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SOME FIXED POINT RESULTS IN PARTIAL S-METRIC SPACES

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ABSTRACT. We introduce in this article a new class of generalized metric spaces, called partial S-metric spaces. In addition, we also give some interesting results on fixed points in the partial S-metric spaces and some applications.

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1. INTRODUCTION

A new structure of generalized metric spaces (Ψ, d) have been introduced by Mustafa and Sims [13]. They are called G-metric spaces. Note that the metric space (Ψ, d) is considered as a generalization of metric space. Many fixed point results for various mappings in this new structure have been developed. Many researchers proved many interesting results in fixed point theory in this metric space; we can refer the reader to consider [2, 4, 14, 23, 16]. Also, various problems of fixed point problems of contractive mappings in metric spaces endowed with a partially order have been studied [3, 5, 9, 10].

Different generalizations of metric spaces were considered by many researchers. For example, the concepts of 2-matric spaces and D-metric spaces, respectively, were introduce by Gähler [8] and Dhage [6], but some authors pointed out that these attempts are not valid (see [11, 19, 17, 18, 15]). A modification of the definition of D-metric introduced recentlyby Dhage [6], called the D^*-metric. For more results, we refer the reader to consider [25, 28, 29].

In this article, we introduce the concept of partial S-metric spaces. Many fixed point theorems in ordered partial S^*-metric Spaces and common fixed point theorem for self-mapping on complete S^*-metric spaces are investigated.

We begin with the following definitions.

Definition 1.1. Let Ψ be a nonempty set and $\mathcal{G}$ be a nonnegative function on the set $\Psi \times \Psi \times \Psi$ that satisfies the following condition: for all $\theta, \upsilon, \omega, \alpha \in \Psi$,

($\mathcal{G}1$) $\mathcal{G}(\theta, \upsilon, \omega) = 0$ if $\theta = \upsilon = \omega$,
($\mathcal{G}2$) $0 < \mathcal{G}(\theta, \theta, \upsilon)$ for all $\theta, \upsilon \in \Psi$ with $\theta \neq \upsilon$,
($\mathcal{G}3$) $\mathcal{G}(\theta, \theta, \upsilon) \leq \mathcal{G}(\theta, \upsilon, \omega)$ for all $\theta, \upsilon, \omega \in \Psi$ with $\theta \neq \upsilon$,
($\mathcal{G}4$) $\mathcal{G}(\theta, \upsilon, \omega) = \mathcal{G}(\theta, \omega, \upsilon) = \mathcal{G}(\upsilon, \omega, \theta) = \cdots$,
($\mathcal{G}5$) $\mathcal{G}(\theta, \upsilon, \omega) \leq \mathcal{G}(\theta, \alpha, \alpha) + \mathcal{G}(\alpha, \upsilon, \omega)$ for all $\theta, \upsilon, \omega, \alpha \in \Psi$.

The function $\mathcal{G}$ is called a *generalized metric* or a G-metric on Ψ and the pair $(\Psi, \mathcal{G})$ is called a G-metric space.

Definition 1.2. A G-metric space $(\Psi, \mathcal{G})$ is symmetric if

(G6) $\mathcal{G}(\theta, \upsilon, \upsilon) = \mathcal{G}(\theta, \theta, \upsilon)$ for all $\theta, \upsilon \in \Psi$.

We can find some examples and basic properties of G-metric spaces in Mustafa and Sims [13].

Definition 1.3. Let Ψ be a nonempty set. A generalized metric (or D^*-metric) on Ψ is a function: $D^* : \Psi^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $\theta, \upsilon, \omega, \alpha \in \Psi$.

(1) $D^*(\theta, \upsilon, \omega) \geq 0$,
(2) $D^*(\theta, \upsilon, \omega) = 0$ if and only if $\theta = \upsilon = \omega$,
(3) $D^*(\theta, \upsilon, \omega) = D^*(p\{\theta, \upsilon, \upsilon\})$,(symmetry) where p is a permutation function,
(4) $D^*(\theta, \upsilon, \omega) \leq D^*(\theta, \upsilon, \alpha) + D^*(\alpha, \omega, \omega)$.

The pair (Ψ, D^*) is called a generalized metric (or D^*-metric) space.

Remark 1.1. It is easy to see that every symmetric G-metric is a D^*-metric.

Proof. For all $\theta, \upsilon, \omega, \alpha \in \Psi$ we have

$$\begin{aligned} \mathcal{G}(\theta, \upsilon, \omega) &= \mathcal{G}(\omega, \theta, \upsilon) \\ &\leq \mathcal{G}(\omega, \alpha, \alpha) + \mathcal{G}(\alpha, \theta, \upsilon) = \mathcal{G}(\theta, \upsilon, \alpha) + \mathcal{G}(\alpha, \omega, \omega). \end{aligned}$$

∎

Remark 1.2. ([30]) In a D^*-metric space, $D^*(\theta, \theta, \upsilon) = D^*(\theta, \upsilon, \upsilon)$.

For more details of D^*-metric see [31, 33]

Note that the converse may not hold as shown in the following example.

Example 1.1. *Let $\Psi = \mathbb{R}$ and define the function D^* such that*

$$D^*(\theta, \upsilon, \omega) = |\theta + y - 2\omega| + |\theta + \omega - 2\upsilon| + |\upsilon + \omega - 2\theta|.$$

Then one can easily verify that $(\mathbb{R}, D^*)$ is a D^*-metric, but it is not G-metric. For, if set $\theta = 5, \upsilon = -5$ and $\omega = 0$ then $\mathcal{G}(\theta, \theta, \upsilon) \leq \mathcal{G}(\theta, \upsilon, \omega)$ is not hold.

Now, we recall the concept of S-metric spaces, which modifies the D^*-metric and G-metric spaces as follows:

Definition 1.4. Let Ψ be a nonempty set. An S-*metric* on Ψ is a function $S : \Psi^3 \to [0, \infty)$ that satisfies the following conditions, for each $\theta, \upsilon, \omega, \alpha \in \Psi$,

(1) $S(\theta, \upsilon, \omega) \geq 0$,
(2) $S(\theta, \upsilon, \omega) = 0$ if and only if $\theta = \upsilon = \omega$,
(3) $S(\theta, \upsilon, \omega) \leq S(\theta, \theta, \alpha) + S(\upsilon, \upsilon, \alpha) + S(\omega, \omega, \alpha)$.

The pair (Ψ, S) is called an S-*metric space*.

Remark 1.3. It is clear that every D^*-metric is an S-metric.

Namely, for all $\theta, \upsilon, \omega, \alpha \in \Psi$ we have

$$\begin{aligned} D^*(\theta, \upsilon, \omega) &\leq D^*(\theta, \upsilon, \alpha) + D^*(\alpha, \omega, \omega) = D^*(\alpha, y, \theta) + D^*(\omega, \omega, \alpha) \\ &\leq D^*(\alpha, y, \alpha) + D^*(\alpha, \theta, \theta) + D^*(\omega, \omega, \alpha) \\ &= D^*(\alpha, \alpha, \upsilon) + D^*(\alpha, \theta, \theta) + D^*(\omega, \omega, \alpha) \\ &= D^*(\theta, \theta, \alpha) + D^*(\upsilon, \upsilon, \alpha) + D^*(\omega, \omega, \alpha). \end{aligned}$$

Note that it is not always true that every S-metric is a D^*-metric.

Example 1.2. *Let $\Psi = \mathbb{R}^n$ and $||\cdot||$ a norm on Ψ, then $S(\theta, \upsilon, \omega) = ||\upsilon + \omega - 2\theta|| + ||\upsilon - \omega||$ is an S-metric on Ψ, but it is not D^*-metric,since it is not symmetry.*

Lemma 1.1. [26, 27, 32] *In an S-metric space, we have $S(\theta, \theta, \upsilon) = S(\upsilon, \upsilon, \theta)$.*

In this section we recall the concept of partial D^*-metric space.

Definition 1.5. [24] A partial D^*-metric on a nonempty set Ψ is a function $p^* : \Psi \times \Psi \times \Psi \to \mathbb{R}^+$ such that for all $\theta, \upsilon, \omega, \alpha \in \Psi$:

(p_1) $\theta = \upsilon = \omega$ if and only if $p^*(\theta, \theta, \theta) = p^*(\theta, \upsilon, \omega) = p^*(\upsilon, \upsilon, \upsilon) = p^*(\omega, \omega, \omega)$,
(p_2) $p^*(\theta, \theta, \theta) \leq p^*(\theta, \upsilon, \omega)$,
(p_3) $p^*(\theta, \upsilon, \omega) = p^*(p\{\theta, \upsilon, \upsilon\})$,(symmetry) where p is a permutation function,
(p_4) $p^*(\theta, \upsilon, \omega) \leq p^*(\theta, \upsilon, \alpha) + p^*(\alpha, \omega, \omega) - p^*(\alpha, \alpha, \alpha)$.

