The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 11, pp. 1-17, 2019

GLOBAL ANALYSIS ON RIEMANNIAN MANIFOLDS

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Received 10 March, 2019; accepted 6 September, 2019; published 4 November, 2019.

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ABSTRACT. In this paper, an exposition of the central concept of global analysis on a Riemannan manifold is given. We extend the theory of smooth vector fields from open subsets of Euclidean space to Riemannan manifolds. Specifically, we prove that a Riemannian manifold admits a unique solution for a system of ordinary differential equations generated by the flow of smooth tangent vectors. The idea of partial differential equations on Riemannian manifold is highlighted on the unit sphere.

Key words and phrases: Global analysis; Riemannian manifold; Vector fields; Tangent bundles; Differential equations; Smooth flow of vector fields.

2010 *Mathematics Subject Classification.* Primary 50J10; Secondary: 35P20; 35R01.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Differential equations introduced by Sir Isaac Newton, have played a decisive role in the mathematical study of natural phenomena. For many decades, researchers have focused on applicability of the qualitative properties of solutions of differential equations. One of such attempts is the study of the geometrical properties as well as the topological properties of the solutions. Global analysis, also called analysis on manifolds, is the study of the geometric and topological properties of differential equations on manifolds and vector bundles, see e.g. [28, 26, 17, 12, 22] and [2].

Central to the field of global analysis is the index theorem propounded by Atiyah and Singer [4] in (1963), which states that analytic and topological index of an elliptic differential operator on a compact manifold are equal. This theorem is one of the main bridges, which stimulated a lot of further research and interplay between geometry, analysis and mathematical physics. To understand the Atiyah-Singer index formula on a Riemannian manifold, one needs to clarify the concept of global analysis on a Riemannian manifold.

Global analysis uses techniques in manifold theory and topological spaces of mappings to classify behaviours of differential equations, particularly nonlinear differential equations, [26]. Global Analysis and the theory of differential equations are classical fields of mathematics that have a wide range of applications within mathematics, for instance in number theory, group theory, geometry and topology, but also have important applications outside of mathematics to physics, engineering and technology.

The field of Global Analysis is rooted in pure mathematics and focuses on geometric and topological aspects of analysis. The interests of the field include spectral and scattering theory on manifolds, regularity and existence of global solutions to pseudo-differential equations and boundary value problems, topological questions related to generalizations of the Atiyah-Singer index theorem among other topics, see e.g. [7, 15, 13] and [6]. More recent works in this field are those of Li [18] in 2016, Lai [16] in 2017, Yiming [31] in 2018 and Zhang [32] in (2019).

An earlier attempt to explain global analysis was made by Smale in [28]. He gave a historical background of the development of global analysis since the 1968 Summer Institute of the American Mathematical Society in global analysis. He highlighted the pioneering activities of Poincar$\acute{e}$ and Birkhoff in the development of the study of the topology of linear elliptic differential operators, especially in the work of Atiyah, Singer, and Bott. Smale also examined the works of Andronov, Pontryagin and Peixoto in dynamical systems on a manifold.

This present paper is centred around demonstrating what global analysis on a Riemannian manifold means. We prove the existence of a unique solution for a system of ordinary differential equations generated by the flow of smooth tangent vectors on a Riemannian manifold. We proceed with a clear discussion of basic concepts such as vector field, Riemannian manifold and the flow of smooth vector fields with diagrammatic illustrations where necessary.

2. PRELIMINARIES

Following Lee [17] and Jost [14], we let M be a topological space. Firstly, we recall some basic topological notions. The topological space M is called Hausdorff if for any two distinct points $x, y \in M$, there exists disjoint open subsets $U, V \subset M$ containing x and y respectively. A covering $(U_\alpha)_{\alpha\in I}$ (I an arbitrary index set) is called locally finite if each $x \in M$ has a neighbourhood that intersects only finitely many U_α. M is said to be paracompact if any open covering possesses a locally finite refinement. This means that for any open covering $(U_\alpha)_{\alpha\in I}$ there exists a locally finite open covering $(U'_\beta)_{\beta\in I'}$ (I' an arbitrary index set) with

$$\forall \beta \in I' \, \exists \alpha \in I : U'_\beta \subset U_\alpha.$$

The space M is said to be connected if there are no two or more disjoint open subsets whose union is M. It is second countable if it admits a countable bases for its topology. M is compact if its every open covering has a subcovering.

