The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 4, pp. 1-10, 2019

AN EXISTENCE OF THE SOLUTION TO NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS UNDER THE HÖLDER CONDITION

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Received 15 March, 2019; accepted 28 May, 2019; published 8 August, 2019.

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ABSTRACT. In this paper, we show the existence and uniqueness of solution of the neutral stochastic functional differential equations under weakened Hölder condition, a weakened linear growth condition, and a contractive condition. Furthermore, in order to obtain the existence of a solution to the equation we used the Picard sequence.

Key words and phrases: Schwarz's inequality, Triangle inequality, Bessel's inequality, Grüss type inequalities, Integral inequalities.

2000 *Mathematics Subject Classification.* 60H05, 60H10, 60H20.

ISSN (electronic): 1449-5910

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2016R1D1A1A09917895).

1. INTRODUCTION

In the study of natural science system, one assumes that the system under consideration is governed by principle causality. A more realistic model would include some of the past states of the system. Stochastic functional differential equation gives a mathematical formulation for such system. One of the special but important class of stochastic functional differential equations is the stochastic differential delay equations.

So, we need an another class of stochastic equations depending on past and present values but that involves derivatives with delays as well as the function itself. Such equations historically have been referred to as *neutral stochastic functional differential equations, or neutral stochastic differential delay equations* (see, [3], [4], [5], [6], [7], [8], [9], [10], [12], [13]).

Such equations are more difficult to motivate but often arise in the study of two or more simple electrodynamics or oscillatory systems with some interconnections between them. Moreover, we can not ignore the effect of the science systems with time delay. For example, In studying the collision problem in electrodynamics, Driver [2] considered the system of neutral type

$$\dot{y}(t) = f_1(y(t), y(\delta(t))) + f_2(y(t), y(\delta(t)))\dot{y}(\delta(t)),$$

where $\delta(t) \leq t$. Generally, a neutral functional differential equation has the form

$$\frac{d}{dt}[y(t) - D(y_t)] = f(y_t, t).$$

Taking into account stochastic perturbations, we are led to a neutral stochastic functional differential equation

$$d[y(t) - D(y_t)] = f(y_t, t)dt + g(y_t, t)dB(t) \tag{1.1}$$

Neutral stochastic functional differential equations(NSDEs) are known to model problems from several areas of science and engineering. For instance, in 2007, Mao [12] published the stochastic differential equations and applications, in 2010, Li and Fu [11] considered the stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks, in 2013, Wei *et al.* [14] considered the existence and uniqueness of the solution to following neutral stochastic functional differential equations with infinite delay. Also, Kim [8] considered the solution to following neutral stochastic functional differential equations with infinite delay

$$d[x(t) - G(t, x_t)] = f(t, x_t)dt + g(t, x_t)dB(t), \tag{1.2}$$

where $x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}$.

Motivated by [4], [8], [14], one of the objectives of this paper is to get one proof to existence and uniqueness theorem for given NSDEs. The other objective of this paper is to estimate on how fast the Picard iterations $x_n(t)$ convergence the unique solution $x(t)$ of the NSDEs.

2. PRELIMINARY

Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\mathrm{trace}(A^T A)}$. And $BC((-\infty, 0]; R^d)$ denote the family of bounded continuous R^d-value functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$. $\mathcal{M}^2((-\infty, T]; R^d)$ denote the family of all $\mathbb{R}^d$-valued measurable $\mathcal{F}_t$-adapted process $\psi(t) = \psi(t, w), t \in (-\infty, T]$ such that $E\int_{-\infty}^{T} |\psi(t)|^2 dt < \infty$.

Let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets).

Let $B(t)$ is a m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$.

For $0 \leq t_0 \leq T < \infty$, we define two Borel measurable mappings $f : [t_0, T]\times BC((-\infty, 0]; R^d) \to R^d$ and $g : [t_0, T] \times BC((-\infty, 0]; R^d) \to R^{d\times m}$ and a continuous mapping $G : [t_0, T] \times BC((-\infty, 0]; R^d) \to R^d$ with $G(t, 0) = 0$.

