (1.2) with non-repetitive sample.

Remark 2.3. (1) When $t_i = 1\,(i = 1, 2, \cdots, n)$, (2.3) becomes refinements of the right end of (1.1).

(2) Replace x_i and x_{i_j} in (2.3) with $x_i^{-1}\,(i = 1, 2, \cdots, n)$ and $x_{i_j}^{-1}\,(i_j = i_1, i_2, \cdots, i_n)$ respectively, then take the reciprocal, the refinements of the left end of (1.2) can be obtained.

Theorem 2.4. *Let L be defined as in the first section. For $1 \leq k \leq n$, we write*

$$\bar{L}(n,k) = \frac{\prod_{i=1}^{n} x_i^{t_i/T_n}}{\sum_{i=1}^{n} t_i x_i^{-1}} \left(L(n,k) + \frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}} \right).$$

We have
(1) $L(n,k)$ is monotonically increasing with respect to k.
(2)

$$T_n \left(\sum_{i=1}^{n} t_i x_i^{-1} \right)^{-1} = \bar{L}(n,1) \leq \bar{L}(n,2) \leq \cdots \leq \bar{L}(n,n) = \prod_{i=1}^{n} x_i^{t_i/T_n}, \tag{2.4}$$

(2.4) is the refinements of the left end of (1.2) with non-repetitive sample.

Remark 2.4. When $t_i = 1\ (i = 1, 2, \cdots, n)$, (2.4) becomes refinements of the left end of (1.1).

3. PROOF OF THEOREMS

Proof. Theorem 2.1. (1) For $k = 1, 2, \cdots, n-1$, from the right end of (1.2), the inequality signs in the following two formulas can be obtained

$$\begin{aligned} F(1,k+1) &= \sum_{i=1}^{k+1} t_i x_i - T_{k+1} \prod_{i=1}^{k+1} x_i^{t_i/T_{k+1}} \\ &= \sum_{i=1}^{k+1} t_i x_i - T_{k+1} \left(\left(\prod_{i=1}^{k} x_i^{t_i/T_k} \right)^{T_k/T_{k+1}} \cdot x_{k+1}^{t_{k+1}/T_{k+1}} \right) \\ &\geq \sum_{i=1}^{k+1} t_i x_i - T_{k+1} \left(\frac{T_k}{T_{k+1}} \prod_{i=1}^{k} x_i^{t_i/T_k} + \frac{t_{k+1}}{T_{k+1}} x_{k+1} \right) \\ &= \sum_{i=1}^{k} t_i x_i - T_k \prod_{i=1}^{k} x_i^{t_i/T_k} = F(1,k), \end{aligned} \tag{3.1}$$

$$\begin{aligned} F(k,n) &= \sum_{i=k}^{n} t_i x_i - (T_n - T_{k-1}) \prod_{i=k}^{n} x_i^{t_i/(T_n - T_{k-1})} \\ &= \sum_{i=k}^{n} t_i x_i - (T_n - T_{k-1}) \left(\left(\prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)} \right)^{(T_n - T_k)/(T_n - T_{k-1})} \cdot x_k^{t_k/(T_n - T_{k-1})} \right) \\ &\geq \sum_{i=k}^{n} t_i x_i - (T_n - T_{k-1}) \left(\frac{T_n - T_k}{T_n - T_{k-1}} \prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)} + \frac{t_k}{T_n - T_{k-1}} x_k \right) \\ &= \sum_{i=k+1}^{n} t_i x_i - (T_n - T_k) \prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)} = F(k+1,n). \end{aligned} \tag{3.2}$$

(3.1) and (3.2) imply that $F(1,k)$ is monotonically increasing with respect to k and $F(k,n)$ is monotonically decreasing with respect to k, respectively.