A partial D^*-metric space is a pair (Ψ, p^*) such that Ψ is a nonempty set and p^* is a partial D^*-metric on Ψ. It is clear that, if $p^*(\theta, \upsilon, \omega) = 0$, then from ($p_1$) and ($p_2$) $\theta = \upsilon = \omega$. But if $\theta = \upsilon = \omega$, $p^*(\theta, \upsilon, \omega)$ may not be 0. We can consider the partial D^*-metric space to be the pair $(\mathbb{R}^+, p^*)$, where $p^*(\theta, \upsilon, \omega) = \max\{\theta, \upsilon, \upsilon\}$ for all $\theta, \upsilon, \upsilon \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p^*)$ is a D^*-metric space.

It is easy to see that every D^*-metric is a partial D^*-metric, but the converse is not necessarily hold.

In the following examples a partial D^*-metric that fails to satisfy properties D^*-metric.

Example 1.3. *Assume that p^* is a nonnegative mapping defined on the set $\Psi = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. Define the mapping*

$$p^*(\theta, \upsilon, \omega) = |\theta - \upsilon| + |\upsilon - \omega| + |\theta - \omega| + \max\{\theta, \upsilon, \upsilon\}.$$

Then (Ψ, p^*) is a partial D^*-metric, but it is not D^*-metric.

Example 1.4. *Let (Ψ, p) be a partial metric space and $p^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a mapping defined as follows:*

$$p^*(\theta, \upsilon, \omega) = p(\theta, \upsilon) + p(\theta, \omega) + p(\upsilon, \omega) - p(\theta, \theta) - p(\upsilon, \upsilon) - p(\omega, \omega).$$

Then clearly p^* is a partial D^*-metric.

Remark 1.4. [24] Let (Ψ, p^*) be a partial D^*-metric space. Then $p^*(\theta, \theta, \upsilon) = p^*(\theta, \upsilon, \upsilon)$.

Lemma 1.2. *Let (Ψ, p^*) be a partial D^*-metric space. If define $p(\theta, \upsilon) = p^*(\theta, \upsilon, \upsilon)$, then (Ψ, p) is a partial metric space*

Proof. (p_1) $\theta = \upsilon$ if and anly if $p^*(\theta, \theta, \theta) = p^*(\theta, \upsilon, \upsilon) = p(\upsilon, \upsilon, \upsilon)$ implies that $p(\theta, \theta) = p(\theta, \upsilon) = p(\upsilon, \upsilon)$,

(p_2) $p^*(\theta, \theta, \theta) \leq p^*(\theta, \upsilon, \upsilon)$ implies that $p(\theta, \theta) \leq p(\theta, \upsilon)$,

(p_3) $p^*(\theta, \upsilon, \upsilon) = p^*(\upsilon, \theta, \theta)$ implies that $p(\theta, \upsilon) = p(\upsilon, \theta)$,

(p_4) $p^*(\upsilon, \upsilon, \theta) \leq p^*(\upsilon, \upsilon, \omega) + p^*(\omega, \theta, \theta) - p^*(\omega, \omega, \omega)$ implies that $p(\theta, \upsilon) \leq p(\upsilon, \omega) + p(\omega, \theta) - p(\omega, \omega)$.

∎

2. PARTIAL S-METRIC SPACE

we introduce in this section the notion partial S-metric space and we prove some properties of this space.

Definition 2.1. A partial S-metric on a nonempty set Ψ is a function $S^* : \Psi \times \Psi \times \Psi \to \mathbb{R}^+$ such that for all $\theta, \upsilon, \omega, \alpha \in \Psi$:

(s_1) $\theta = \upsilon = \omega$ if and only if $S^*(\theta, \theta, \theta) = S^*(\theta, \upsilon, \omega) = S^*(\upsilon, \upsilon, \upsilon) = S^*(\omega, \omega, \omega)$,

(s_2) $S^*(\theta, \theta, \theta) \leq S^*(\theta, \theta, \upsilon)$,

(s_3) $S^*(\theta, \upsilon, \omega) \leq S^*(\theta, \theta, \alpha) + S^*(\upsilon, \upsilon, \alpha) + S^*(\omega, \omega, \alpha) - 2S^*(\alpha, \alpha, \alpha)$.

A partial S-metric space is a pair (Ψ, S^*) such that Ψ is a nonempty set and S^* is a partial S-metric on Ψ. A basic example of a partial S-metric space is the pair $(\mathbb{R}^+, S^*)$, where $S^*(\theta, \upsilon, \omega) = \max\{\theta, \upsilon, \upsilon\}$ for all $\theta, \upsilon, \omega, \alpha \in \Psi \in \mathbb{R}^+$.

Remark 2.1. It is easy to see that every partial D^*-metric is a partial S-metric. Namely, for all $\theta, \upsilon, \omega, \alpha \in \Psi$ we have

$$\begin{aligned}
p^*(\theta, \upsilon, \omega) &\leq p^*(\theta, \upsilon, \alpha) + p^*(\alpha, \omega, \omega) - p^*(\alpha, \alpha, \alpha) \\
&= p^*(\alpha, y, \theta) + p^*(\omega, \omega, \alpha) - p^*(\alpha, \alpha, \alpha) \\
&\leq p^*(\alpha, y, \alpha) + p^*(\alpha, \theta, \theta) + p^*(\omega, \omega, \alpha) - p^*(\alpha, \alpha, \alpha) - p^*(\alpha, \alpha, \alpha) \\
&= p^*(\alpha, \alpha, \upsilon) + p^*(\alpha, \theta, \theta) + p^*(\omega, \omega, \alpha) - 2p^*(\alpha, \alpha, \alpha) \\
&= p^*(\theta, \theta, \alpha) + p^*(\upsilon, \upsilon, \alpha) + p^*(\omega, \omega, \alpha) - 2p^*(\alpha, \alpha, \alpha).
\end{aligned}$$

But in general the converse is not hold (see the following example).

It is easy to see that every S-metric is a partial S-metric, but the converse it is not necessarily hold.

In general example of a partial S-metric space is the pair $(\mathbb{R}^+, S^*)$, where $S^*(\theta, \upsilon, \omega) = \max\{a\theta + b\upsilon, (\alpha + b)\omega\}$ for all $\theta, \upsilon, \omega \in \mathbb{R}^+$ and $a, b > 0$. This partial S-metric is not only $S-$metric but also it is not partial D^*-metric.

In the following example a partial S-metric fails to satisfy properties S-metric.

Example 2.1. *(a) Assume that (Ψ, S) is an S-metric space and a be a positive real number and let $S^* : \Psi \times \Psi \times \Psi \longrightarrow \Psi$ be a mapping defined as follows:*

$$S^*(\theta, \upsilon, \omega) = S(\theta, \upsilon, \omega) + a.$$

It is clear that (Ψ, S) is a partial S-metric, but it is not S-metric.

(b) Let $(\Psi, ||, ||)$ be a norm space and let $S^* : \Psi \times \Psi \times \Psi \longrightarrow \Psi$ be a mapping defined as follows:

$$S^*(\theta, \upsilon, \omega) = ||\theta - \omega|| + ||\upsilon - \omega|| + \max\{||\theta||, ||\upsilon||, ||\omega||\}.$$

Then it is easy to see that it is a partial S-metric, but it is not S-metric.

Lemma 2.1. *For partial S-metric S^*, we have*

(1) $S^*(\theta, \theta, \upsilon) = S^*(\upsilon, \upsilon, \theta)$.
(2) *If* $S^*(\theta, \theta, \upsilon) = 0$ *then* $\theta = \upsilon$.

Proof. (1) (i)

$$\begin{aligned} S^*(\theta, \theta, \upsilon) &\leq S^*(\theta, \theta, \theta) + S^*(\theta, \theta, \theta) + S^*(\upsilon, \upsilon, \theta) - 2S^*(\theta, \theta, \theta) \\ &= S^*(\upsilon, \upsilon, \theta), \end{aligned}$$

and similarly

(ii)

$$\begin{aligned} S^*(\upsilon, \upsilon, \theta) &\leq S^*(\upsilon, \upsilon, \upsilon) + S^*(\upsilon, \upsilon, \upsilon) + S^*(\theta, \theta, \upsilon) - 2S^*(\upsilon, \upsilon, \upsilon) \\ &= S^*(\theta, \theta, \upsilon). \end{aligned}$$

Hence by (i),(ii) we get $S^*(\theta, \theta, \upsilon) = S^*(\upsilon, \upsilon, \theta)$.

(2) (i) $S^*(\theta, \theta, \theta) \leq S^*(\theta, \theta, \upsilon) = 0$ and similarly
(ii) $S^*(\upsilon, \upsilon, \upsilon) \leq S^*(\upsilon, \upsilon, \theta) = S^*(\theta, \theta, \upsilon) = 0$. Hence by (i),(ii) we get $S^*(\theta, \theta, \upsilon) = S^*(\theta, \theta, \theta) = S^*(\upsilon, \upsilon, \upsilon) = 0$, that is $\theta = \upsilon$.

∎

Lemma 2.2. *Let (Ψ, p) be a partial metric space and S^* be a nonnegative mapping define on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that*

$$S^*(\theta, \upsilon, \omega) = \max\{p(\theta, \upsilon), p(\theta, \omega), p(\upsilon, \omega)\}.$$

Then S^* is a partial S-metric.