With all the above preparation, consider the following d-dimensional neutral SFDEs:

$$\mathrm{d}[x(t) - G(t, x_t)] = f(t, x_t)\mathrm{d}t + g(t, x_t)\mathrm{d}B(t), \quad t_0 \leq t \leq T, \tag{2.1}$$

where $x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}$ can be considered as a $BC((-\infty, 0]; R^d)$-value stochastic process. The initial value of the system (2.1)

$$x_{t_0} = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\} \tag{2.2}$$

is an $\mathcal{F}_{t_0}$measurable, $BC((-\infty, 0]; R^d)$- value random variable such that $\xi \in \mathcal{M}^2((-\infty, 0]; R^d)$.

To be more precise, we give the definition of the solution of the equation (2.1) with initial data (2.2).

Definition 2.1. [14] R^d-value stochastic process $x(t)$ defined on $-\infty < t \leq T$ is called the solution of (2.1) with initial data (2.2), if $x(t)$ has the following properties:
(i) $x(t)$ is continuous and $\{x(t)\}_{t_0\leq t\leq T}$ is $\mathcal{F}_t$-adapted;
(ii) $\{f(t, x_t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ and $\{g(t, x_t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$;
(iii) $x_{t_0} = \xi$, for each $t_0 \leq t \leq T$,

$$x(t) = \xi(0) + G(t, x_t) - G(t_0, \xi) + \int_{t_0}^{t} f(s, x_s)ds + \int_{t_0}^{t} g(s, x_s)dB(s) \quad \text{a.s.} \tag{2.3}$$

The $x(t)$ is called as a unique solution, if any other solution $\overline{x}(t)$ is indistinguishable with $x(t)$, that is

$$P\{x(t) = \overline{x}(t), \text{ forany } -\infty < t \leq T\} = 1.$$

The following lemmas are known as special name for stochastic integrals which was appear in [1], [12] and will play an important role in next section.

Lemma 2.1. *(Stachurska's inequality)* [1] *Let $u(t)$ and $k(t)$ be nonnegative continuous functions for $t \geq \alpha$, and let $u(t) \leq a(t) + b(t)\int_\alpha^t k(s)u^p(s)\mathrm{d}s$, $t \in J = [\alpha, \beta)$, where a/b is nondecreasing function and $0 < p < 1$. Then*

$$u(t) \leq a(t)\left\{1 - (p-1)\left[\frac{a(t)}{b(t)}\right]^{p-1}\int_\alpha^t k(s)b^p(s)\mathrm{d}s\right\}^{-1/(p-1)}.$$

Lemma 2.2. *(Hölder's inequality)* [12] *If $\frac{1}{p} + \frac{1}{q} = 1$ for any $p, q > 1$, $f \in L^p$, and $g \in L^q$, then $fg \in L^1$ and $\int_a^b fg\,\mathrm{dx} \leq (\int_a^b |f|^p\,\mathrm{dx})^{1/p}(\int_a^b |g|^q\,\mathrm{dx})^{1/q}$.*

Lemma 2.3. *(Moment inequality)* [12] *If $p \geq 2$, $g \in \mathcal{M}^2([0, T]; R^{d\times m})$ such that $E\int_0^T |g(s)|^p\,\mathrm{d}s < \infty$, then*

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\,\mathrm{d}s.$$

In order to attain the solution of equation (2.1) with initial data (2.2), we propose the following assumptions:

(H1) (Hölder condition) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, we assume that

$$|f(t,\varphi) - f(t,\psi)|^2 \vee |g(t,\varphi) - g(t,\psi)|^2 \leq C\|\varphi - \psi\|^{2\alpha}$$

where C is a positive constant and $0 < \alpha \leq 1$ is a constant.

(H2) (weakened linear growth condition) For any $t \in [t_0, T]$, it follows that $f(t, 0), g(t, 0) \in L^2$ such that

$$|f(t,0)|^2 \vee |g(t,0)|^2 \leq C_1,$$

where C_1 is a positive constant.

(H3) (contractive condition) Assuming that there exists a positive number C_0 such that $0 < C_0 < 1$ and for any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|G(t,\varphi) - G(t,\psi)| \leq C_0\|\varphi - \psi\|.$$

3. MAIN RESULTS

In order to obtain the existence of solutions to neutral SFDEs, let $y^0_{t_0} = \xi$ and $y^0(t) = \xi(0)$, for $t_0 \leq t \leq T$. For each $n = 1, 2, \ldots$, set $y^n_{t_0} = \xi$ and define the following Picard sequence

$$\begin{aligned} &y^n(t) - \xi(0) \\ &= G(t, y^{n-1}_t) - G(t_0, y^{n-1}_{t_0}) + \int_{t_0}^t f(s, y^{n-1}_s)ds + \int_{t_0}^t g(s, y^{n-1}_s)dB(s). \end{aligned} \tag{3.1}$$

Now we give the existence theorem to the solution of equation (2.1) with initial data (2.2) by approximate solutions by means of Picard sequence.

