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ON AN EXTENSION OF EDWARDS'S DOUBLE INTEGRAL WITH APPLICATIONS

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ABSTRACT. The aim of this note is to provide an extension of the well known and useful Edwards's double integral. As an application, new class of twelve double integrals involving hypergeometric function have been evaluated in terms of gamma function. The results are established with the help of classical summation theorems for the series ${}_3F_2$ due to Watson, Dixon and Whipple. Several new and interesting integrals have also been obtained from our main findings.

Key words and phrases: Generalized hypergeometric function, Watson, Dixon and Whipple summation theorems, Edwards's double integral.

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1. INTRODUCTION

To justify our doing, we must quote Sylvester [14]:

" It seems to be expected of every pilgrim up the slope of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock"

We start with the following well known and useful double integral due to Edwards [3], viz

$$\int_0^1 \int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.1}$$

provided $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$.

Several new class of double integrals involving hypergeometric and generalized hypergeometric functions have been evaluated in terms of gamma functions in recent years. For this, we refer the works [4, 5, 6, 10] and the references cited there.

This note is organised as follows. In Section 2, we consider a natural extension of Edwards's integral (1.1). We also mention two special cases. In Section 3, four double integrals involving hypergeometric functions have been evaluated in terms of generalized hypergeometric functions. In Section 4, we discussed some applications of the results derived in Section 3 by applying classical summation theorems due to Watson, Dixon and Whipple. In Section 5, we mention some of the very interesting special cases of the results given in Section 4.

We conclude this section by mentioning that whenever a double integral reduces to gamma function, the results are very important from the application point of view. Thus the results established in this may be useful potentially.

2. AN EXTENTION OF EDWARDS'S DOUBLE INTEGRAL

In this section, we shall establish an extension of Edwards's double integral (1.1). For this, we consider the follwing double integral

$$I = \int_0^1 \int_0^1 x^{\gamma-1}y^{\epsilon-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}dxdy. \tag{2.1}$$

Using the Binomial series

$$(1-x)^{-a} = {}_1F_0\left[\begin{matrix} a \\ - \end{matrix}; x\right]$$

where ${}_1F_0$ is a special case of the generalized hypergeometric function ${}_pF_q$ given in the Appendix at the end of this note, we have by (2.1),

$$I = \int_0^1 \int_0^1 x^{\gamma-1}y^{\epsilon-1}(1-x)^{\alpha-1}(1-y)^{\beta-1} {}_1F_0\left[\begin{matrix} \alpha+\beta-\delta \\ - \end{matrix}; xy\right] dxdy.$$

Now, expressing the Binomial function ${}_1F_0$ with the help of the definition, changing the order of integration and series, which is clearly seen to be justified due to the uniform convergence of the sereis involved in the process, then separating two integrals, we have

$$I = \sum_{n=0}^{\infty} \frac{(\alpha+\beta-\delta)_n}{n!} \int_0^1 x^{\gamma+n-1}(1-x)^{\alpha-1}dx \cdot \int_0^1 y^{\epsilon+n-1}(1-y)^{\beta-1}dy.$$

Evaluating the beta-integral and using the result

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

we have, after some simplification,

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\epsilon)}{\Gamma(\gamma+\alpha)\Gamma(\epsilon+\beta)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta-\delta)_n(\gamma)_n(\epsilon)_n}{(\gamma+\alpha)_n(\epsilon+\beta)_n n!}.$$

Summing up the series, we have

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\epsilon)}{\Gamma(\gamma+\alpha)\Gamma(\epsilon+\beta)} \cdot \ {}_3F_2\left[\begin{matrix}\alpha+\beta-\delta,\ \gamma,\ \epsilon\\ \gamma+\alpha,\ \epsilon+\beta\end{matrix};1\right],$$

valid for Re$(\alpha) > 0,$ Re$(\beta) > 0,$ Re$(\gamma) > 0,$ Re$(\epsilon) > 0$ and Re$(\delta) > 0.$
Now select $\epsilon = \alpha + \gamma$, we get

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta} dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \cdot \ {}_2F_1\left[\begin{matrix}\alpha+\beta-\delta,\ \gamma\\ \alpha+\beta+\gamma\end{matrix};1\right].$$

Using the well known Gauss's summation theorem [1, 9], viz

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

provided Re$(c-a-b) > 0$, we arrive at the following result

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta} dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \tag{2.2}$$

provided Re$(\alpha) > 0,$ Re$(\beta) > 0,$ Re$(\gamma) > 0$ and Re$(\delta) > 0.$

Remark 2.1.