Proof. (s_1) It is easy to see that: $\theta = \upsilon = \omega \Longleftrightarrow S^*(\theta, \theta, \theta) = S^*(\theta, \upsilon, \omega) = S^*(\upsilon, \upsilon, \upsilon) = S^*(\omega, \omega, \omega)$.

Since $p(\theta, \theta) \leq p(\theta, \upsilon)$ hence

(s_2) $S^*(\theta, \theta, \theta) = p(\theta, \theta) \leq p(\theta, \upsilon) = S^*(\theta, \theta, \upsilon)$.

Since

$$\begin{aligned} p(\theta, \upsilon) &\leq p(\theta, \alpha) + p(\alpha, \upsilon) - p(\alpha, \alpha) \\ &< p(\theta, \alpha) + p(y, \alpha) + p(\omega, \alpha) - 2p(\alpha, \alpha), \end{aligned}$$

$$p(\theta, \omega) < p(\theta, \alpha) + p(y, \alpha) + p(\omega, \alpha) - 2p(\alpha, \alpha),$$

and

$$p(\upsilon, \omega) < p(\theta, \alpha) + p(\upsilon, \alpha) + p(\omega, \alpha) - 2p(\alpha, \alpha),$$

we obtain

(s_3)

$$\begin{aligned} S^*(\theta,\upsilon,\omega) &= \max\{p(\theta,\upsilon),p(\theta,\omega),p(y,\omega)\} \\ &\leq p(\theta,\alpha)+p(y,\alpha)+p(\omega,\alpha)-2p(\alpha,\alpha) \\ &= S^*(\theta,\theta,\alpha)+S^*(\upsilon,\upsilon,\alpha)+S^*(\omega,\omega,\alpha)-2S^*(\alpha,\alpha,\alpha). \end{aligned}$$

∎

Lemma 2.3. *Let (Ψ,p) be a complete partial metric space and $S^*:\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be a mapping defined as follows*

$$S^*(\theta,\upsilon,\omega)=\max\{p(\theta,\upsilon),p(\theta,\omega),p(y,\omega)\}.$$

Then (Ψ,S^*) is a complete partial S-metric.

Proof. Let $\{\theta_n\}$ be a Cauchy sequence in (Ψ,S^*), since (Ψ,p) is complete space, hence there exists $\theta\in\Psi$ such that $p(\theta_n,\theta)\longrightarrow p(\theta,\theta)$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta) &= \lim_{n\to\infty}\max\{p(\theta_n,\theta_n),p(\theta_n,\theta),p(\theta_n,\theta)\} \\ &= \lim_{n\to\infty} p(\theta_n,\theta)=p(\theta,\theta)=S^*(\theta,\theta,\theta)\}. \end{aligned}$$

That is $\{\theta_n\}$ is a Cauchy sequence in (Ψ,S^*). ∎

Definition 2.2. Let (Ψ,S^*) be a partial S-metric space. For $r>0$ define

$$B_{S^*}(\theta,r)=\{y\in\Psi : S^*(\theta,\theta,\upsilon)<S^*(\theta,\theta,\theta)+r\}$$

Lemma 2.4. *Let (Ψ,S^*) be a partial S-metric space. If $r>0$, then ball $B_{S^*}(\theta,r)$ with center $\theta\in\Psi$ and radius r is open ball.*

Proof. Let $\upsilon\in B_{S^*}(\theta,r)$, hence $S^*(\theta,\theta,\upsilon)<S^*(\theta,\theta,\theta)+r$. Set $\frac{S^*(\theta,\theta,\theta)-S^*(\theta,\theta,\upsilon)+r}{2}=\delta$. Let $z\in B_{S^*}(y,\delta)$, hence

$$S^*(\upsilon,\upsilon,\omega) < S^*(\upsilon,\upsilon,\upsilon)+\delta=S^*(\upsilon,\upsilon,\upsilon)+\frac{S^*(\theta,\theta,\theta)-S^*(\theta,\theta,\upsilon)+r}{2}.$$

By triangular inequality we have

$$\begin{aligned} S^*(\theta,\theta,\omega)=S^*(\omega,\omega,\theta) &\leq 2S^*(\omega,\omega,\upsilon)+S^*(\theta,\theta,\upsilon)-2S^*(\upsilon,\upsilon,\upsilon) \\ &= 2S^*(\upsilon,\upsilon,\omega)+S^*(\theta,\theta,\upsilon)-2S^*(\upsilon,\upsilon,\upsilon) \\ &< 2S^*(\upsilon,\upsilon,\upsilon)+S^*(\theta,\theta,\theta)+r-2S^*(\upsilon,\upsilon,\upsilon) \\ &= S^*(\theta,\theta,\theta)+r. \end{aligned}$$

Hence $B_{S^*}(\upsilon,\delta)\subseteq B_{S^*}(\theta,r)$. That is the ball $B_{S^*}(\theta,r)$ is an open ball. ∎

Each partial S-metric S^* on Ψ generates a topology τ_{S^*} on Ψ which has as a base the family of open S^*-balls $\{B_{S^*}(\theta,\epsilon):\theta\in\Psi,\ \epsilon>0\}$.

Let (Ψ,S^*) be a partial S-metric space and $A\subset\Psi$.

(1) If for every $\theta\in A$ there exists $r>0$ such that $B_{p^*}(\theta,r)\subset A$, then subset A is called an open subset of Ψ.

(2) a sequence $\{\theta_n\}$ in a partial S-metric space (Ψ,S^*) converges to Ψ if and only if $S^*(\theta,\theta,\theta)=\lim_{n\to\infty}S^*(\theta_n,\theta_n,\theta)=\lim_{n\to\infty}S^*(\theta_n,\theta_n,\theta_n)$. That is for each $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$|S^*(\theta_n,\theta_n,\theta)-S^*(\theta,\theta,\theta)|<\epsilon \text{ and } |S^*(\theta_n,\theta_n,\theta_n)-S^*(\theta,\theta,\theta)|<\epsilon,$$

for all $n \geq n_0$. In these cases we conclude for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(\theta_n, \theta_n, \theta_n) - S^*(\theta_n, \theta_n, \theta)| < \epsilon,$$

for all $n \geq n_0$.

(3) A sequence $\{\theta_n\}$ in a partial S-metric space (Ψ, S^*) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} S^*(\theta_n, \theta_n, \theta_m)$.

Let τ_{S^*} be the set of all open subsets of Ψ , then τ_{S^*} is a topology on Ψ (induced by the partial S-metric S^*).

A partial S-metric space (Ψ, S^*) is said to be complete if every Cauchy sequence $\{\theta_n\}$ in Ψ converges, with respect to τ_{S^*}, to a point $\theta \in \Psi$.

Lemma 2.5. *Let (Ψ, S^*) be a partial S-metric space. If sequence $\{\theta_n\}$ in Ψ converges to θ, then θ is unique.*

Proof. Let $\{\theta_n\}$ converges to θ and υ, then we have

$$\lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_n) = \lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta) = S^*(\theta, \theta, \theta)$$

and

$$\lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_n) = \lim_{n\to\infty} S^*(\theta_n, \theta_n, \upsilon) = S^*(\upsilon, \upsilon, \upsilon).$$

Then by third condition partial S-metric we have:

$$\begin{array}{rcl} S^*(\theta, \theta, \upsilon) & \leq & 2S^*(\theta, \theta, \theta_n) + S^*(\upsilon, \upsilon, \theta_n) - 2S^*(\theta_n, \theta_n, \theta_n) \\ & \leq & 2(S^*(\theta_n, \theta_n, \theta) - S^*(\theta_n, \theta_n, \theta_n)) \\ & + & S^*(\theta_n, \theta_n, \upsilon) - S^*(\upsilon, \upsilon, \upsilon) + S^*(\upsilon, \upsilon, \upsilon). \end{array}$$

By taking limit as $n \to \infty$ we have $S^*(\theta, \theta, \upsilon) \leq S^*(\upsilon, \upsilon, \upsilon)$. Hence, $S^*(\theta, \theta, \upsilon) = S^*(\upsilon, \upsilon, \upsilon)$. Similarly, we can show that $S^*(\theta, \theta, \upsilon) \leq S^*(\theta, \theta, \theta)$. That is $S^*(\theta, \theta, \upsilon) = S^*(\theta, \theta, \theta) = S^*(\upsilon, \upsilon, \upsilon)$. Therefore, $\theta = \upsilon$. ∎

Lemma 2.6. *Let (Ψ, S^*) be a partial S-metric space. Then the convergent sequence $\{\theta_n\}$ in Ψ is Cauchy.*

Proof. Let $\{\theta_n\}$ converges to θ, that is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(\theta_n, \theta_n, \theta) - S^*(\theta, \theta, \theta)| < \epsilon \; \forall n \geq n_0,$$

$$|S^*(\theta_n, \theta_n, \theta_n) - S^*(\theta, \theta, \theta)| < \epsilon \; \forall n \geq n_0,$$

and

$$|S^*(\theta_n, \theta_n, \theta_n) - S^*(\theta_n, \theta_n, \theta)| < \epsilon \; \forall n \geq n_0.$$