Theorem 3.1. *Assume that (H1)-(H3) hold. Then, there exists a unique solution to the neutral SFDEs (2.1) with initial data (2.2) and with $\gamma_1 \geq 1$. Moreover, the solution belongs to $\mathcal{M}^2((-\infty, T]; R^d)$.*

We prepare two lemmas in order to prove this theorem.

Lemma 3.2. *Let $u(t)$ and $a(t)$ be continuous functions on $[0, T]$. Let $k \geq 1$ and $0 < p \leq 1$ be constants. If $u(t) \leq k + \int_{t_0}^t a(s)u^p(s)\, ds$ for $t \in I$, then*

$$u(t) \leq k \exp\left(\int_{t_0}^t a(s)\, ds\right)$$

for $t \in I$.

Proof. For $t \in I$, let $k \geq 1$ and define a function $z(t) = k + \int_{t_0}^t a(s)u^p(s)\, ds$. Then, $z(t) \geq 1, z(t_0) = k, u(t) \leq z(t)$, and

$$z'(t) = a(t)u^p(t) \leq a(t)z^p(t) \leq a(t)z(t).$$

As on I, we deduce that

$$\frac{z'(t)}{z(t)} \leq a(t).$$

Integrating both sides from t_0 to t, where $t \in I$, and applying some change of variables yields

$$z(t) \leq k \exp\left(\int_{t_0}^t a(s)\, ds\right)$$

for $t \in I$. Using inequality $u(t) \leq z(t)$, we get the required inequality. ∎

Lemma 3.3. *Let the assumption (H1) and (H3) hold. If $x(t)$ is a solution of (2.1) with initial data (2.2), then*

$$\begin{aligned}&E\Big(\sup_{-\infty<t\le T}|x(t)|^2\Big)\\ &\le\bigg(\frac{4+\alpha\sqrt{\alpha}}{(1-\alpha)(1-\sqrt{\alpha})}E\|\xi\|^2+\eta(T-t_0)C_1\bigg)\exp(\eta C(T-t_0)),\end{aligned}$$

where C and C_1 are positive constants, $\eta=6(T-t_0+4)/(1-\alpha)(1-\sqrt{\alpha})$ with $4E\|\xi\|^2/(1-\alpha)(1-\sqrt{\alpha})+\eta(T-t_0)C_1\ge 1$. In particular, $x(t)$ belong to $\mathcal{M}^2((-\infty,T];R^d)$.

Proof. For each number $n\ge 1$, define the stopping time

$$\tau_n=T\wedge\inf\{t\in[t_0,T]:\|x(t)\|\ge n\}.$$

Obviously, as $n\to\infty,\tau_n\uparrow T$ a.s. Let $x^n(t)=x(t\wedge\tau_n),t\in(-\infty,T]$. Then, for $t_0\le t\le T$, $x^n(t)$ satisfy the following equation

$$x^n(t)=G(t,x_t^n)-G(t_0,x_{t_0}^n)+J^n(t),$$

where

$$J^n(t)=\xi(0)+\int_{t_0}^t f(s,x_s^n)I_{[t_0,\tau_n]}(s)\,\mathrm{d}s+\int_{t_0}^t g(s,x_s^n)I_{[t_0,\tau_n]}(s)\,\mathrm{d}B(s).$$

Applying the elementary inequality $(a+b)^2\le\frac{a^2}{\alpha}+\frac{b^2}{1-\alpha}$ when $a,b>0,0<\alpha<1$, and from the condition (H3), we have

$$\begin{aligned}|x^n(t)|^2&\le\frac{1}{\alpha}|G(t,x_t^n)-G(t_0,x_{t_0}^n)|^2+\frac{1}{1-\alpha}|J^n(t)|^2\\ &\le\sqrt{\alpha}\|x_t^n\|^2+\frac{\alpha}{1-\sqrt{\alpha}}\|\xi\|^2+\frac{1}{1-\alpha}|J^n(t)|^2.\end{aligned}$$