A map between topological spaces is called continuous if the preimage of any open set is again open. A bijective map which is continuous in both directions is called a homeomorphism. If the map is bijective and of class C^∞ with a differentiable inverse of class C^∞ then we say it is a diffeomorphism. For more details, one may see Jost [14].

Definition 2.1. An n-dimensional chart on M is any pair (U, φ), where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$ called the image of the chart. In diagram we have Figure (1).

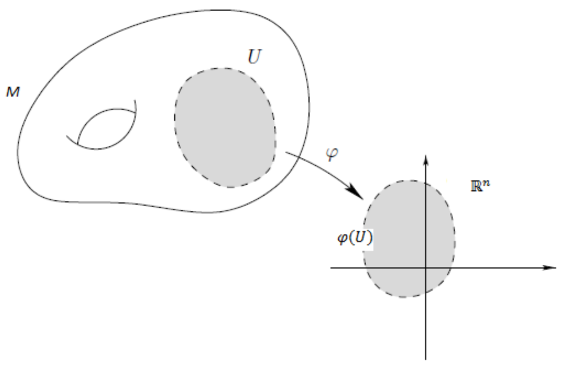


Figure 1: Coordinate chart

Definition 2.2. A Hausdorff, second countable, connected topological space M is called an n-dimensional topological manifold (with a countable basis) if any point of M belongs to an n-dimensional chart.

Definition 2.3. Let M and N be topological spaces. Let $\varphi : M \to N$ be a smooth map. Suppose $f : N \to \mathbb{R}$ is a smooth function on N. The pullback of f by φ is a smooth function $\varphi^* f$ on M defined by $(\varphi^* f)(x) = f(\varphi(x))$ for $x \in M$. In diagram, this is shown as Figure (2).

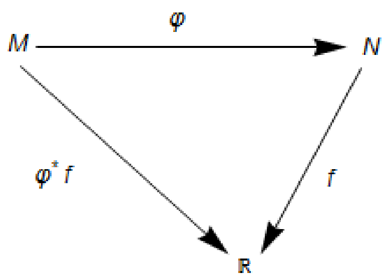


Figure 2: Pullback function.

Let M be an n-dimensional manifold. For any chart (U, φ) on M, the local coordinate system $(x^1, x^2, \cdots, x^n)$ is defined in U by taking the φ-pullback of the Cartesian coordinate system in $\mathbb{R}^n$. Consequently, one can say that a chart is an open set $U \subset M$ with a local coordinate system.

For any two charts (U, φ) and (V, ψ) on M for which the intersection $U \cap V$ is not empty, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the chart transition from one chart to the other. In diagram, see Figure (3).

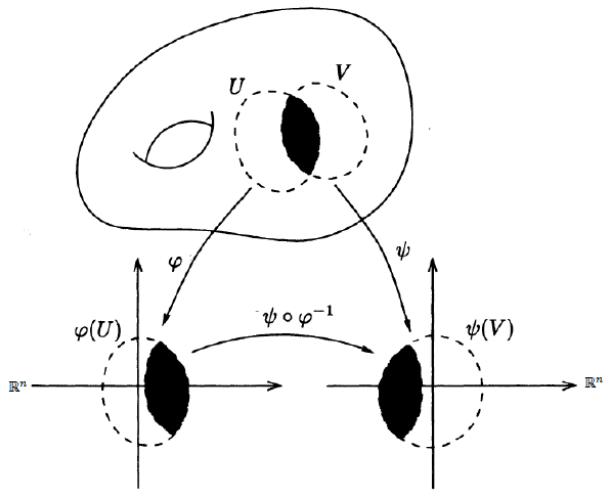


Figure 3: Chart transition.

A chart transition map on an n-dimensional manifold is called C^k if its k^{th} derivatives exist and are continuous for a positive integer $k \leq n$. When this condition holds for all positive integers, we say the map is C^∞ or simply "smooth". A family of charts on the manifold M is called a C^k-atlas when the associated chart transition is C^k and if it covers all of M. Two charts (U, φ) and (V, ψ) are said to be compatible if $U \cap V \neq \emptyset$ and the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Similarly, two C^k-atlases are called compatible if their union is again a C^k-atlas. The union of all compatible C^k-atlases determines a C^k-structure on M. We collect these notions together as the following definition.