Then we have:

$$\begin{array}{rcl} S^*(\theta_n, \theta_n, \theta_m) & \leq & 2S^*(\theta_n, \theta_n, \theta) + S^*(\theta_m, \theta_m, \theta) - 2S^*(\theta, \theta, \theta) \\ & < & 2\epsilon + \epsilon + S^*(\theta, \theta, \theta). \end{array}$$

Similarly,

$$\begin{array}{rcl} S^*(\theta, \theta, \theta) & \leq & 3S^*(\theta, \theta, \theta_n) - 2S^*(\theta_n, \theta_n, \theta_n) \\ & = & 2(S^*(\theta_n, \theta_n, \theta) - S^*(\theta_n, \theta_n, \theta_n)) + S^*(\theta, \theta, \theta_n) \\ & \leq & 2(S^*(\theta_n, \theta_n, \theta) - S^*(\theta_n, \theta_n, \theta_n)) \\ & + & 2S^*(\theta, \theta, \theta_m) + S^*(\theta_n, \theta_n, \theta_m) - 2S^*(\theta_m, \theta_m, \theta_m). \end{array}$$

Hence,

$$S^*(\theta, \theta, \theta) < 2\epsilon + 2\epsilon + S^*(\theta_n, \theta_n, \theta_m).$$

By above inequalities we have

$$|S^*(\theta_n, \theta_n, \theta_m) - S^*(\theta, \theta, \theta)| < 4\epsilon.$$

That is

$$\lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_m) = S^*(\theta, \theta, \theta).$$

∎

Lemma 2.7. *Let* (Ψ, S^*) *be a partial* S*-metric space. If there exist sequences* $\{\theta_n\}$ *and* $\{\upsilon_n\}$ *such that* $\lim_{n\to\infty} \theta_n = \theta$ *and* $\lim_{m\to\infty} \upsilon_m = \upsilon$*, then*

$$\lim_{m,n\to\infty} S^*(\theta_n, \theta_n, \upsilon_m) = S^*(\theta, \theta, \upsilon).$$

Proof. Since $\lim_{n\to\infty} \theta_n = \Psi$ then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(\theta_n, \theta_n, \theta) - S^*(\theta, \theta, \theta)| < \epsilon, |S^*(\theta_n, \theta_n, \theta_n) - S^*(\theta, \theta, \theta)| < \epsilon$$

and

$$|S^*(\theta_n, \theta_n, \theta_n) - S^*(\theta_n, \theta_n, \theta)| < \epsilon,$$

for all $n \geq n_0$. Similarly, since $\lim_{m\to\infty} \upsilon_m = \upsilon$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(\upsilon_m, \upsilon_m, \upsilon) - S^*(\upsilon, \upsilon, \upsilon)| < \epsilon, |S^*(\upsilon_m, \upsilon_m, \upsilon_m) - S^*(\upsilon, \upsilon, \upsilon)| < \epsilon$$

and

$$|S^*(\upsilon_m, \upsilon_m, \upsilon_m) - S^*(\upsilon_m, \upsilon_m, \upsilon)| < \epsilon,$$

for all $m \geq n_0$.

Then we have

$$\begin{aligned} S^*(\theta_n, \theta_n, \upsilon_m) &\leq 2S^*(\theta_n, \theta_n, \theta) + S^*(\upsilon_m, \upsilon_m, \theta) - 2S^*(\theta, \theta, \theta) \\ &\leq 2(S^*(\theta_n, \theta_n, \theta) - S^*(\theta, \theta, \theta)) + 2S^*(\upsilon_m, \upsilon_m, \upsilon) \\ &+ S^*(\theta, \theta, \upsilon) - S^*(\upsilon, \upsilon, \upsilon) \\ &< 2\epsilon + 2\epsilon + S^*(\theta, \theta, \upsilon), \end{aligned}$$

hence we obtain

$$S^*(\theta_n, \theta_n, \upsilon_m) - S^*(\theta, \theta, \upsilon) < 4\epsilon. \tag{2.1}$$

On the other hand, we get

$$\begin{aligned} S^*(\theta, \theta, \upsilon) &\leq 2S^*(\theta, \theta, \theta_n) + S^*(\upsilon, \upsilon, \theta_n) - 2S^*(\theta_n, \theta_n, \theta_n) \\ &\leq 2(S^*(\theta_n, \theta_n, \theta) - S^*(\theta_n, \theta_n, \theta_n)) + 2S^*(\upsilon, \upsilon, \upsilon_m) \\ &- S^*(\upsilon_m, \upsilon_m, \upsilon_m) + S^*(\theta_n, \theta_n, \upsilon_m) \\ &< 4\epsilon + S^*(\theta_n, \theta_n, \upsilon_m), \end{aligned}$$

that is

$$S^*(\theta, \theta, \upsilon) - S^*(\theta_n, \theta_n, \upsilon_m) < 4\varepsilon. \tag{2.2}$$

Therefore by relations (2.1) and (2.2) we have

$$|S^*(\theta_n, \theta_n, \upsilon_m) - S^*(\theta, \theta, \upsilon)| < 4\epsilon,$$

that is

$$\lim_{m,n\to\infty} S^*(\theta_n, \theta_n, \upsilon_m) = S^*(\theta, \theta, \upsilon).$$

∎

Lemma 2.8. *Let (Ψ, S^*) be a partial S-metric space then*

$$\begin{aligned} S^s(\theta, \upsilon, \omega) &= S^*(\theta, \theta, \upsilon) + S^*(\upsilon, \upsilon, \omega) + S^*(\omega, \omega, \theta) - S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon) \\ &- S^*(\omega, \omega, \omega), \end{aligned}$$

is an S_b-metric on Ψ for $b \geq 2$.

Proof. First we show that $S^s(\theta, \theta, \upsilon) = S^s(\upsilon, \upsilon, \theta)$. Since,

$$\begin{aligned} S^s(\theta, \theta, \upsilon) &= S^*(\theta, \theta, \theta) + S^*(\theta, \theta, \upsilon) + S^*(\upsilon, \upsilon, \theta) - 2S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon) \\ &= 2S^*(\theta, \theta, \upsilon) - S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon). \end{aligned}$$

Similarly, we can show that

$$S^s(\upsilon, \upsilon, \theta) = 2S^*(\theta, \theta, \upsilon) - S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon).$$

Therefore, $S^s(\theta, \theta, \upsilon) = S^s(\upsilon, \upsilon, \theta)$. Also, always we have that

$$S^*(\theta, \theta, \upsilon) - S^*(\theta, \theta, \theta) \leq S^s(\theta, \theta, \upsilon).$$

(i) If $S^s(\theta, \upsilon, \omega) = 0$ then it follows that $S^*(\theta, \upsilon, \omega) = S^*(\theta, \theta, \theta) = S^*(\upsilon, \upsilon, \upsilon) = S^*(\omega, \omega, \omega)$. That is $\theta = \upsilon = \omega$. Conversely, if $\theta = \upsilon = \omega$ then we have $S^s(\theta, \upsilon, \omega) = 0$.

(ii) Since

$$S^*(\theta, \upsilon, \omega) \leq S^*(\theta, \theta, \alpha) + S^*(\upsilon, \upsilon, \alpha) + S^*(\omega, \omega, \alpha) - 2S^*(\alpha, \alpha, \alpha).$$

Therefore,

$$\begin{aligned} S^s(\theta, \upsilon, \omega) &= S^*(\theta, \theta, \upsilon) + S^*(\upsilon, \upsilon, \omega) + S^*(\omega, \omega, \theta) \\ &- S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon) - S^*(\omega, \omega, \omega) \\ &\leq 2S^*(\theta, \theta, \alpha) - 2S^*(\alpha, \alpha, \alpha) + S^*(\upsilon, \upsilon, \alpha) \\ &+ 2S^*(\upsilon, \upsilon, \alpha) - 2S^*(\alpha, \alpha, \alpha) + S^*(\omega, \omega, \alpha) \\ &+ 2S^*(\omega, \omega, \alpha) - 2S^*(\alpha, \alpha, \alpha) + S^*(\theta, \theta, \alpha) \\ &- S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon) - S^*(\omega, \omega, \omega) \\ &= 3S^*(\alpha, \alpha, \theta) - 2S^*(\alpha, \alpha, \alpha) - S^*(\theta, \theta, \theta) \\ &+ 3S^*(\alpha, \alpha, \upsilon) - 2S^*(\alpha, \alpha, \alpha) - S^*(\upsilon, \upsilon, \upsilon) \\ &+ 3S^*(\alpha, \alpha, \omega) - 2S^*(\alpha, \alpha, \alpha) - S^*(\omega, \omega, \omega) \\ &\leq 2[2S^*(\alpha, \alpha, \theta) - S^*(\alpha, \alpha, \alpha) - S^*(\theta, \theta, \theta)] \\ &+ 2[2S^*(\alpha, \alpha, \upsilon) - S^*(\alpha, \alpha, \alpha) - S^*(\upsilon, \upsilon, \upsilon)] \\ &+ 2[2S^*(\alpha, \alpha, \omega) - S^*(\alpha, \alpha, \alpha) - S^*(\omega, \omega, \omega)] \\ &= 2[S^s(\alpha, \alpha, \theta) + S^s(\alpha, \alpha, \upsilon) + S^s(\alpha, \alpha, \omega)] \\ &= 2[S^s(\theta, \theta, \alpha) + S^s(\upsilon, \upsilon, \alpha) + S^s(\omega, \omega, \alpha)]. \end{aligned}$$

∎

The following lemma plays an important role to give fixed point results on a partial S-metric space.