Taking the expectation on both sides, one sees that

$$\begin{aligned}&E\Big(\sup_{t_0<s\le t}|x^n(s)|^2\Big)\\ &\le\sqrt{\alpha}E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)+\frac{\alpha}{1-\sqrt{\alpha}}E\|\xi\|^2+\frac{1}{1-\alpha}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).\end{aligned}$$

Noting that $\sup_{-\infty<s\le t}|x^n(s)|^2\le\|\xi\|^2+\sup_{t_0\le s\le t}|x^n(s)|^2$, we get

$$\begin{aligned}&E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)\\ &\le\sqrt{\alpha}E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)+\frac{\alpha}{1-\sqrt{\alpha}}E\|\xi\|^2+\frac{1}{1-\alpha}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).\end{aligned}$$

Consequently

$$\begin{aligned}&E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)\\ &\le\frac{1+\alpha-\sqrt{\alpha}}{(1-\sqrt{\alpha})^2}E\|\xi\|^2+\frac{1}{(1-\alpha)(1-\sqrt{\alpha})}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).\end{aligned}\tag{3.2}$$

On the other hand, by the elementary inequality $(a+b+c)^2\le 3a^2+3b^2+3c^2$, one can show that

$$|J^n(s)|^2\le 3\bigg[E\|\xi\|^2+\bigg|\int_{t_0}^t|f(s,x_s^n)|^2ds\bigg|^2+\bigg|\int_{t_0}^s g(r,x_r^n)\,\mathrm{d}B(r)\bigg|^2\bigg].$$

By Hölder's inequality and Lemma (2.3), one can show that

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big)\\ &\le 3\bigg[E\|\xi\|^2+(T-t_0)\int_{t_0}^t E|f(s,x_s^n)|^2ds+4\int_{t_0}^t E|g(s,x_s^n)|^2\,ds\bigg]. \end{aligned}$$

By the condition (H1) and (H2), one can show that

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big)\\ &\le 3E\|\xi\|^2+6(T-t_0+4)C_1(T-t_0)+6(T-t_0+4)C\int_{t_0}^t E\|x_s^n\|^{2\alpha}ds. \end{aligned}$$

Substituting this into (3.2) yields that

$$\begin{aligned} &E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)\\ &\le \frac{4+\alpha\sqrt{\alpha}}{(1-\alpha)(1-\sqrt{\alpha})}E\|\xi\|^2+\eta(T-t_0)C_1+\eta C\int_{t_0}^t E\|x_s^n\|^{2\alpha}ds, \end{aligned}$$

where $\eta=6(T-t_0+4)/(1-\alpha)(1-\sqrt{\alpha})$. Therefore, we have

$$\begin{aligned} &E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)\\ &\le \frac{4+\alpha\sqrt{\alpha}}{(1-\alpha)(1-\sqrt{\alpha})}E\|\xi\|^2+\eta(T-t_0)C_1+\eta C\int_{t_0}^t \sup_{-\infty<r\le s}E|x^n(r)|^{2\alpha}ds. \end{aligned}$$

The Lemma (3.2) then yields that

$$\begin{aligned} &E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)\\ &\le \bigg(\frac{4+\alpha\sqrt{\alpha}}{(1-\alpha)(1-\sqrt{\alpha})}E\|\xi\|^2+\eta(T-t_0)C_1\bigg)\exp(\eta C(T-t_0)) \end{aligned}$$

with $(4+\alpha\sqrt{\alpha})E\|\xi\|^2/(1-\alpha)(1-\sqrt{\alpha})+\eta(T-t_0)C_1\ge 1$. For all $n=0,1,2,\dots$, we deduce that

$$\begin{aligned} &E\Big(\sup_{-\infty<s\le t}|x(s\wedge\tau_n)|^2\Big)\\ &\le \bigg(\frac{4+\alpha\sqrt{\alpha}}{(1-\alpha)(1-\sqrt{\alpha})}E\|\xi\|^2+\eta(T-t_0)C_1\bigg)\exp(\eta C(T-t_0)). \end{aligned}$$