Definition 2.4. A differentiable n-dimensional manifold M is a connected paracompact Hausdorff topological space for which every point has a neighbourhood U that is homeomorphic to an open subset $\Omega \subset \mathbb{R}^n$. Such a homeomorphism $\varphi : U \to \Omega$ is called a chart. Again, a family $\{U_\alpha, \varphi_\alpha\}$ of charts for which the U_α constitute an open covering of M is called an atlas. The atlas $\{U_\alpha, \varphi_\alpha\}$ of M is called differentiable if all charts transitions

$$\psi_\alpha \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\alpha) \to \psi_\alpha(U_\alpha \cap V_\alpha)$$

are differentiable of class $C^\infty(M)$. A maximal differentiable atlas is called a differentiable structure and a manifold with differentiable structure is called a differentiable manifold; see e.g. [10, 27, 11] and [14].

We also need the concepts of tangent space and Riemannian metric to define Riemannian Manifold. Following B$\ddot{a}$r [5] and Chavel [10], we let a linear map

$$\xi : C^\infty(M) \to \mathbb{R}$$

be such that at a point $x \in M$, $\xi(fg)(x) = \xi(f)(g(x)) + \xi(g)(f(x))$ for all $f, g \in C^\infty(M)$.

We denote the set of all such maps by T_xM. For all $\xi, \eta \in T_xM$ and $\lambda \in \mathbb{R}$, we define the sum and scalar multiplication $\xi + \eta$ and $\lambda\xi$ so that T_xM is a linear space over $\mathbb{R}$.

Definition 2.5. The linear space T_xM is called the tangent space of M at a point x and we denote by $TM := \sqcup_x T_xM$ the disjoint union of all the tangent spaces at the point x. TM is called the tangent bundle of M. A tangent space is illustrated in Figure (4).

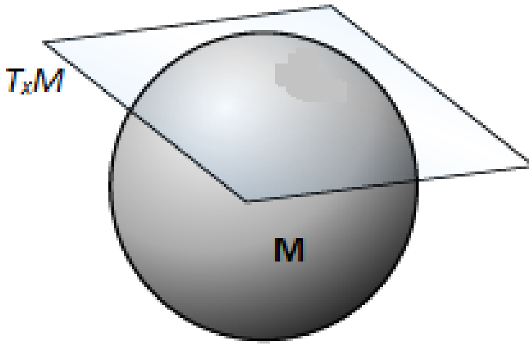


Figure 4: tangent space.

We will denote the components of ξ in local coordinate chart $(x^1, \cdots, x^n)$ by ξ^i and write

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i} \; \forall f \in C^\infty(M). \tag{2.1}$$

An alternative notation for (2.1) which we will also adopt in this paper is

$$\xi(f) = \frac{\partial f}{\partial \xi} \text{ so that } \frac{\partial f}{\partial \xi} = \xi^i \frac{\partial f}{\partial x^i}$$

which allows to think of ξ as a directional derivative at x and to interpret $\frac{\partial f}{\partial \xi}$ as a directional derivative; see e.g Grigor'yam [12].

Definition 2.6. (Vector field): A vector field on a smooth manifold M is a family $\{v(x)\}_{x\in M}$ of tangent vectors such that $v(x) \in T_xM$ for any $x \in M$. In local coordinates, it can be represented in the form $v(x) = v^i \frac{\partial}{\partial x^i}$.

A Riemannian metric (also called Riemannian metric tensor) on M is a family of symmetric positive definite bilinear forms $g = \{g(x)\}$ on T_xM which depends on $x \in M$ smoothly. The metric enables to define an inner product $\langle ., . \rangle_{g(x)}$ on TM by $\langle \xi, \eta \rangle_{g(x)} \; \forall \; \xi, \eta \in T_xM$. Hence, T_xM becomes an Euclidean space. For any $\xi \in T_xM$, its length $||\xi|| = \sqrt{\langle \xi, \xi \rangle}$.

In local coordinates, the inner product has the form $\langle \xi, \eta \rangle_{g(x)} = g_{ij}(x)\xi^i\eta^j$ where $(g_{ij})_{ij=1}^n$ is a square symmetric positive-definite matrix expressing the metric in the local coordinates.