Lemma 2.9. *Let (Ψ, S^*) be a partial S-metric space and $b \geq 2$.*

(a) $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^*) if and only if it is a Cauchy sequence in the S_b-metric space (Ψ, S^s).

(b) A partial S-metric space (Ψ, S^*) is complete if and only if the S_b-metric space (Ψ, S^s) is complete. Furthermore, $\lim_{n\to\infty} S(\theta_n, \theta_n, \theta) = 0$ if and only if

$$S^*(\theta, \theta, \theta) = \lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta) = \lim_{n,m\to\infty} S^*(\theta_n, \theta_n, \theta_m).$$

Proof. First we show that every Cauchy sequence in (Ψ, S^*) is a Cauchy sequence in (Ψ, S^s). To this end let $\{\theta_n\}$ be a Cauchy sequence in (Ψ, S^*). Then there exists $\lim_{n,m\to\infty} S^*(\theta_n,\theta_n,\theta_m) = \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_n)$. Since

$$S^s(\theta_n,\theta_n,\theta_m) = 2S^*(\theta_n,\theta_n,\theta_m) - S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_m,\theta_m,\theta_m).$$

Hence, we have

$$\begin{aligned}&\lim_{n,m\to\infty} S^s(\theta_n,\theta_n,\theta_m)\\ =\ & 2\lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_m) - \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_n) - \lim_{m\to\infty} S^*(\theta_m,\theta_m,\theta_m) = 0.\end{aligned}$$

We conclude that $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^s). Next we prove that completeness of (Ψ, S^s) implies completeness of (Ψ, S^*). Indeed, if $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^*) then it is also a Cauchy sequence in (Ψ, S^s). Since the S_b-metric space (Ψ, S^s) is complete we deduce that there exists $\upsilon \in \Psi$ such that $\lim_{n\to\infty} S^s(\theta_n,\theta_n,\upsilon) = 0$. Since,

$$S^s(\theta_n,\theta_n,\upsilon) = 2S^*(\theta_n,\theta_n,\upsilon) - S^*(\upsilon,\upsilon,\upsilon) - S^*(\theta_n,\theta_n,\theta_n).$$

Also,

$$0 \le S^*(\theta_n,\theta_n,\upsilon) - S^*(\upsilon,\upsilon,\upsilon) \le S^s(\theta_n,\theta_n,\upsilon),$$

and

$$0 \le S^*(\theta_n,\theta_n,\upsilon) - S^*(\theta_n,\theta_n,\theta_n) \le S^*(\theta_n,\theta_n,\upsilon).$$

Therefore,

$$\lim_{n\to\infty} S^*(\theta_n,\theta_n,\upsilon) = \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_n) = \lim_{n\to\infty} S^*(\upsilon,\upsilon,\upsilon).$$

Hence we follow that $\{\theta_n\}$ is a convergent sequence in (Ψ, S^*).

Now we prove that every Cauchy sequence $\{\theta_n\}$ in (Ψ, S^s) is a Cauchy sequence in (Ψ, S^*). Let $\epsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $S^s(\theta_n,\theta_n,\theta_m) < \frac{1}{2}$ for all $n, m \ge n_0$. Since

$$\begin{aligned}&S^*(\theta_n,\theta_n,\theta_n)\\ \le\ & 4S^*(\theta_{n_0},\theta_{n_0},\theta_n) - 3S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_{n_0},\theta_{n_0},\theta_{n_0}) + S^*(\theta_n,\theta_n,\theta_n)\\ \le\ & 2S^s(\theta_n,\theta_n,\theta_{n_0}) + S^*(\theta_{n_0},\theta_{n_0},\theta_{n_0}).\end{aligned}$$

Thus, we have

$$\begin{aligned}S^*(\theta_n,\theta_n,\theta_n) &\le 2S^s(\theta_n,\theta_n,\theta_{n_0}) + S^*(\theta_{n_0},\theta_{n_0},\theta_{n_0})\\ &\le 1 + S^*(\theta_{n_0},\theta_{n_0},\theta_{n_0}).\end{aligned}$$

Consequently the sequence $\{S^*(\theta_n,\theta_n,\theta_n)\}$ is bounded in $\mathbb{R}$, and so there exists a $a \in \mathbb{R}$ such that a subsequence $\{S^*(\theta_{n_k},\theta_{n_k},\theta_{n_k})\}$ is convergent to a, i.e. $\lim_{k\to\infty} S^*(\theta_{n_k},\theta_{n_k},\theta_{n_k}) = a$.

It remains to prove that $\{S^*(\theta_n,\theta_n,\theta_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^s), for given $\epsilon > 0$, there exists n_ϵ such that $S^s(\theta_n,\theta_n,\theta_m) < \frac{\epsilon}{2}$ for all $n, m \ge n_\epsilon$. Thus, for all $n, m \ge n_\epsilon$,

$$\begin{aligned}&|S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_m,\theta_m,\theta_m)|\\ \le\ & 4S^*(\theta_n,\theta_n,\theta_m) - 3S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_m,\theta_m,\theta_m)\\ +\ & S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_m,\theta_m,\theta_m)\\ \le\ & 2S^s(\theta_n,\theta_n,\theta_m) < \epsilon\end{aligned}$$

On the other hand,

$$\begin{aligned}&|S^*(\theta_n,\theta_n,\theta_n) - a|\\ \le\ & |S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_{n_k},\theta_{n_k},\theta_{n_k})| + |S^*(\theta_{n_k},\theta_{n_k},\theta_{n_k}) - a|\\ <\ & \epsilon + \epsilon = 2\epsilon\end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_n) = a$.

Now, we show that $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^*). We have,

$$\begin{aligned} & |2S^*(\theta_n, \theta_n, \theta_m) - 2a| \\ = \; & |S^s(\theta_n, \theta_n, \theta_m) + S^*(\theta_n, \theta_n, \theta_n) - a + S^*(\theta_m, \theta_m, \theta_m) - a| \\ \leq \; & S^s(\theta_n, \theta_n, \theta_m) + |S^*(\theta_n, \theta_n, \theta_n) - a| + |S^*(\theta_m, \theta_m, \theta_m) - a| \\ < \; & \frac{\epsilon}{2} + 2\epsilon + 2\epsilon = \frac{9}{2}\epsilon. \end{aligned}$$

That is, $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^*).

We shall have established the lemma if we prove that (Ψ, S^s) is complete if so is (Ψ, S^*). Let $\{\theta_n\}$ be a Cauchy sequence in (Ψ, S^s). Then $\{\theta_n\}$ is a Cauchy sequence in (Ψ, S^*), and so it is convergent to a point $\upsilon \in \Psi$ with

$$\lim_{n,m\to\infty} S^*(\theta_n, \theta_n, \theta_m) = \lim_{n\to\infty} S^*(\upsilon, \upsilon, \theta_n) = S^*(\upsilon, \upsilon, \upsilon).$$

Thus, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$S^*(\upsilon, \upsilon, \theta_n) - S^*(\upsilon, \upsilon, \upsilon) < \frac{\epsilon}{2} \; and \quad |S^*(\upsilon, \upsilon, \upsilon) - S^*(\theta_n, \theta_n, \theta_n)| < \frac{\epsilon}{2}$$

whenever $n \geq n_\epsilon$. Hence, we have

$$\begin{aligned} S^s(\upsilon, \upsilon, \theta_n) &= 2S^*(\upsilon, \upsilon, \theta_n) - S^*(\theta_n, \theta_n, \theta_n) - S^*(\upsilon, \upsilon, \upsilon) \\ &\leq |S^*(\upsilon, \upsilon, \theta_n) - S^*(\upsilon, \upsilon, \upsilon)| + |S^*(\upsilon, \upsilon, \theta_n) - S^*(\theta_n, \theta_n, \theta_n)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore (Ψ, S^s) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty} S^s(\alpha, \alpha, \theta_n) = 0$ if and only if

$$S^*(\alpha, \alpha, \alpha) = \lim_{n\to\infty} S^*(\alpha, a, \theta_n) = \lim_{n,m\to\infty} S^*(\theta_n, \theta_n, \theta_m).$$

∎

Lemma 2.10. *If S^* is a partial S-metric on Ψ, then the functions $S^s, S^m : \Psi \times \Psi \times \Psi \to \mathbb{R}^+$ given by*

$$\begin{aligned} & S^s(\theta, \upsilon, \omega) \\ = \; & S^*(\theta, \theta, \upsilon) + S^*(\upsilon, \upsilon, \omega) + S^*(\omega, \omega, \theta) - S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon) - S^*(\omega, \omega, \omega) \end{aligned}$$

and

$$S^m(\theta, \upsilon, \omega) = \max \left\{ \begin{array}{l} 2S^*(\theta, \theta, \upsilon) - S^*(\theta, \theta, \theta) - S^*(\upsilon, \upsilon, \upsilon), \\ 2S^*(\upsilon, \upsilon, \omega) - S^*(\upsilon, \upsilon, \upsilon) - S^*(\omega, \omega, \omega), \\ 2S^*(\omega, \omega, \theta) - S^*(\omega, \omega, \omega) - S^*(\theta, \theta, \theta) \end{array} \right\}$$

for every $\theta, \upsilon, \upsilon \in \Psi$, are equivalent S-metrics on Ψ.