(2) From the definition of F, we can get $F(1,1) = F(n,n) = 0$ and $F(1,n) = \sum_{i=1}^{n} t_i x_i - T_n \prod_{i=1}^{n} {x_i}^{t_i/T_n}$, from the construction of $\bar{F}$, we can get $\bar{F}(1,1) = \bar{F}(n,n) = \prod_{i=1}^{n} {x_i}^{t_i/T_n}$ and $\bar{F}(1,n) = \sum_{i=1}^{n} \frac{t_i}{T_n} x_i$. Then from the structure of $\bar{F}$ and (3.1), (3.2), it is obvious that $\bar{F}(1,k)$ is monotonically increasing with respect to k and $\bar{F}(k,n)$ is monotonically decreasing with respect to k. Thus the following two formulas hold

$$\prod_{i=1}^{n} {x_i}^{t_i/T_n} = \bar{F}(1,1) \le \bar{F}(1,2) \le \cdots \le \bar{F}(1,n) = \sum_{i=1}^{n} \frac{t_i}{T_n} x_i, \tag{3.3}$$

$$\prod_{i=1}^{n} {x_i}^{t_i/T_n} = \bar{F}(n,n) \le \bar{F}(n-1,n) \le \cdots \le \bar{F}(1,n) = \sum_{i=1}^{n} \frac{t_i}{T_n} x_i. \tag{3.4}$$

For $\alpha \in [0,1]$ and $\beta = 1-\alpha$, formula (3.3) multiplied by α plus formula (3.4) multiplied by β yield (2.1).

This completes the proof of Theorem 2.1. ∎

Proof. Theorem 2.2. (1) The left end of formula (1.2) can be rewritten as the following inequality

$$\frac{T_n}{\prod_{i=1}^{n} {x_i}^{t_i/T_n}} \le \sum_{i=1}^{n} \frac{t_i}{x_i}. \tag{3.5}$$

For $k = 1, 2, \cdots, n-1$, from (3.5), the inequality signs in the following two formulas can be obtained

$$\begin{aligned} G(1,k+1) &= \sum_{i=1}^{k+1} \frac{t_i}{x_i} - \frac{T_{k+1}}{\prod_{i=1}^{k+1} {x_i}^{t_i/T_{k+1}}} \\ &= \sum_{i=1}^{k+1} \frac{t_i}{x_i} - \left(\frac{T_k + t_{k+1}}{\left(\prod_{i=1}^{k} {x_i}^{t_i/T_k}\right)^{T_k/T_{k+1}} \cdot {x_{k+1}}^{t_{k+1}/T_{k+1}}} \right) \\ &\ge \sum_{i=1}^{k+1} \frac{t_i}{x_i} - \left(\frac{T_k}{\prod_{i=1}^{k} {x_i}^{t_i/T_k}} + \frac{t_{k+1}}{x_{k+1}} \right) \\ &= \sum_{i=1}^{k} \frac{t_i}{x_i} - \frac{T_k}{\prod_{i=1}^{k} {x_i}^{t_i/T_k}} = G(1,k), \end{aligned} \tag{3.6}$$

$$G(k,n) = \sum_{i=k}^{n} \frac{t_i}{x_i} - \frac{T_n - T_{k-1}}{\prod_{i=k}^{n} x_i^{t_i/(T_n - T_{k-1})}}$$

$$(3.7)\quad = \sum_{i=k}^{n} \frac{t_i}{x_i} - \left(\frac{(T_n - T_k) + t_k}{\left(\prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)} \right)^{(T_n - T_k)/(T_n - T_{k-1})} \cdot x_k^{t_k/(T_n - T_{k-1})}} \right)$$

$$\geq \sum_{i=k}^{n} \frac{t_i}{x_i} - \left(\frac{T_n - T_k}{\prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)}} + \frac{t_k}{x_k} \right)$$

$$= \sum_{i=k+1}^{n} \frac{t_i}{x_i} - \frac{T_n - T_k}{\prod_{i=k+1}^{n} x_i^{t_i/(T_n - T_k)}} = G(k+1,n).$$

(3.6) and (3.7) imply that $G(1,k)$ is monotonically increasing with respect to k and $G(k,n)$ is monotonically decreasing with respect to k, respectively.