These can be summarised in form of definition thus:

Definition 2.7. (Riemannian metric): A Riemannian metric on a smooth manifold M is an assignment of an inner product $\langle ., . \rangle_{g(x)} : T_xM \times T_xM \to \mathbb{R}$ for all $x \in M$ such that

(1.) $\langle \xi, \xi \rangle_{g(x)} > 0$ for $\xi \neq 0$ (positive definite);
(2.) $\langle \xi, \eta \rangle_{g(x)} = \langle \eta, \xi \rangle_{g(x)} \; \forall \eta, \xi \in T_xM$ (symmetric);
(3.) $\langle a_i\xi_i + a_j\xi_j, b_i\eta_i + b_j\eta_j \rangle_{g(x)} = \sum_{i,j=1}^n a_ib_j \langle \xi_i, \eta_j \rangle_{g(x)} \forall \; \xi_i, \eta_j \in T_xM$ and $a_i, a_j, b_i, b_j \in \mathbb{R}$ (bilinear);
(4.) $\langle \xi, \eta \rangle_{g(x)} = 0$ if and only if either $\xi = 0$ or $\eta = 0$ or $\xi = 0 = \eta$ (non-degenerate); and

(5.) for all $x \in M$, there exist local coordinates $\{x^i\}$ such that $g_{ij}(x) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{g(x)}$ are smooth functions.

From the foregoing concepts, we make the following definition.

Definition 2.8. (Riemannian Manifold): A Riemannian manifold is a pair (M, g), where g is a Riemannian metric on the smooth manifold M.

3. MAIN RESULTS

A vector field usually identified with a linear differential operator of the form

$$\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$$

where $x_i, x_2, \cdots, x_n$ are coordinates on a manifold M and $\xi_i \in C^\infty(M)$ are handy to define the directional derivative of $f \in C^\infty(M)$ as

$$df \cdot \xi = \xi \cdot f = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}.$$

So, the directional derivative of f in the direction of ξ is the action of the differential operator ξ on f, compare with e.g. [19, 6, 16] and [29].

In n dimensional manifold, we can therefore write the vector field in the form

$$(3.1) \qquad \begin{cases} \mathcal{L}_n &= a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n} \\ &= \varphi_1(x_1, x_2, \cdots, x_n) \frac{\partial}{\partial x_1} + \varphi_2(x_1, x_2, \cdots, x_n) \frac{\partial}{\partial x_2} \\ & \quad + \cdots + \varphi_n(x_1, x_2, \cdots, x_n) \frac{\partial}{\partial x_n} \end{cases}$$

where $a_1 = \varphi_1, \cdots, a_n = \varphi_n \in C^\infty(M)$. For instance in a 2-dimensional manifold,

$$\mathcal{L}_2 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} = \varphi(x, y) \frac{\partial}{\partial x} + \psi(x, y) \frac{\partial}{\partial y}.$$

where x and y are coordinates on the manifold, and $a = \varphi(x, y)$, $b = \psi(x, y) \in C^\infty(M)$. If $f \in C^\infty(M)$ then

$$df \cdot \xi = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}.$$

If the same vector field is expressed in terms of coordinates s, r then by the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial f}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial f}{\partial r} \frac{\partial r}{\partial y}$$

so that

$$df \cdot \xi = \left(a \frac{\partial s}{\partial x} + b \frac{\partial s}{\partial y}\right) \frac{\partial f}{\partial s} + \left(a \frac{\partial r}{\partial x} + b \frac{\partial r}{\partial y}\right) \frac{\partial f}{\partial r}.$$

Here, the linear partial differential operator is

$$\xi = \left(a \frac{\partial s}{\partial x} + b \frac{\partial s}{\partial y}\right) \frac{\partial}{\partial s} + \left(a \frac{\partial r}{\partial x} + b \frac{\partial r}{\partial y}\right) \frac{\partial}{\partial r}.$$

We have the following preliminary result.

Lemma 3.1. *A constant coefficient linear ordinary differential equation*

$$(3.2) \qquad a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a(t) \dot{y} + a_0(t) y = g(t)$$

where $a_n(t) \neq 0$ on a Riemannian manifold M is generated by a vector field.