1. In (2.2), if we take $\gamma = \delta = 1$, we get at once, the Edwards's integral (1.1). Hence our result (2.2) may be regarded as an extension of (1.1).
2. In (2.2), if we take $\alpha = \beta = 1$, we get the following interesting and new (presumable) double integral

$$\int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma}(1-xy)^{\delta-2} dxdy = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}$$

provided Re$(\gamma) > 0$ and Re$(\delta) > 0.$

3. Altenatively, the integral (2.2) can also be derived from the following result recorded in Copson [2, p.228, Problem 9]:

$$\int_0^1 dx \int_0^1 dy f(xy)(1-x)^{\alpha-1}(1-y)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^1 f(t)(1-t)^{\alpha+\beta-1} dt$$

by selecting $f(t) = t^{\gamma-1}(1-t)^{\delta-\alpha-\beta}$ and then applying the usual Beta integral to evaluate the integral appearing on the right-hand side.

We conclude this section by remarking that in Section 3, we shall obtain four general double integrals by employing (2.2).

3. FOUR DOUBLE INTEGRALS OF GENERAL TYPE

In this section, we shall evaluate the following four double integrals involving hypergeometric function, which are of general in nature and each is valid for Re$(\alpha) > 0$, Re$(\beta) > 0$, Re$(\gamma) > 0$ and Re$(\delta) > 0$. These are

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}\cdot {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};xy\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\beta)\Gamma(\gamma+\delta)}\cdot\ {}_3F_2\left[\begin{matrix}a, & b, & \gamma\\ c, & \gamma+\delta\end{matrix};1\right].\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}\cdot {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};1-xy\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\beta)\Gamma(\gamma+\delta)}\cdot\ {}_3F_2\left[\begin{matrix}a, & b, & \delta\\ c, & \gamma+\delta\end{matrix};1\right].\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}\cdot {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{y(1-x)}{1-xy}\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\beta)\Gamma(\gamma+\delta)}\cdot\ {}_3F_2\left[\begin{matrix}a, & b, & \alpha\\ c, & \alpha+\beta\end{matrix};1\right].\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}\cdot {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\frac{1-y}{1-xy}\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\beta)\Gamma(\gamma+\delta)}\cdot\ {}_3F_2\left[\begin{matrix}a, & b, & \beta\\ c, & \alpha+\beta\end{matrix};1\right].\end{aligned} \tag{3.4}$$

Proof. In order to derive (3.1), we proceed as follows. Denoting the left-hand side of (3.1) by I, we have

$$\begin{aligned}I = &\int_0^1\int_0^1 x^{\gamma-1}y^{\alpha+\gamma-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}\\ &\cdot {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};xy\right]dxdy.\end{aligned}$$

Now, expressing the ${}_2F_1$ hypergeometric function as a series, changing the order of integration and summation, separating the two integrals, evaluating the resulting beta-integrals, we have, after some simplification

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\beta)\Gamma(\gamma+\delta)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(\gamma)_n}{(c)_n(\gamma+\delta)_n\, n!}.$$

Summing up the series, we easily arrive at the right hand side of (3.1). This completes the proof of (3.1). In exactly the same manner, the integrals (3.2) to (3.4) can be derived. So we prefer to omit the details. ∎

We conclude this section by remarking that by suitably applying certain classical summation theorems, the results (3.1) to (3.4) can be put in terms of gamma function. For this, we require the following classical summation theorems.

Watson's Theorem [1, 12].

$$(3.5)\quad {}_3F_2\left[\begin{matrix} a, & b, & d \\ \frac{1}{2}(a+b+1), & 2d \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(d+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(d-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(d-\frac{1}{2}a+\frac{1}{2})\Gamma(d-\frac{1}{2}b+\frac{1}{2})} = \Omega_1$$

provided $\mathrm{Re}(2c-a-b) > -1$.

Dixon's Theorem [1, 12].

$$(3.6)\quad {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)} = \Omega_2$$

provided $\mathrm{Re}(a-2b-2c) > -2$.

Whipple's Theorem [1, 12].

$$(3.7)\quad {}_3F_2\left[\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}; 1\right] = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma(\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}a+\frac{1}{2}f)\Gamma(\frac{1}{2}b+\frac{1}{2}e)\Gamma(\frac{1}{2}b+\frac{1}{2}f)} = \Omega_3$$

provided $a+b=1$, $e+f=2c+1$ and $\mathrm{Re}(e+f-a-b-c) > 0$.