(2) From the definition of G, we can get $G(1,1) = G(n,n) = 0$ and $G(1,n) = \sum_{i=1}^{n} \frac{t_i}{x_i} - \frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}}$, from the construction of $\bar{G}$, we can get $\bar{G}(1,1) = \bar{G}(n,n) = T_n \left(\sum_{i=1}^{n} t_i x_i^{-1} \right)^{-1}$ and $\bar{G}(1,n) = \prod_{i=1}^{n} x_i^{t_i/T_n}$. Then from the structure of $\bar{G}$ and (3.6), (3.7), it is obvious that $\bar{G}(1,k)$ is monotonically increasing with respect to k and $\bar{G}(k,n)$ is monotonically decreasing with respect to k. Thus the following two formulas hold

$$(3.8)\quad T_n \left(\sum_{i=1}^{n} t_i x_i^{-1} \right)^{-1} = \bar{G}(1,1) \leq \bar{G}(1,2) \leq \cdots \leq \bar{G}(1,n) = \prod_{i=1}^{n} x_i^{t_i/T_n},$$

$$(3.9)\quad T_n \left(\sum_{i=1}^{n} t_i x_i^{-1} \right)^{-1} = \bar{G}(n,n) \leq \bar{G}(n-1,n) \leq \cdots \leq \bar{G}(1,n) = \prod_{i=1}^{n} x_i^{t_i/T_n}.$$

For $\alpha \in [0,1]$ and $\beta = 1 - \alpha$, formula (3.8) multiplied by α plus formula (3.9) multiplied by β yield (2.2).

This completes the proof of Theorem 2.2. ∎

Proof. Theorem 2.3. (1) Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n\,(k \geq 2)$, arbitrarily take $k-1$ elements from $i_1, i_2, \cdots, i_k$ and mark them as $r_1, r_2, \cdots, r_{k-1}$, respectively. Then mark the remaining one as r_k. From the right end of (1.2), the inequality sign in the following formula can be obtained

$$
\begin{aligned}
&\sum_{j=1}^{k} t_{i_j} x_{i_j} - \left(\sum_{j=1}^{k} t_{i_j}\right) \prod_{j=1}^{k} x_i^{t_{i_j} \Big/ \sum\limits_{j=1}^{k} t_{i_j}} \\
&= \sum_{j=1}^{k} t_{i_j} x_{i_j} - \left(\sum_{j=1}^{k} t_{i_j}\right) \left(\left(\prod_{l=1}^{k-1} x_{r_l}^{t_{r_l} \Big/ \sum\limits_{l=1}^{k-1} t_{r_l}} \right)^{\sum\limits_{l=1}^{k-1} t_{r_l} \Big/ \sum\limits_{j=1}^{k} t_{i_j}} \cdot x_{r_k}^{t_{r_k} \Big/ \sum\limits_{j=1}^{k} t_{i_j}} \right) \\
&\geq \sum_{j=1}^{k} t_{i_j} x_{i_j} - \left(\sum_{j=1}^{k} t_{i_j}\right) \left(\frac{\sum\limits_{l=1}^{k-1} t_{r_l}}{\sum\limits_{j=1}^{k} t_{i_j}} \prod_{l=1}^{k-1} x_{r_l}^{t_{r_l} \Big/ \sum\limits_{l=1}^{k-1} t_{r_l}} + \frac{t_{r_k}}{\sum\limits_{j=1}^{k} t_{i_j}} x_{r_k} \right) \\
&= \sum_{l=1}^{k-1} t_{r_l} x_{r_l} - \left(\sum_{l=1}^{k-1} t_{r_l}\right) \prod_{l=1}^{k-1} x_{r_l}^{t_{r_l} \Big/ \sum\limits_{l=1}^{k-1} t_{r_l}} .
\end{aligned} \tag{3.10}
$$