Proof. Rewrite the system (3.2) as follows. Define n new variables, for example:

$$y = y_1,\ \frac{dy}{dt} = y_2,\ \frac{d^2y}{dt^2} = y_3,\ \frac{d^3y}{dt^3} = y_4, \cdots, \frac{d^{n-1}y}{dt^{n-1}} = y_n.$$

Substitute the new variables into (3.2) to obtain the system of equations

$$\left\{\begin{array}{rcl} \dot{y}_1 &=& y_2 \\ \dot{y}_2 &=& y_3 \\ \dot{y}_3 &=& y_4 \\ &\vdots& \\ \dot{y}_{n-1} &=& y_n \\ \dot{y}_n &=& b_{n-1}(t)y_n + \cdots + b_1(t)y_2 + b_0(t)y_1 + f(t). \end{array}\right. \tag{3.3}$$

Equation (3.3) is the required system of first-order equations in $(y_1(t),\ y_2(t), \cdots, y_n(t)) \in M$.

These can be displayed in matrix form as

$$\dot{y} = Ay + F(t) \tag{3.4}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b_0(t) & b_1(t) & b_2(t) & b_3(t) & \cdots & b_{n-1}(t) \end{pmatrix},\ y = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \text{ and } F(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}.$$

Therefore, $\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial y_i}$ is the generator of (3.2) for $y \in M$ where,

$$\xi_1 = y_2,\ \xi_2 = y_3, \cdots, \xi_n = b_{n-1}(t)y_n + \cdots + b_1(t)y_2 + b_0(t)y_1 + f(t).$$

∎

The intuitive meaning of such an equation (3.2) is that a point on the manifold is a function of time t and its coordinates are changing according to the system of ordinary differential equations (3.3). Hence, the system of the ordinary differential equations and the vector fields are effectively the same, see e.g [29].

Consequently, given a vector field $v \in T_xM$ for x in a Riemannian manifold M, the basic problem of the theory of ordinary differential equation is to find a smooth map $\Psi : I \to M$ for some interval $I \ni 0$ such that

$$\left\{\begin{array}{rcl} \dot{\Psi}(t) &=& v_{\Psi(t)})\ \forall t \in I \text{ and} \\ \Psi(0) &=& x. \end{array}\right. \tag{3.5}$$

We have the following result.

Theorem 3.2. *Let $v \in T_xM$ be a vector field for x in a Riemannian M and suppose the flow of the vector field $\Psi : U \times I \to M$ towards $y \in M$ satisfies*

$$\left\{\begin{array}{rcl} \partial_t\Psi(y,t) &=& v_{\Psi(y,t)} \text{ and} \\ \Psi(0) &=& y \end{array}\right. \tag{3.6}$$

for each $(y,t) \in U \times I$.

Let $\psi : W \to \Omega \subset \mathbb{R}^n$ be a chart for M such that $\Psi(U \times I) \subset W$. Then $u = \psi \circ \Psi(\psi^{-1}(z), t)$ defines a map $\psi(U \times I) \mapsto \Omega$ satisfying

$$\partial_t u^k = \tilde{v}^k((u^1, u^2, \cdots, u^n), 0) = z^k \tag{3.7}$$

for $k = 1, 2, \cdots, n$ *and all* $(z^1, z^2, \cdots, z^n) \in \psi(U)$ *and* $t \in I$. *The operator*

$$v(\psi^{-1}(z^1, z^2, \cdots, z^n)) = \sum_{k=1}^{n} \tilde{v}^k \partial_k.$$

If u *satisfies (3.7) then* $\Psi(y,t) = \psi^{-1}(u(\psi(y), t))$ *solves (3.6).*

Proof. We have

$$\xi_{(z,t)}(u(\partial_t)) = \xi\Psi(\psi^{-1}(z), t)\psi \circ \xi_{\psi^{-1}(z,t)}\Psi(\partial_t) = \xi\psi(v).$$

Since $\xi\psi(\partial_j) = e_j$ for $j = 1, 2, \cdots, n$, it follows that if

$$v = \sum_{k=1}^{n} \tilde{v}^k \partial_k$$

then

$$\xi\psi(v) = \sum_{k=1}^{n} \tilde{v}^k e_k$$

which completes the proof. ∎

The main result of this work is a proof that a Riemannian manifold M admits a unique solution for a system of differential equations generated by the flow of smooth tangent vectors. This is presented as the next theorem.