4. DOUBLE INTEGRAL INVOVING HYPERGEOMETRIC FUNCTION

In this section, we shall establish twelve double integrals invoving hypergeometric function in terms of gamma function. These are obtained with the help of the results given in Section 3 together with applications of classical Watson, Dixon and Whipple summation theorems. These are

$$(4.1)\quad \int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta} \times {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) \end{matrix}; xy\right] dxdy = c_1\Omega_1$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(d) > 0$. Ω_1 is the same as given in (3.5) and c_1 is given by

$$(4.1.1)\quad c_1 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}.$$

$$\begin{aligned}
(4.2) \qquad & \int_0^1 \int_0^1 x^{d-1} y^{\alpha+d-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{d-\alpha-\beta} \\
& \quad \times {}_2F_1 \left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; 1-xy \right] dxdy \\
& = c_1 \Omega_1
\end{aligned}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(d) > 0$. Ω_1 is the same as given in (3.5) and c_1 is given by (4.1.1).

$$\begin{aligned}
(4.3) \qquad & \int_0^1 \int_0^1 x^{\gamma-1} y^{d+\gamma-1} (1-x)^{d-1} (1-y)^{d-1} (1-xy)^{\delta-2d} \\
& \quad \times {}_2F_1 \left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{y(1-x)}{1-xy} \right] dxdy \\
& = c_2 \Omega_1
\end{aligned}$$

provided $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\delta) > 0$ and $\mathrm{Re}(d) > 0$. Ω_1 is the same as given in (3.5) and c_2 is given by

$$(4.3.1) \qquad c_2 = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \cdot \frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}.$$

$$\begin{aligned}
(4.4) \qquad & \int_0^1 \int_0^1 x^{\gamma-1} y^{d+\alpha-1} (1-x)^{d-1} (1-y)^{d-1} (1-xy)^{\delta-2d} \\
& \quad \times {}_2F_1 \left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1-y}{1-xy} \right] dxdy \\
& = c_2 \Omega_1
\end{aligned}$$

provided $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\delta) > 0$ and $\mathrm{Re}(d) > 0$. Ω_1 is the same as given in (3.5) and c_2 is given by (4.3.1).

$$\begin{aligned}
(4.5) \qquad & \int_0^1 \int_0^1 x^{c-1} y^{\alpha+c-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1+a-2c-\alpha-\beta} \\
& \quad \times {}_2F_1 \left[\begin{matrix} a, \quad b \\ 1+a-b \end{matrix} ; xy \right] dxdy \\
& = c_3 \Omega_2
\end{aligned}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(c) > 0$. Ω_2 is the same as given in (3.6) and c_3 is given by

$$(4.5.1) \qquad c_3 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\Gamma(1+a-2c)}{\Gamma(1+a-c)}.$$

(4.6)
$$\begin{aligned}&\int_0^1\int_0^1 x^{a-2c}y^{\alpha+a-2c}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta}\\&\qquad\times {}_2F_1\left[\begin{matrix}a, & b\\ 1+a-b\end{matrix}; 1-xy\right]dxdy\\&= c_3\Omega_2\end{aligned}$$

provided Re$(\alpha) > 0$, Re$(\beta) > 0$ and Re$(1+a-2c) > 0$. Ω_2 is the same as given in (3.6) and c_3 is given by (4.5.1).

(4.7)
$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{c+\gamma-1}(1-x)^{c-1}(1-y)^{a-2c}(1-xy)^{\delta-a+c-1}\\&\qquad\times {}_2F_1\left[\begin{matrix}a, & b\\ 1+a-b\end{matrix}; \frac{y(1-x)}{1-xy}\right]dxdy\\&= c_4\Omega_2\end{aligned}$$

provided Re$(\gamma) > 0$, Re$(c) > 0$ and Re$(1+a-2c) > 0$. Ω_2 is the same as given in (3.6) and c_4 is given by

(4.7.1)
$$c_4 = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(c)\Gamma(1+a-2c)}{\Gamma(1+a-c)}.$$

(4.8)
$$\begin{aligned}&\int_0^1\int_0^1 x^{\gamma-1}y^{a-2c+\gamma}(1-x)^{a-2c}(1-y)^{c-1}(1-xy)^{\delta-a+c-1}\\&\qquad\times {}_2F_1\left[\begin{matrix}a, & b\\ 1+a-b\end{matrix}; \frac{1-y}{1-xy}\right]dxdy\\&= c_4\Omega_2\end{aligned}$$

provided Re$(\gamma) > 0$, Re$(c) > 0$ and Re$(1+a-2c) > 0$. Ω_2 is the same as given in (3.6) and c_4 is given by (4.7.1).