From the definition of H, (3.10) and combinatorial knowledge, we can get

$$
\begin{aligned}
&H(n,k) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{C_n^k} \left(\sum_{j=1}^{k} t_{i_j} x_{i_j} - \left(\sum_{j=1}^{k} t_{i_j}\right) \prod_{j=1}^{k} x_{i_j}^{t_{i_j} \Big/ \sum\limits_{j=1}^{k} t_{i_j}} \right) \\
&\geq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \sum_{\{r_1, r_2, \cdots, r_{k-1}\} \subset \{i_1, i_2, \cdots, i_k\}} \frac{1}{k} \\
&\quad \cdot \frac{1}{C_n^k} \left(\sum_{l=1}^{k-1} t_{r_l} x_{r_l} - \left(\sum_{l=1}^{k-1} t_{r_l}\right) \prod_{l=1}^{k-1} x_{r_l}^{t_{r_l} \Big/ \sum\limits_{l=1}^{k-1} t_{r_l}} \right) \\
&= \sum_{1 \leq j_1 < j_2 < \cdots < j_{k-1} \leq n} \frac{1}{C_n^{k-1}} \left(\sum_{m=1}^{k-1} t_{j_m} x_{j_m} - \left(\sum_{m=1}^{k-1} t_{j_m}\right) \prod_{m=1}^{k-1} x_{j_m}^{t_{j_m} \Big/ \sum\limits_{m=1}^{k-1} t_{j_m}} \right) \\
&= H(n, k-1) .
\end{aligned} \tag{3.11}
$$

(3.11) implies that $H(n,k)$ is monotonically increasing with respect to k.

(2) From $H(n,1) = 0$ and the structure of $\bar{H}$, we have

$$
\bar{H}(n,1) = \frac{1}{T_n} H(n,1) + \prod_{i=1}^{n} x_i^{t_i/T_n} = \prod_{i=1}^{n} x_i^{t_i/T_n}. \tag{3.12}
$$

From $H(n,n) = \sum\limits_{i=1}^{n} t_i x_i - T_n \prod\limits_{i=1}^{n} x_i^{t_i/T_n}$ and the structure of $\bar{H}$, we have

$$
\bar{H}(n,n) = \frac{1}{T_n} H(n,n) + \prod_{i=1}^{n} x_i^{t_i/T_n} = \sum_{i=1}^{n} \frac{t_i}{T_n} x_i. \tag{3.13}
$$

From (3.11) and the structure of $\bar{H}$, it is obvious that $\bar{H}\left(n,k\right)$ is monotonically increasing with respect to k. Therefore, the following formula holds

$$\bar{H}\left(n,1\right)\le\bar{H}\left(n,2\right)\le\cdots\le\bar{H}\left(n,n\right). \tag{3.14}$$

Combination of (3.12), (3.13) and (3.14) yields (2.3).

The proof of Theorem 2.3 is completed. ∎

Proof. Theorem 2.4. (1) Similar to the proof of (1) in Theorem 2.3, from (3.5), the inequality sign in the following formula can be obtained

$$\begin{aligned}
&\sum_{j=1}^{k}\frac{t_{i_j}}{x_{i_j}}-\frac{\sum\limits_{j=1}^{k}t_{i_j}}{\prod\limits_{j=1}^{k}x_{i_j}^{t_{i_j}\big/\sum\limits_{j=1}^{k}t_{i_j}}}\\
&=\sum_{j=1}^{k}\frac{t_{i_j}}{x_{i_j}}-\frac{\sum\limits_{l=1}^{k-1}t_{r_l}+t_{r_k}}{\left(\prod\limits_{l=1}^{k-1}x_{r_l}^{t_{r_l}\big/\sum\limits_{l=1}^{k-1}t_{r_l}}\right)^{\sum\limits_{l=1}^{k-1}t_{r_l}\big/\sum\limits_{j=1}^{k}t_{i_j}}\cdot x_{r_k}^{t_{r_k}\big/\sum\limits_{j=1}^{k}t_{i_j}}}\\
&\ge\sum_{j=1}^{k}\frac{t_{i_j}}{x_{i_j}}-\left(\frac{\sum\limits_{l=1}^{k-1}t_{r_l}}{\prod\limits_{l=1}^{k-1}x_{r_l}^{t_{r_l}\big/\sum\limits_{l=1}^{k-1}t_{r_l}}}+\frac{t_{r_k}}{x_{r_k}}\right)=\sum_{l=1}^{k-1}\frac{t_{r_l}}{x_{r_l}}-\frac{\sum\limits_{l=1}^{k-1}t_{r_l}}{\prod\limits_{l=1}^{k-1}x_{r_l}^{t_{r_l}\big/\sum\limits_{l=1}^{k-1}t_{r_l}}}.
\end{aligned}\tag{3.15}$$