Proof. It is easy to see that S^s and S^m are S-metrics on Ψ. Let $\theta, \upsilon, \omega \in \Psi$. It is obvious that

$$S^m(\theta, \upsilon, \omega) \leq 2S^s(\theta, \upsilon, \omega).$$

On the other hand, since $a+b+c \leq 3\max\{a,b,c\}$, it provides that

$$\begin{aligned}
& S^s(\theta,\upsilon,\omega)\\
= \;& S^*(\theta,\theta,\upsilon)+S^*(\upsilon,\upsilon,\omega)+S^*(\omega,\omega,\theta)-S^*(\theta,\theta,\theta)-S^*(\upsilon,\upsilon,\upsilon)-S^*(\omega,\omega,\omega)\\
= \;& \frac{1}{2}[2S^*(\theta,\theta,\upsilon)-S^*(\theta,\theta,\theta)-S^*(\upsilon,\upsilon,\upsilon)]\\
+ \;& \frac{1}{2}[2S^*(\upsilon,\upsilon,\omega)-S^*(\upsilon,\upsilon,\upsilon)-S^*(\omega,\omega,\omega)]\\
+ \;& \frac{1}{2}[2S^*(\omega,\omega,\theta)-S^*(\omega,\omega,\omega)-S^*(\theta,\theta,\theta)]\\
\leq \;& \frac{3}{2}\max\left\{\begin{array}{l} 2S^*(\theta,\theta,\upsilon)-S^*(\theta,\theta,\theta)-S^*(\upsilon,\upsilon,\upsilon),\\ 2S^*(\upsilon,\upsilon,\omega)-S^*(\upsilon,\upsilon,\upsilon)-S^*(\omega,\omega,\omega),\\ 2S^*(\omega,\omega,\theta)-S^*(\omega,\omega,\omega)-S^*(\theta,\theta,\theta)\end{array}\right\}\\
= \;& \frac{3}{2}S^m(\theta,\upsilon,\omega).
\end{aligned}$$

Thus,we have

$$\frac{1}{2}S^m(\theta,\upsilon,\omega) \leq S^s(\theta,\upsilon,\omega) \leq \frac{3}{2}S^m(\theta,\upsilon,\omega),$$

these inequalities implies that S^s and S^n are equivalent. ∎

Remark 2.2. Note that:

$$S^s(\theta,\theta,\upsilon)=2S^*(\theta,\theta,\upsilon)-S^*(\theta,\theta,\theta)-S^*(\upsilon,\upsilon,\upsilon)=S^m(\theta,\theta,\upsilon).$$

A mapping $F:\Psi\to\Psi$ is said to be continuous at $\theta_0\in\Psi$, if for every $\epsilon>0$, there exists $\delta>0$ such that $F(B_{S^*}(\theta_0,\delta))\subseteq B_{S^*}(F\theta_0,\epsilon)$.

3. FIXED POINT RESULT

In this section we prove some fixed point theorems in ordered partial S^*-metric spaces.

Theorem 3.1. *Let $(\Psi,\preceq)$ be a partially ordered set and suppose that there is a partial S-metric S^* on Ψ such that (Ψ,S^*) is a complete partial S-metric space. Suppose $F:\Psi\to\Psi$ is a continuous and nondecreasing mapping such that*

$$S^*(F\theta,F\upsilon,F\omega)\leq$$

$$k\max\left\{\begin{array}{l} S^*(\theta,\upsilon,\omega),S^*(\theta,\theta,F\theta),S^*(\upsilon,\upsilon,F\upsilon),S^*(\omega,\omega,F\omega)\\ \frac{1}{2}\left[S^*(\theta,\theta,F\upsilon)+S^*(\theta,\theta,F\omega)\right]\end{array}\right\} \tag{3.1}$$

for all $\theta,\upsilon,\omega\in\Psi$ with $\omega\preceq\upsilon\preceq\theta$, where $0<k<1$. If there exists an $\theta_0\in\Psi$ with $\theta_0\preceq F\theta_0$, then there exists $\theta\in\Psi$ such that $\theta=F\theta$. Moreover, $S^*(\theta,\theta,\theta)=0$.

Proof. If $F\theta_0=\theta_0$, then the proof is complete, so suppose $\theta_0\neq F\theta_0$. Now let $\theta_n=F\theta_{n-1}$ for $n=1,2,\cdots$. If $\theta_{n_0}=\theta_{n_0+1}$ for some $n_0\in\mathbb{N}$, then it is clear that θ_{n_0} is a fixed point of F. Thus assume $\theta_n\neq\theta_{n+1}$ for all $n\in\mathbb{N}$. Notice that, since $\theta_0\preceq F\theta_0$ and F is nondecreasing, we have

$$\theta_0\preceq\theta_1\preceq\theta_2\preceq\cdots\preceq\theta_n\preceq\theta_{n+1}\preceq\cdots.$$

By inequality (3.1) for these points we have

$$
\begin{array}{rcl}
\multicolumn{3}{l}{S^*(\theta_{n+1},\theta_{n+1},\theta_n)} \\
& = & S^*(F\theta_n, F\theta_n, F\theta_{n-1}) \\
& \leq & k\max\left\{\begin{array}{l} S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,F\theta_n), \\ S^*(\theta_n,\theta_n,F\theta_n), S^*(\theta_{n-1},\theta_{n-1},F\theta_{n-1}) \\ \frac{1}{2}[S^*(\theta_n,\theta_n,F\theta_n)+S^*(\theta_n,\theta_n,F\theta_{n-1})] \end{array}\right\} \\
& \leq & k\max\left\{\begin{array}{l} S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,\theta_{n+1}), \\ S^*(\theta_n,\theta_n,\theta_{n+1}), S^*(\theta_{n-1},\theta_{n-1},\theta_n), \\ \frac{1}{2}[S^*(\theta_n,\theta_n,\theta_{n+1})+S^*(\theta_n,\theta_n,\theta_n)] \end{array}\right\} \\
& \leq & k\max\left\{\begin{array}{l} S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,\theta_{n+1}) \\ S^*(\theta_n,\theta_n,\theta_{n+1}) \end{array}\right\} \\
(3.2) \qquad & = & k\max\{S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,\theta_{n+1})\}.
\end{array}
$$

since $S^*(\theta_n,\theta_n,\theta_n) \leq S^*(\theta_n,\theta_n,\theta_{n+1})$. Now if

$$\max\{S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,\theta_{n+1})\} = S^*(\theta_n,\theta_n,\theta_{n+1})$$

for some n, since $0 < k < 1$ by (3.2) we have

$$
\begin{array}{rcl}
S^*(\theta_{n+1},\theta_{n+1},\theta_n) & \leq & kS^*(\theta_n,\theta_n,\theta_{n+1}) \\
& < & S^*(\theta_n,\theta_n,\theta_{n+1}) = S^*(\theta_{n+1},\theta_{n+1},\theta_n)
\end{array}
$$

which is a contradiction because $S^*(\theta_{n+1},\theta_{n+1},\theta_n) > 0$. Thus

$$\max\{S^*(\theta_n,\theta_n,\theta_{n-1}), S^*(\theta_n,\theta_n,\theta_{n+1})\} = S^*(\theta_n,\theta_n,\theta_{n-1})$$

for all n. Therefore we have

$$S^*(\theta_{n+1},\theta_{n+1},\theta_n) \leq kS^*(\theta_n,\theta_n,\theta_{n-1})$$

and so

$$S^*(\theta_{n+1},\theta_{n+1},\theta_n) \leq k^n S^*(\theta_1,\theta_1,\theta_0). \tag{3.3}$$

Therefore

$$
\begin{array}{rcl}
\multicolumn{3}{l}{S^s(\theta_{n+1},\theta_{n+1},\theta_n)} \\
& = & 2S^*(\theta_{n+1},\theta_{n+1},\theta_n) - S^*(\theta_n,\theta_n,\theta_n) - S^*(\theta_{n+1},\theta_{n+1},\theta_{n+1}) \\
& \leq & 2S^*(\theta_{n+1},\theta_{n+1},\theta_n) \\
& \leq & 2k^n S^*(\theta_1,\theta_1,\theta_0).
\end{array}
$$

This show that $\lim\limits_{n\to\infty} S^s(\theta_{n+1},\theta_{n+1},\theta_n) = 0$. Now we have

$$
\begin{array}{rcl}
S^s(\theta_m,\theta_m,\theta_n) & \leq & 2S^s(\theta_m,\theta_m,\theta_{m-1}) + \cdots + 2S^s(\theta_{n+1},\theta_{n+1},\theta_n) \\
& \leq & 4k^{m-1}S(\theta_1,\theta_1,\theta_0) + \cdots + 4k^n S^*(\theta_1,\theta_1,\theta_0) \\
& = & \dfrac{4k^n - 4k^m}{1-k} S^*(\theta_1,\theta_1,\theta_0) \\
& \leq & \dfrac{4k^n}{1-k} S^*(\theta_1,\theta_1,\theta_0) \longrightarrow 0.
\end{array}
$$