(4.9)
$$\begin{aligned}&\int_0^1\int_0^1 x^{c-1}y^{\alpha+c-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1+c-e-\alpha-\beta}\\&\qquad\times {}_2F_1\left[\begin{matrix}a, \ b\\ e\end{matrix}; xy\right]dxdy\\&= c_5\Omega_3\end{aligned}$$

provided Re$(\alpha) > 0$, Re$(\beta) > 0$ and Re$(c) > 0$. Ω_3 is the same as given in (3.7) and c_5 is given by

(4.9.1)
$$c_5 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(1+c-e)}{\Gamma(1+2c-e)}.$$

(4.10)
$$\int_0^1\int_0^1 x^{c-e}y^{\alpha+c-e}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times {}_2F_1\left[\begin{matrix}a,\ b\\ e\end{matrix};1-xy\right]dxdy = c_5\Omega_3$$

provided Re$(\alpha)>0$, Re$(\beta)>0$ and Re$(1+c-e)>0$. Ω_3 is the same as given in (3.7) and c_5 is given by (4.9.1).

(4.11)
$$\int_0^1\int_0^1 x^{\gamma-1}y^{c+\gamma-1}(1-x)^{c-1}(1-y)^{c-e}(1-xy)^{\delta-2c+e-1} \times {}_2F_1\left[\begin{matrix}a,\ b\\ e\end{matrix};\frac{y(1-x)}{1-xy}\right]dxdy = c_6\Omega_3$$

provided Re$(\gamma)>0$, Re$(c)>0$ and Re$(1+c-e)>0$. Ω_3 is the same as given in (3.7) and c_6 is given by

(4.11.1)
$$c_6=\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\cdot\frac{\Gamma(c)\Gamma(1+c-e)}{\Gamma(1+2c-e)}.$$

(4.12)
$$\int_0^1\int_0^1 x^{\gamma-1}y^{e-c+\gamma}(1-x)^{c-e}(1-y)^{c-1}(1-xy)^{\delta-2c+e-1} \times {}_2F_1\left[\begin{matrix}a,\ b\\ e\end{matrix};\frac{1-y}{1-xy}\right]dxdy = c_6\Omega_3$$

provided Re$(\gamma)>0$, Re$(c)>0$ and Re$(1+c-e)>0$. Ω_3 is the same as given in (3.7) and c_6 is given by (4.11.1).

Proof. The derivations of the results (4.1) to (4.12) are quite straight forward. In order to establish (4.1), we proceed as follows. In (3.1), if we set $\gamma=\delta=d$ and $c=\frac{1}{2}(a+b+1)$, we get

(4.13)
$$\int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta} \times {}_2F_1\left[\begin{matrix}a,\quad b\\ \frac{1}{2}(a+b+1)\end{matrix};xy\right]dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}\ {}_3F_2\left[\begin{matrix}a, & b, & d\\ \frac{1}{2}(a+b+1), & 2d\end{matrix};1\right].$$

We now observe that the ${}_3F_2$ appearing on the right hand side of (4.13) can be evaluated with the help of classical Watson's summation theorem (3.5) and we easily arrive at the result (4.1).

In exactly the same manner, the results (4.2) to (4.12) can be established. So we prefer to omit the details.

∎

5. SPECIAL CASES

In this section, we shall mention some of the very interesting special cases of our main results. The conditions of convergence of the integrals can be easily obtained from the respective integrals.