Consequently the required inequality follows by letting $n\to\infty$. ∎

Proof of Theorem 3.1. To check the uniqueness, let $x(t)$ and $\overline{x}(t)$ be any two solutions of (2.1) with initial data (2.2). By Lemma 3.3, $x(t),\overline{x}(t)\in\mathcal{M}^2((-\infty,T];R^d)$. Note that

$$x(t)-\overline{x}(t)=G(t,x_t)-G(t,\overline{x}_t)+J(t),$$

where $J(t)=\int_{t_0}^t[f(s,x_s)-f(s,\overline{x}_s)]ds+\int_{t_0}^t[g(s,x_s)-g(s,\overline{x}_s)]dB(s)$. One then gets

$$|x(t)-\overline{x}(t)|^2\le\frac{1}{\alpha}|G(t,x_t)-G(t,\overline{x}_t)|^2+\frac{1}{1-\alpha}|J(t)|^2,$$

where $0<\alpha<1$. We derive that

$$|x(t)-\overline{x}(t)|^2\le\alpha\|x_t-\overline{x}_t\|^2+\frac{1}{1-\alpha}|J(s)|^2.$$

Therefore

$$E\Big(\sup_{t_0\le s\le t}|x(s)-\overline{x}(s)|^2\Big)\le \alpha E\Big(\sup_{t_0\le s\le t}|x(s)-\overline{x}(s)|^2\Big)+\frac{1}{(1-\alpha)}E\Big(\sup_{t_0\le s\le t}|J(s)|^2\Big).$$

Consequently

$$E\Big(\sup_{t_0\le s\le t}|x(s)-\overline{x}(s)|^2\Big)\le \frac{1}{(1-\alpha)^2}E\Big(\sup_{t_0\le s\le t}|J(s)|^2\Big).$$

On the other hand, one can show that

$$\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|J(s)|^2\Big)\\
&\le 2\Big[(T-t_0)E\int_{t_0}^t|f(s,x_s)-f(s,\overline{x}_s)|^2ds+4E\int_{t_0}^t|g(s,x_s)-g(s,\overline{x}_s)|^2\,\mathrm{ds}\Big]\\
&\le 2(T-t_0+4)C\int_{t_0}^t E\|x_s-\overline{x}_s\|^{2\alpha}\,ds.
\end{aligned}$$

For any $\varepsilon>0$, by the condition (H1), this yields that

$$E\Big(\sup_{t_0\le s\le t}|J(s)|^2\Big)\le \varepsilon+2(T-t_0+4)C\int_{t_0}^t E\sup_{t_0<r\le s}|x(r)-\overline{x}(r)|^{2\alpha}\,ds.$$

Therefore, by the Stachurska's inequality and letting $\epsilon\to 0$, we have $E\sup_{t_0\le s\le t}|J(s)|^2=0$. This implies that

$$E\Big(\sup_{t_0<s\le t}|x(t)-\overline{x}(t)|^2\Big)=0.$$

Hence, we get $x(t)=\overline{x}(t)$ for $t_0\le t\le T$ a.s. The uniqueness has been proved.

Now we check the existence of the solution using the Picard sequence (3.1). Obviously, from the Picard iterations, we have $x^0(t)\in\mathcal{M}^2((-\infty,T]:R^d)$. Moreover, one can show the boundedness of the sequence $\{x^n(t),n\ge 0\}$ that $x^n(t)\in\mathcal{M}^2((-\infty,T]:R^d)$, in fact

$$x^n(t)=G(t,x_t^{n-1})-G(t_0,x_{t_0}^{n-1})+J^{n-1}(t),$$

where

$$J^{n-1}(t)=\xi(0)+\int_{t_0}^t f(s,x_s^{n-1})ds+\int_{t_0}^t g(s,x_s^{n-1})dB(s).$$

Applying the elementary inequality $(a+b)^2\le\frac{a^2}{\alpha}+\frac{b^2}{1-\alpha}$ when $a,b>0, 0<\alpha<1$, we have

$$\begin{aligned}
|x^n(t)|^2 &\le \frac{1}{\alpha}|G(t,x_t^{n-1})-G(t_0,\xi)|^2+\frac{1}{1-\alpha}|J^{n-1}(t)|^2\\
&\le \sqrt{\alpha}\|x_t^{n-1}\|^2+\frac{\alpha}{1-\sqrt{\alpha}}\|\xi\|^2+\frac{1}{1-\alpha}|J^{n-1}(t)|^2,
\end{aligned}$$

where condition (H3) has also been used. Taking the expectation on both sides, one sees that

$$\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x^n(s)|^2\Big)-\sqrt{\alpha}E\sup_{-\infty<s\le t}|x^{n-1}(s)|^2\\
&\le\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}E\|\xi\|^2+\frac{1}{1-\alpha}E\Big(\sup_{t_0\le s\le t}|J^{n-1}(s)|^2\Big).
\end{aligned}\tag{3.3}$$