Then $\{\theta_n\}$ is a Cauchy sequence in the S-metric space (Ψ, S^s). Since (Ψ, S^*) is complete then from Lemma 2.9, the sequence $\{\theta_n\}$ converges in the S-metric space (Ψ, S^s), say $\lim\limits_{n\to\infty} S^s(\theta_n,\theta,\theta) = 0$. Again from Lemma 2.9, we have

$$S^*(\theta,\theta,\theta) = \lim_{n\to\infty} S^*(\theta_n,\theta,\theta) = \lim_{n,m\to\infty} S^*(\theta_n,\theta_m,\theta_m). \tag{3.4}$$

Moreover since $\{\theta_n\}$ is a Cauchy sequence in the S-metric space (Ψ, S^s), we have $\lim_{n,m\to\infty} S^s(\theta_n, \theta_m, \theta_m) = 0$ and by (3.3) we have $\lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_n) = 0$, thus by definition S^s we have $\lim_{n,m\to\infty} S^*(\theta_n, \theta_m, \theta_m) = 0$. Therefore by (3.4), we have

$$S^*(\theta, \theta, \theta) = \lim_{n\to\infty} S^*(\theta_n, \theta, \theta) = \lim_{n,m\to\infty} S^*(\theta_n, \theta_m, \theta_m) = 0.$$

Now we claim that $F\theta = \theta$. Suppose $S^*(\theta, F\theta, F\theta) > 0$. Since F is continuous, then for given $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_{S^*}(\theta, \delta)) \subseteq B_{S^*}(F\theta, \varepsilon)$. Since $S^*(\theta, \theta, \theta) = \lim_{n\to\infty} S^*(\theta_n, \theta, \theta) = 0$, then there exists $N \in \mathbb{N}$ such that $S^*(\theta_n, \theta, \theta) < S^*(\theta, \theta, \theta) + \delta$ for all $n \geq N$. Therefore, we have $\theta_n \in B_{S^*}(\theta, \delta)$ for all $n \geq N$. Thus $F(\theta_n) \in F(B_{S^*}(\theta, \delta)) \subseteq B_{S^*}(F\theta, \varepsilon)$ and so $S^*(F\theta_n, F\theta, F\theta) < S^*(F\theta, F\theta, F\theta) + \varepsilon$ for all $n \geq N$. This shows that $S^*(F\theta, F\theta, F\theta) = \lim_{n\to\infty} S^*(\theta_{n+1}, F\theta, F\theta)$. Now we use the inequality (3.1) for $\theta = \upsilon = \omega$, then we have

$$S^*(F\theta, F\theta, F\theta) \leq k \max \{S^*(\theta, \theta, \theta), S^*(\theta, F\theta, F\theta)\} = kS^*(\theta, F\theta, F\theta).$$

Therefore, we obtain

$$\begin{aligned}
& S^*(\theta, F\theta, F\theta) \\
\leq \ & S^*(\theta, \theta_{n+1}, \theta_{n+1}) + 2S^*(\theta_{n+1}, F\theta, F\theta) - 2S^*(\theta_{n+1}, \theta_{n+1}, \theta_{n+1}) \\
\leq \ & S^*(\theta, \theta_{n+1}, \theta_{n+1}) + S^*(\theta_{n+1}, F\theta, F\theta)
\end{aligned}$$

and letting $n \to \infty$, we have

$$\begin{aligned}
S^*(\theta, F\theta, F\theta) &\leq \lim_{n\to\infty} S^*(\theta, \theta_{n+1}, \theta_{n+1}) + \lim_{n\to\infty} S^*(\theta_{n+1}, F\theta, F\theta) \\
&= S^*(F\theta, F\theta, F\theta) \\
&\leq kS^*(\theta, F\theta, F\theta) \\
&< S^*(\theta, F\theta, F\theta),
\end{aligned}$$

which is a contradiction because $S^*(\theta, F\theta, F\theta) > 0$. Thus $S^*(\theta, F\theta, F\theta) = 0$ and so $\theta = F\theta$. ∎

Similarly, we can show that if instead of υ by θ and set $\omega = \upsilon$ in Theorem 3.1 then we have the following Corollary.

Corollary 3.2. *Let $(\Psi, \preceq)$ be a partially ordered set and suppose that there is a partial S-metric S^* on Ψ such that (Ψ, S^*) is a complete partial S-metric space. Suppose $F : \Psi \to \Psi$ is a continuous and nondecreasing mapping such that*

$$S^*(F\theta, F\theta, F\upsilon) \leq$$

$$k \max \left\{ \begin{array}{l} S^*(\theta, \theta, \upsilon), S^*(\theta, \theta, F\theta), S^*(\upsilon, \upsilon, F\upsilon), \\ \frac{1}{2} [S^*(\theta, \theta, F\theta) + S^*(\theta, \theta, F\upsilon)] \end{array} \right\}$$

for all $\theta, \upsilon \in \Psi$ with $\upsilon \preceq \theta$, where $0 < k < 1$. If there exists an $\theta_0 \in \Psi$ with $\theta_0 \preceq F\theta_0$, then there exists $\theta \in \Psi$ such that $\theta = F\theta$. Moreover, $S^*(\theta, \theta, \theta) = 0$.

Since every D^*-metric is a S^*-metric. Hence we have the following Corollary.

Corollary 3.3. *Let $(\Psi, \preceq)$ be a partially ordered set and (Ψ, D^*) be a complete D^*-metric space. Suppose $F : \Psi \to \Psi$ is a continuous and nondecreasing mapping such that*

$$D^*(F\theta, F\theta, F\upsilon) \leq$$

$$k \max \left\{ \begin{array}{l} D^*(\theta, \theta, \upsilon), D^*(\theta, \theta, F\theta), D^*(\upsilon, \upsilon, F\upsilon), \\ \frac{1}{2} [D^*(\theta, \theta, F\theta) + D^*(\theta, \theta, F\upsilon)] \end{array} \right\}$$

for all $\theta, \upsilon \in \Psi$ *with* $\upsilon \preceq \theta$, *where* $0 < k < 1$. *If there exists an* $\theta_0 \in \Psi$ *with* $\theta_0 \preceq F\theta_0$, *then there exists* $\theta \in \Psi$ *such that* $\theta = F\theta$. *Moreover,* $D^*(\theta, \theta, \theta) = 0$.

Also, we have the following Corollary.

Corollary 3.4. *Let* $(\Psi, \preceq)$ *be a partially ordered set and suppose that there is a partial metric* p *on* Ψ *such that* (Ψ, p) *is a complete partial metric space. Suppose* $F : \Psi \to \Psi$ *is a continuous and nondecreasing mapping such that*

$$p(F\theta, F\upsilon) \leq k \max \left\{ \begin{array}{l} p(\theta, \upsilon), p(\theta, F\theta), p(\upsilon, F\upsilon), \\ \frac{1}{2}\left[p(\theta, F\theta) + p(\theta, F\upsilon)\right] \end{array} \right\}$$

for all $\theta, \upsilon \in \Psi$ *with* $\upsilon \preceq \theta$, *where* $0 < k < 1$. *If there exists an* $\theta_0 \in \Psi$ *with* $\theta_0 \preceq F\theta_0$, *then there exists* $\theta \in \Psi$ *such that* $\theta = F\theta$. *Moreover,* $p(\theta, \theta) = 0$.

Proof. By Lemma 2.2 if define

$$S^*(\theta, \upsilon, \omega) = \max\{p(\theta, \upsilon), p(\theta, \omega), p(y, \omega)\},$$

then $S^*(\theta, \theta, \upsilon) = p(\theta, \upsilon)$ is a partial S-metric and by Lemma 2.3 and Corollary 3.2 the proof is complete. ∎

In the following theorem we remove the continuity of F. Also, the contractive condition (3.1) do not have to satisfied for $\theta = \upsilon = \omega$.

Theorem 3.5. *Let* $(\Psi, \preceq)$ *be a partially ordered set and suppose that there is a partial* S*-metric* S^* *on* Ψ *such that* (Ψ, S^*) *is a complete partial* S*-metric space. Suppose* $F : \Psi \to \Psi$ *is a nondecreasing mapping such that*

$S^*(F\theta, F\upsilon, F\omega) \leq$

$$k \max \left\{ \begin{array}{l} S^*(\theta, \upsilon, \omega), S^*(\theta, \theta, F\theta), S^*(\upsilon, \upsilon, F\upsilon), S^*(\omega, \omega, F\omega), \\ \frac{1}{2}[S^*(\theta, \theta, F\upsilon) + S^*(\theta, \theta, F\omega)] \end{array} \right\} \tag{3.5}$$

for all $\theta, \upsilon, \omega \in \Psi$ *with* $\omega \preceq \upsilon \prec \theta$ *(that is,* $\omega \preceq \upsilon \preceq \theta$ *and* $\upsilon \neq \theta$ *), where* $0 < k < 1$. *Also, the condition*

$$\left\{ \begin{array}{l} \text{If } \{\theta_n\} \subset \Psi \text{ is an increasing sequence} \\ \text{with } \theta_n \to \theta \text{ in } \Psi, \text{then } \theta_n \prec \theta \text{ for all } n \end{array} \right. \tag{3.6}$$

holds. If there exists an $\theta_0 \in \Psi$ *with* $\theta_0 \preceq F\theta_0$, *then there exists* $\theta \in \Psi$ *such that* $\theta = F\theta$. *Moreover,* $S^*(\theta, \theta, \theta) = 0$.