(i) In (4.1), if we take $b = -2n$ and replace a by $a + 2n$, where n is zero or a positive integer, we get the following result.

$$\begin{aligned}(5.1)\qquad &\int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta}\\ &\quad\times {}_2F_1\left[\begin{matrix}-2n, & a+2n\\ \multicolumn{2}{c}{\frac{1}{2}(a+1)}\end{matrix};xy\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}\,\frac{(\frac{1}{2})_n(\frac{1}{2}+\frac{1}{2}a-d)_n}{(d+\frac{1}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}.\end{aligned}$$

(ii) In (4.1), if we take $b = -2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer, we get the following elegant result.

$$\begin{aligned}(5.2)\qquad &\int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta}\\ &\quad\times {}_2F_1\left[\begin{matrix}-2n-1, & a+2n+1\\ \multicolumn{2}{c}{\frac{1}{2}(a+1)}\end{matrix};xy\right]dxdy\\ &= 0.\end{aligned}$$

(iii) In (5.1), if we take $\alpha = \beta = 1$, we get

$$\begin{aligned}(5.3)\qquad &\int_0^1\int_0^1 x^{d-1}y^{d}(1-xy)^{d-2}\,{}_2F_1\left[\begin{matrix}-2n, & a+2n\\ \multicolumn{2}{c}{\frac{1}{2}(a+1)}\end{matrix};xy\right]dxdy\\ &= \frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}\,\frac{(\frac{1}{2})_n(\frac{1}{2}+\frac{1}{2}a-d)_n}{(d+\frac{1}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}.\end{aligned}$$

(iv) In (5.2), if we take $\alpha = \beta = 1$, we get

$$\begin{aligned}(5.4)\qquad &\int_0^1\int_0^1 x^{d-1}y^{d}(1-xy)^{d-2}\,{}_2F_1\left[\begin{matrix}-2n-1, & a+2n+1\\ \multicolumn{2}{c}{\frac{1}{2}(a+1)}\end{matrix};xy\right]dxdy\\ &= 0.\end{aligned}$$

(v) In (5.1), if we take $d = 1$, we get

$$\begin{aligned}(5.5)\qquad &\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\,{}_2F_1\left[\begin{matrix}-2n, & a+2n\\ \multicolumn{2}{c}{\frac{1}{2}(a+1)}\end{matrix};xy\right]dxdy\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\frac{(\frac{1}{2})_n(\frac{1}{2}a-\frac{1}{2})_n}{(\frac{3}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}.\end{aligned}$$

(vi) In (5.2), if we take $d = 1$, we get

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1} \times {}_2F_1\left[\begin{matrix}-2n-1, \quad a+2n+1\\ \frac{1}{2}(a+1)\end{matrix}; xy\right]dxdy = 0. \tag{5.6}$$

Several other integrals of the type (5.1) to (5.6) can also be obtained from the integrals (4.2) to (4.4). These are left as an exercise to the interested researchers.

(vii) In (4.1), if we set $a = b = \frac{1}{2}$ and use the result [8, p. 473, Eq.(75)]

$${}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}; z\right] = \frac{2}{\pi}K(\sqrt{z}), \tag{5.7}$$

where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2sin^2t}}, \tag{5.8}$$

we get the following interesting integral

$$\int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta} \times K(\sqrt{xy})dxdy = \frac{\pi^{3/2}}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}\frac{\Gamma(d+\frac{1}{2})\Gamma(d)}{\Gamma^2(\frac{3}{4})\Gamma(d+\frac{1}{4})\Gamma(d+\frac{1}{4})}. \tag{5.9}$$

(viii) In (4.1), if we set $a = b = 1$ and use the result [8, p. 476, Eq.(147)]

$${}_2F_1\left[\begin{matrix}1, 1\\ \frac{3}{2}\end{matrix}; z\right] = \frac{\sin^{-1}(\sqrt{z})}{\sqrt{z(1-z)}}, \tag{5.10}$$

we get the following interesting integral

$$\int_0^1\int_0^1 x^{d-\frac{3}{2}}y^{\alpha+d-\frac{3}{2}}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta-\frac{1}{2}} \times \sin^{-1}(\sqrt{xy})dxdy = \frac{\pi}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(d+\frac{1}{2})\Gamma(d-\frac{1}{2})}{\Gamma(2d)}. \tag{5.11}$$

(ix) In (4.1), if we take $b = -a$ and use the result [8, p. 459, Eq.(83)]

$${}_2F_1\left[\begin{matrix}a, \ -a\\ \frac{1}{2}\end{matrix}; z\right] = \cos(2a\sin^{-1}(\sqrt{z})) \tag{5.12}$$

we get the following interesting integral

$$\int_0^1\int_0^1 x^{d-1}y^{\alpha+d-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{d-\alpha-\beta} \times \cos(2a\sin^{-1}(\sqrt{xy}))dxdy = 2^{1-2d}\pi^{\frac{3}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(d)\Gamma(d+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}a)\Gamma(d-\frac{1}{2}a+\frac{1}{2})\Gamma(d+\frac{1}{2}a+\frac{1}{2})}. \tag{5.13}$$