On the other hand, by elementary inequality, Hölder's inequality and moment inequality, one can show that

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|J^{n-1}(s)|^2\Big)\\
&\le 3\Big[E\|\xi\|^2+(T-t_0)E\int_{t_0}^t|f(s,x_s^{n-1})|^2ds+4E\int_{t_0}^t|g(s,x_s^{n-1})|^2\,\mathrm{d}s\Big].
\end{aligned}
$$

Therefore, using the conditions (H1) and (H2), we have

$$
E\Big(\sup_{t_0\le s\le t}|J^{n-1}(s)|^2\Big)\le 3E\|\xi\|^2+\beta+6C(T-t_0+4)\int_{t_0}^t E\|x_s^{n-1}\|^2\,ds,
$$

where $\beta=6(T-t_0+4)(T-t_0)C_1$. Substituting this into (3.3) yields that

$$
\begin{aligned}
&E\sup_{t_0\le s\le t}|x^n(s)|^2\\
&\le \gamma+\sqrt{\alpha}E\sup_{t_0\le s\le t}|x^{n-1}(s)|^2+\frac{6(T-t_0+4)C}{1-\alpha}\int_{t_0}^t E\sup_{t_0\le r\le s}|x^{n-1}(r)|^{2\alpha}ds.
\end{aligned}
$$

where $\gamma=\frac{\beta}{1-\alpha}+\frac{3-(2+\alpha)\sqrt{\alpha}}{(1-\sqrt{\alpha})(1-\alpha)}E\|\xi\|^2+\frac{C}{(1-\alpha)}6(T-t_0+4)(T-t_0)E\|\xi\|^{2\alpha}$. It also follows note that for any $k\ge 1$,

$$
\begin{aligned}
&\max_{1\le n\le k}E\Big(\sup|x^{n-1}(u)|^{2\alpha}\Big)\\
&=\max\Big\{E\|\xi\|^{2\alpha},E(\sup|x^1(u)|^{2\alpha}),\dots,E(\sup|x^{k-1}(u)|^{2\alpha})\Big\}\\
&\le\max\Big\{E\|\xi\|^{2\alpha},E(\sup|x^1(u)|^{2\alpha}),\dots,E(\sup|x^{k-1}(u)|^{2\alpha}),E(\sup|x^k(u)|^{2\alpha})\Big\}\\
&\le E\|\xi\|^{2\alpha}+\max_{1\le n\le k}E(\sup|x^n(u)|^{2\alpha}).
\end{aligned}
$$

Therefore, one can derive that

$$
\begin{aligned}
&\max_{1\le n\le k}E\Big(\sup_{t_0\le s\le t}|x^n(s)|^2\Big)\\
&\le\gamma_1+\frac{6(T-t_0+4)C}{(1-\alpha)(1-\sqrt{\alpha})}\int_{t_0}^t\max_{1\le n\le k}E\Big(\sup_{t_0\le r\le s}|x^n(r)|^{2\alpha}\Big)\,ds,
\end{aligned}
$$

where $\gamma_1=\frac{\gamma}{1-\sqrt{\alpha}}+\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})}E\|\xi\|^2+\frac{6C(T-t_0+4)(T-t_0)}{(1-\sqrt{\alpha})(1-\alpha)}E\|\xi\|^{2\alpha}$. By Lemma 3.2, we have

$$
\max_{1\le n\le k}E\Big(\sup_{t_0\le s\le t}|x^n(s)|^2\Big)\le\gamma_1\exp\Big(\frac{6C(T-t_0+4)(T-t_0)}{(1-\sqrt{\alpha})(1-\alpha)}\Big)
$$

with $\gamma_1\ge 1$. Since k is arbitrary, for all $n=0,1,2,\dots,$ we deduce that

$$
E\Big(\sup_{t_0\le s\le t}|x^n(s)|^2\Big)\le\gamma_1\exp\Big(\frac{6C(T-t_0+4)(T-t_0)}{(1-\sqrt{\alpha})(1-\alpha)}\Big),
$$

which shows the boundedness of the sequence $\{x^n(t),n\ge 0\}$.