Proof. As in the proof of Theorem 3.1, we can construct a sequence $\{\theta_n\}$ in Ψ by $\theta_n = F\theta_{n-1}$ for $n = 1, 2, \ldots$ Also we can assume that the consequtive terms of $\{\theta_n\}$ are different. Otherwise we are finished. Therefore we have

$$\theta_0 \prec \theta_1 \prec \theta_2 \prec \cdots \prec \theta_n \prec \theta_{n+1} \prec \cdots.$$

Again, as in the proof of Theorem 3.1, we can show that $\{\theta_n\}$ is a Cauchy sequence in the S-metric space (Ψ, S^s) and therefore there exists $\theta \in \Psi$ such that

$$\begin{aligned} S^*(\theta, \theta, \theta) &= \lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta) = \lim_{n,m\to\infty} S^*(\theta_n, \theta_n, \theta_m) \\ &= \lim_{n\to\infty} S^*(\theta_n, \theta_n, \theta_m) = 0. \end{aligned}$$

Now we claim that $F\theta = x$. Suppose $S^*(\theta,\theta,F\theta) > 0$. Since the condition (3.6) is satisfied, then we can use the (3.5) for $\theta = \upsilon = \theta_n$ and $\upsilon = \theta$. Therefore, we obtain

$$\begin{aligned} S^*(\theta_{n+1},\theta_{n+1},F\theta) &= S^*(F\theta_n,F\theta_n,F\theta) \\ &\leq k\max\left\{\begin{array}{l} S^*(\theta_n,\theta_n,\theta), S^*(\theta_n,\theta_n,F\theta_n), \\ S^*(\theta_n,\theta_n,F\theta_n), S^*(\theta,\theta,F\theta), \\ \frac{1}{2}[S^*(\theta_n,\theta_n,F\theta_n) + S^*(\theta_n,\theta_n,F\theta)] \end{array}\right\} \\ &\leq k\max\left\{\begin{array}{l} S^*(\theta_n,\theta_n,\theta), S^*(\theta_n,\theta_n,\theta_{n+1}), \\ S^*(\theta_n,\theta_n,\theta_{n+1}), S^*(\theta,\theta,F\theta), \\ \frac{1}{2}[S^*(\theta_n,\theta_n,\theta_{n+1}) + S^*(\theta_n,\theta_n,F\theta)] \end{array}\right\}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} S^*(\theta,\theta,F\theta) &= \lim_{n\to\infty} S^*(\theta_{n+1},\theta_{n+1},F\theta) \\ &\leq k\max\left\{\begin{array}{l} \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta), \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_{n+1}), \\ \lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_{n+1}), \lim_{n\to\infty} S^*(\theta,\theta,F\theta), \\ \frac{1}{2}[\lim_{n\to\infty} S^*(\theta_n,\theta_n,\theta_{n+1}) + \lim_{n\to\infty} S^*(\theta_n,\theta_n,F\theta)] \end{array}\right\} \\ &\leq kS^*(\theta,\theta,F\theta) \\ &< S^*(\theta,\theta,F\theta), \end{aligned}$$

which is a contradiction. Thus $S^*(\theta,\theta,F\theta) = 0$ and so $\theta = F\theta$. ∎

Similarly we can show that if instead of υ by Ψ and set $\omega = \upsilon$ in Theorem 3.5 then we have the following Corollary.

Corollary 3.6. *Let $(\Psi,\preceq)$ a partially ordered set and suppose that there is a partial S-metric S^* on Ψ such that (Ψ,S^*) is a complete partial S-metric space. Suppose $F:\Psi\to\Psi$ is a nondecreasing mapping such that*

$$S^*(F\theta,F\theta,F\upsilon) \leq k\max\left\{\begin{array}{l} S^*(\theta,\theta,\upsilon), S^*(\theta,\theta,F\theta), S^*(\upsilon,\upsilon,F\upsilon), \\ \frac{1}{2}[S^*(\theta,\theta,F\theta) + S^*(\theta,\theta,F\upsilon)] \end{array}\right\}$$

for all $\theta,\upsilon \in \Psi$ with $\upsilon \prec \theta$ (that is, $\upsilon \preceq \theta$ and $\upsilon \neq \theta$), where $0 < k < 1$. Also, the condition

$$\left\{\begin{array}{l} \text{If } \{\theta_n\} \subset \Psi \text{ is a increasing sequence} \\ \text{with } \theta_n \to \theta \text{ in } \Psi, \text{then } \theta_n \prec \theta \text{ for all } n \end{array}\right.$$

hold. If there exists an $\theta_0 \in \Psi$ with $\theta_0 \preceq F\theta_0$, then there exists $\theta \in \Psi$ such that $\theta = F\theta$. Moreover, $S^*(\theta,\theta,\theta) = 0$.

Also, we have the following Corollary.

Corollary 3.7. *Let $(\Psi,\preceq)$ a partially ordered set and suppose that there is a partial metric p on Ψ such that (Ψ,p) is a complete partial metric space. Suppose $F:\Psi\to\Psi$ is a nondecreasing mapping such that*

$$p(F\theta,F\upsilon) \leq k\max\left\{\begin{array}{l} p(\theta,\upsilon), p(\theta,F\theta), p(\upsilon,F\upsilon), \\ \frac{1}{2}\,[p(\theta,F\theta) + p(\theta,F\upsilon)] \end{array}\right\}$$

for all $\theta,\upsilon \in \Psi$ with $\upsilon \prec \theta$ (that is, $\upsilon \preceq \theta$ and $\upsilon \neq \theta$), where $0 < k < 1$. Also, the condition

$$\left\{\begin{array}{l} \text{If } \{\theta_n\} \subset \Psi \text{ is a increasing sequence} \\ \text{with } \theta_n \to \theta \text{ in } \Psi, \text{then } \theta_n \prec \theta \text{ for all } n \end{array}\right.$$

hold. If there exists an $\theta_0 \in \Psi$ with $\theta_0 \preceq F\theta_0$, then there exists $\theta \in \Psi$ such that $\theta = F\theta$. Moreover, $p(\theta,\theta) = 0$.

Proof. By Lemma 2.2 it is enough define

$$S^*(\theta, \upsilon, \omega) = \max\{p(\theta, \upsilon), p(\theta, \omega), p(\upsilon, \omega)\},$$

then $S^*(\theta, \theta, \upsilon) = p(\theta, \upsilon)$ is a partial S-metric. Hence by Lemma 2.3 and Corollary 3.7 the proof is complete. ∎

Example 3.1. *Let $\Psi = [0, \infty)$ and $S^*(\theta, \upsilon, \omega) = \max\{\theta, \upsilon, \upsilon\}$, then it is clear that (Ψ, S^*) is a complete partial S-metric space. We can define a partial order on Ψ as follows:*

$$\theta \preceq \upsilon \Leftrightarrow (\theta = \upsilon) \text{ or } (\theta, \upsilon \in [0, 1] \text{ with } \theta \leq \upsilon).$$

Let $F : \Psi \to \Psi$,

$$F\theta = \begin{cases} \frac{\theta^2}{1+\theta} \quad , & \theta \in [0, 1] \\ \\ \frac{1}{2}\theta \quad , & \theta \in (1, \infty) \end{cases}$$

and F is nondecreasing with respect to $\preceq$ and for $\upsilon \prec \Psi$ and $k \geq \frac{1}{2}$, we have

$$\begin{aligned} S^*(F\theta, F\theta, F\upsilon) &= \max\{\frac{\theta^2}{1+\theta}, \frac{\upsilon^2}{1+\upsilon}\} \\ &= \frac{\theta^2}{1+\theta} \leq \frac{1}{2}\theta = \frac{1}{2}S^*(\theta, \theta, \upsilon) \\ &\leq k(S^*(\theta, \theta, \upsilon)) \\ &\leq k \max\left\{ \begin{array}{l} S^*(\theta, \theta, \upsilon), S^*(\theta, \theta, F\theta), S^*(\upsilon, \upsilon, F\upsilon), \\ \frac{1}{2}[S^*(\theta, \theta, F\theta) + S^*(\theta, \theta, F\upsilon)] \end{array} \right\}, \end{aligned}$$

that is, the condition (3.5) of Theorem 3.5 is satisfied. Also, it is clear that the condition (3.6) is satisfied and for $\theta_0 = 0$, we have $\theta_0 \preceq F\theta_0$. Therefore all conditions of Theorem 3.5 are satisfied and so F has a fixed point in Ψ.

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The authors declare no conflict of interest.

Competing interest
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