(x) In (4.6), if we take $a=b=\frac{1}{2}$ and use the result (5.7), we get the following interesting result

$$\int_0^1\int_0^1 x^{\frac{1}{2}-2c}y^{\alpha+\frac{1}{2}-2c}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times K(\sqrt{1-xy})dxdy = \frac{\pi^{\frac{3}{2}}}{2\sqrt{2}\Gamma^2(\frac{3}{4})}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(c)\Gamma(\frac{3}{2}-2c)\Gamma(\frac{3}{4}-c)}{\Gamma(1-c)\Gamma(\frac{5}{4}-c)}. \tag{5.14}$$

(xi) In (4.6), if we take $a=1, b=\frac{1}{2}$ and use the result [8, p. 473, Eq.(83)]

$$ {}_2F_1\left[\begin{matrix}1, \frac{1}{2}\\ \frac{3}{2}\end{matrix}; z\right] = \frac{1}{2\sqrt{z}}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) \tag{5.15}$$

we get the following interesting integral

$$\int_0^1\int_0^1 x^{1-2c}y^{\alpha+1-2c}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta-\frac{1}{2}} \times \log\left(\frac{1+\sqrt{1-xy}}{1-\sqrt{1-xy}}\right)dxdy = \frac{\pi}{4}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(c)\Gamma(1-c)\Gamma(2-2c)}{\Gamma^2(\frac{3}{2}-c)}. \tag{5.16}$$

(xii) In (4.6), if we take $a=\frac{3}{2}, b=\frac{1}{2}$ and use the result [8, p. 473, Eq.(91)]

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{3}{2}\\ 2\end{matrix}; z\right] = \frac{4}{\pi}D(\sqrt{z}) \tag{5.17}$$

where $D(k)$ is the complete Elliptic integral defined by by

$$D(k) = \int_0^\pi \frac{\sin^2 t\, dt}{\sqrt{1-k^2\sin^2 t}}, \tag{5.18}$$

we get the following interesting integral

$$\int_0^1\int_0^1 x^{\frac{3}{2}-2c}y^{\alpha+\frac{3}{2}-2c}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times D(\sqrt{1-xy})dxdy = \frac{1}{\sqrt{2\pi}}\Gamma^2\left(\frac{3}{4}\right)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{\Gamma(c)\Gamma(\frac{5}{2}-2c)\Gamma(\frac{5}{4}-c)}{\Gamma(2-c)(\frac{7}{4}-c)}. \tag{5.19}$$

Similarly, other results can be obtained.

Remark : For other integrals of this type, we refer [4, 6, 13]

6. CONCLUDING REMARK

In this paper, we have provided an interesting extension of the well known and useful Edwards's double integral. As an application, a new class of twelve double integrals involving hypergeometric function have been evaluated in terms of gamma function. The results have been established with the help of classical Watson's, Dixon's and Whipple's theorems for the series ${}_3F_2$ with unit argument.

Recently good deal of progress has been done in the direction of generalizing and extending almost all classical summation theorems by Rakha and Rathie [11] and Kim, et al. [7]

We conclude this paper by remarking that the generalizations of the results (4.1) to (4.12) by employing the generalized classical summation theorems are under investigations and the same will form a subsequent paper in this direction.

7. APPENDIX

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined as follows:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_p \\ b_q \end{matrix}; z\right] &= {}_pF_q\left[\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \end{aligned}$$

where $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $j \in \overline{1,q} = \{1, 2, \cdots, q\}$.

Here, in what follows, we denote $\mathbb{C}, \mathbb{R}$ and $\mathbb{Z}_0^-$ respectively be the sets of complex numbers, real numbers and non-positive integers.

The series ${}_pF_q$ converges for all $z \in \mathbb{C}$ if $p \leq q$. It diverges for all $z \neq 0$ when $p > q+1$, unless at least one numerator parameter is a negative integer in which the series becomes a polymonial i.e. contains a finite number of terms. Also, if $p = q+1$, the series converges on the unit circle $|z| = 1$ when Re$(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j) > 0$.

It is not out of place to mention here that almost all the elementary function appeared in the literature are special cases of the generalized hypergeometric function.

Also, whenever a generalized hypergeometric function reduces to gamma function, the result are important from application point of view. For a detailed theory of generalized hypergeometric function and its applications, we refer standard texts [1, 9, 12].

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