Next, we check that the sequence $\{x^n(t)\}$ is Cauchy sequence. For all $n\ge 0$ and $t_0\le t\le T$, we have

$$
\begin{aligned}
&x^{n+1}(t)-x^n(t)=G(t,x_t^n)-G(t,x_t^{n-1})\\
&+\int_{t_0}^t[f(s,x_s^n)-f(s,x_s^{n-1})]ds+\int_{t_0}^t[g(s,x_s^n)-g(s,x_s^{n-1})]dB(s).
\end{aligned}
$$

Using an elementary inequality $(u+v)^2 \leq \frac{1}{\alpha}u^2 + \frac{1}{1-\alpha}v^2$ and the condition (H3), we derive that

$$E\Big(\sup_{t_0<s\leq t}|x^{n+1}(s)-x^n(s)|^2\Big) \leq \alpha E\Big(\sup_{t_0<s\leq t}|x^n(s)-x^{n-1}(s)|^2\Big)$$
$$+\frac{2(T-t_0+4)C}{1-\alpha}E\int_{t_0}^{t}\sup_{t_0\leq r\leq s}|x^n(r)-x^{n-1}(r)|^{2\alpha}\,ds.$$

This yields that

$$\limsup_{n\to\infty} E\Big(\sup_{t_0<s\leq t}|x^{n+1}(s)-x^n(s)|^2\Big)$$
$$\leq \limsup_{n\to\infty}\frac{2(T-t_0+4)C}{(1-\alpha)^2}\int_{t_0}^{t}E\Big(\sup_{t_0\leq u\leq s}|x^{n+1}(u)-x^n(u)|^{2\alpha}\Big)\,ds.$$

Let $Z(t)=\limsup_{n\to\infty}E\Big(\sup_{t_0\leq s\leq t}|x^{n+1}(s)-x^n(s)|^2\Big)$, we get

$$Z(t)\leq \epsilon + \frac{2(T-t_0+4)C}{(1-\alpha)^2}\int_{t_0}^{t}Z^{\alpha}(s)\,ds.$$

By Stachurska's inequality, we get $Z(t)=0$. This shows the sequence $\{x^n(t), n\geq 0\}$ is a Cauchy sequence in L^2. Hence, as $n\to\infty, x^n(t)\to x(t)$, that is $E|x^n(t)-x(t)|^2\to 0$. Therefore, we obtain that $x(t)\in\mathcal{M}^2((-\infty,T];R^d)$. Now to show that $x(t)$ satisfy (2.3).

$$E\bigg|\int_{t_0}^{t}[f(s,x_s^n)-f(s,x_s)]ds+\int_{t_0}^{t}[g(s,x_s^n)-g(s,x_s)]\,\mathrm{d}B(s)\bigg|^2$$
$$\leq 2\bigg[(T-t_0)E\int_{t_0}^{t}|f(s,x_s^n)-f(s,x_s)|^2ds+4E\int_{t_0}^{t}|g(s,x_s^n)-g(s,x_s)|^2\,\mathrm{d}s\bigg]$$
$$\leq 2(T-t_0+4)\int_{t_0}^{t}E\Big(\sup_{t_0\leq u\leq s}|x^n(u)-x(u)|^{2\alpha}\Big)\,\mathrm{d}s.$$

Noting that sequence $x^n(t)$ is uniformly converge on $(-\infty,T]$, it means that

$$E\Big(\sup_{t_0\leq u\leq s}|x^n(u)-x(u)|^2\Big)\to 0$$

as $n\to\infty$. Hence, taking limits on both sides in the Picard sequence, we obtain that

$$x(t)=\xi(0)+G(t,x_t)-G(t_0,x_{t_0})+\int_{t_0}^{t}f(s,x_s)ds+\int_{t_0}^{t}g(s,x_s)dB(s).$$

The above expression demonstrates that $x(t)$ is a solution of equation (2.1) satisfying the initial condition (2.2). So far, the existence of theorem is complete. ∎

Remark 3.1. Using the weakened Hölder's condition, in the Theorem 3.1, we have shown that the Picard iterations $x^n(t)$ converge to the unique solution $x(t)$ of equation (2.1). In the next, we should gives an estimate on the difference between $x^n(t)$ and $x(t)$ under the weakened Hölder's condition, and it clearly shows that one can use the Picard iteration procedure to obtain the approximate solutions to equations (2.1).

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