Theorem 3.3. *Let* $f \in C^\infty(\psi(U))$ *in a neighbourhood* $\psi(U)$ *of* $x \in M$ *and let* $f : \Omega \to \mathbb{R}^n$ *where* $\Omega \subset \mathbb{R}^n$ *is open. Assume* $||f||_\infty = M_0 < 0$ *and* $||\xi(f)||_\infty = M_1 < 0$. *Then there exists a unique smooth map* $\Psi : I \to \Omega$ *satisfying*

$$\begin{cases} \frac{d}{dt}\Psi^i(t) &= f(\Psi^1(t), \Psi^2(t), \cdots, \Psi^n(t)) \text{ and} \\ \Psi(0) &= z; \ \forall z \in \Omega, \text{ and } t \in I; \text{ for } i = 1, 2, \cdots, n. \end{cases} \tag{3.8}$$

The integral curves Ψ^i *are functions of time* $t \in I$. *In particular we take* $\dfrac{d(z, \partial\Omega)}{M_0} \in I$.

Proof. By Picard's iteration technique of successive approximation, since $\Psi^{(0)}(t) = z$ for all $t \in I$, we seek improved approximations using the formula

$$\Psi^{(k+1)}(t) = z + \int_0^{(t)} f(\Psi^k(s))ds. \tag{3.9}$$

The formula (3.9) is valid since it solves

$$\frac{d}{dt}\Psi^{(k+1)} = f(\Psi^k).$$

Moreover,

$$||\Psi^{(k)}(t) - z|| \le |t|M_0 < d(z, \partial\Omega).$$

Thus, $\Psi^{(k)} \in \Omega \ \forall z \in \Omega$.

To show convergence, we need to prove that

$$||\Psi^{(k+1)}(t) - \Psi^{(k)}(t)|| \le \frac{M_0 M_1^k |t|^{k+1}}{(n+1)!}, \ \forall k > 0; \text{ and } t \in I. \tag{3.10}$$

For $k = 0$,

$$||\Psi^{(1)}(t) - \Psi^{(0)}(t)|| = ||\int_0^t f(z)ds|| \le |t|M_0.$$

Assume by induction that (3.10) holds for $k-1$. Then

$$\begin{aligned} ||\Psi^{(k+1)}(t) - \Psi^{(k)}(t)|| &= ||\int_0^t f(\Psi^{(k)}(s)) - f(\Psi^{(k-1)}(s))ds|| \\ &\leq \Big|\int_0^t M_1||\Psi^{(k)}(s)) - \Psi^{(k-1)}(s))||ds\Big| \\ &\leq M_1\Big|\int_0^t \frac{M_0M_1^{k-1}|s|^k}{k!}ds\Big| \leq \frac{M_0M_1^k|t|^{k+1}}{(k+1)!}. \end{aligned}$$

Thus $\{\Psi^{(k)}\}$ is a Cauchy sequence in a complete space of continuous maps and hence convergent uniformly to a continuous function Ψ. So,

$$\Psi(t) = z + \int_0^{(t)} f(\Psi(s))ds \;\forall t \in I$$

and differentiable. Consequently,

$$\dot{\Psi}(t) = f(\Psi)(t)$$

for each $t \in I$ and $\Psi(0) = z$. Moreover, $\Psi \in C^\infty(\Omega)$.

Furthermore suppose σ is another such solution then

$$\begin{aligned} ||\Psi(t) - \sigma(t)|| &= ||\int_0^t f(\Psi(s)) - f(\sigma(s))ds|| \\ &\leq M_1\Big|\int_0^t ||\Psi(s)) - \sigma(s))||ds\Big| \\ &\leq c\frac{M_1^k|t|^k}{k!} \to 0 \text{ as } k \to \infty \end{aligned}$$

where $c = ||\Psi - \sigma||_\infty$. Therefore $\Psi = \sigma$, proving uniqueness. ∎

Further results can be obtained for global analysis on a Riemannian manifold M by studying a partial differential operator on M. An example is the Laplacian on M. This leads to the theory of partial differential equations on M. We now give an explicit definition of the Laplacian on a Riemannian manifold with particular reference to the n-dimensional unit sphere.

Let M be a smooth, compact and connected n-dimensional Riemannian manifold without boundary and let g be a smooth Riemannian metric on M. For a coordinate chart on M,

$$(x^1, x^2, \cdots, x^n) : U \to \mathbb{R}^n, \; (U \subset M \text{ open}),$$

we represent g by the matrix-valued function (g_{ij}) where $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_g$ and $\langle \cdot, \cdot \rangle_g$ is the inner product on the tangent space T_xM.