From the definition of L, (3.15) and combinatorial knowledge, we can get

$$\begin{aligned}
L\left(n,k\right)&=\sum_{1\le i_1<i_2<\cdots<i_k\le n}\frac{1}{C_n^k}\left(\sum_{j=1}^{k}\frac{t_{i_j}}{x_{i_j}}-\frac{\sum\limits_{j=1}^{k}t_{i_j}}{\prod\limits_{j=1}^{k}x_{i_j}^{t_{i_j}\big/\sum\limits_{j=1}^{k}t_{i_j}}}\right)\\
&\ge\sum_{1\le i_1<i_2<\cdots<i_k\le n}\ \sum_{\{r_1,r_2,\cdots,r_{k-1}\}\subset\{i_1,i_2,\cdots,i_k\}}\frac{1}{k}\cdot\frac{1}{C_n^k}\left(\sum_{l=1}^{k-1}\frac{t_{r_l}}{x_{r_l}}-\frac{\sum\limits_{l=1}^{k-1}t_{r_l}}{\prod\limits_{l=1}^{k-1}x_{r_l}^{t_{r_l}\big/\sum\limits_{l=1}^{k-1}t_{r_l}}}\right)\\
&=\sum_{1\le j_1<j_2<\cdots<j_{k-1}\le n}\frac{1}{C_n^{k-1}}\left(\sum_{m=1}^{k-1}\frac{t_{j_m}}{x_{j_m}}-\frac{\sum\limits_{m=1}^{k-1}t_{j_m}}{\prod\limits_{m=1}^{k-1}x_{j_m}^{t_{j_m}\big/\sum\limits_{m=1}^{k-1}t_{j_m}}}\right)\\
&=L\left(n,k-1\right).
\end{aligned}\tag{3.16}$$

(3.16) implies that $L(n,k)$ is monotonically increases with respect to k.

(2) From $L(n,1)=0$ and the structure of $\bar{L}$, we have

$$\bar{L}(n,1)=\frac{\prod_{i=1}^{n} x_i^{t_i/T_n}}{\sum_{i=1}^{n} t_i x_i^{-1}}\left(L(n,1)+\frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}}\right)=T_n\left(\sum_{i=1}^{n} t_i x_i^{-1}\right)^{-1}, \tag{3.17}$$

From $L(n,n)=\sum_{i=1}^{n}\frac{t_i}{x_i}-\frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}}$ and the structure of $\bar{L}$, we have

$$\bar{L}(n,n)=\frac{\prod_{i=1}^{n} x_i^{t_i/T_n}}{\sum_{i=1}^{n} t_i x_i^{-1}}\left(L(n,n)+\frac{T_n}{\prod_{i=1}^{n} x_i^{t_i/T_n}}\right)=\prod_{i=1}^{n} x_i^{t_i/T_n}, \tag{3.18}$$

From (3.16) and the structure of $\bar{L}$, it is obvious that $\bar{L}(n,k)$ is monotonically increasing with respect to k. Therefore, the following formula holds

$$\bar{L}(n,1)\le\bar{L}(n,2)\le\cdots\le\bar{L}(n,n). \tag{3.19}$$

Combination of (3.17), (3.18) and (3.19) yields (2.4).

The proof of Theorem 2.4 is completed. ∎

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