The volume form dV_g of (M, g) is defined as $dV_g = \sqrt{|g|}dx$; with $|g| = \det(g_{ij})$ and

$$dx = dx^1 \wedge \cdots \wedge dx^n.$$

To give definition of the Laplacian on smooth functions over the Riemannian manifold M, we need the following definitions.

Definition 3.1. (Differential). For a fixed $x \in M$, let f be a smooth function in a neighbourhood of x. The differential df of f at x is a linear functional on T_xM given by the pairing $\langle df, \xi \rangle = \xi(f)$ for any $\xi \in T_xM$.

Hence, df is an element of the dual space T_m^*M, (called the cotangent space). Elements of T_x^*M are called covectors. A basis $\{e_1, \cdots, e_n\}$ in T_xM has dual basis $\{e^1, \cdots, e^n\}$ in T_x^*M which is defined by

$$\langle e^i, e_j\rangle = \delta^i_j = \begin{cases} 1; \ j = i \\ 0; \ j \neq i. \end{cases}$$

For example, the basis $\{\frac{\partial}{\partial x^i}\}$ has dual basis $\{dx^i\}$ because $\langle dx^i, \frac{\partial}{\partial x_j}\rangle = \delta^i_j$.

The covector df can be represented in the basis $\{dx^i\}$ as follows:

$$df = \frac{\partial f}{\partial x^i} dx^i$$

that is, the partial derivatives $\frac{\partial f}{\partial x_i}$ are the components of the differential df. Indeed,

$$\langle \frac{\partial f}{\partial x^i} dx^i, \frac{\partial}{\partial x_j}\rangle = \frac{\partial f}{\partial x^i}\langle dx^i, \frac{\partial}{\partial x_j}\rangle = \frac{\partial f}{\partial x^i}\delta^i_j = \langle df, \frac{\partial}{\partial x_j}\rangle.$$

Definition 3.2. (Gradient). For any smooth function f on M, its gradient $\nabla_g f$ at a point $x \in M$ is defined by

$$(\nabla_g f)(x) = g^{-1}(x) df(x)$$

where $g^{-1}(x) := g^{ij}(x)$ is the inverse of g. If we let $\xi = \nabla_g f(x)$ then for any $\eta \in T_xM$ one writes

$$\langle \nabla_g f, \eta\rangle_g = \langle df, \eta\rangle = \frac{\partial f}{\partial \eta}$$

which is another way of defining the gradient. In local coordinates,

$$(\nabla_g f)^i = g^{ij}\frac{\partial f}{\partial x^j}.$$

If f_1 on M is another smooth function, set $\eta = \nabla f_1$ then,

$$\langle \nabla_g f, \nabla_g f_1\rangle = \langle df, \nabla_g f_1\rangle = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f_1}{\partial x^j}.$$

At this point, we may recall again that the Riemannian measure (volume form) $dV_g(x)$ on (M, g) can be represented in local coordinates as

$$\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n \text{ where } |g| = \det(g_{ij}).$$

Theorem 3.4. *(Divergence theorem). Let v be any smooth vector field on M and dV a Riemannian measure. Then, there exists a unique smooth function on M called divergence and denoted by* $\operatorname{div} v$ *such that the following identity holds:*

$$\int_M (\operatorname{div} v) f dV = -\int_M \langle v, \nabla_g f\rangle dV \ \forall f \in C_0^\infty(M). \tag{3.11}$$

In local coordinates,

$$\operatorname{div} v = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^j}\big(\sqrt{|g|}\, v^j\big). \tag{3.12}$$

Note also that for any continuous function f on M if

$$\int_M f\psi dV = 0 \ \forall \psi \in C_0^\infty(M)$$

then $f \equiv 0$. For proof, one may see e.g. [12].

Theorem 3.5. *([12]). Let (M, g) be a Riemannian manifold without boundary. Then for every smooth vector field v on M,*

$$\int_M \mathrm{div} v \, dV_g = 0;$$

where dV_g is the volume form on M induced by the metric g.

The Laplacian on smooth functions on (M, g) is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{3.13}$$

defined in local coordinates by

$$\Delta_g = -\,\mathrm{div}(\mathrm{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i}\big(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}\big) \tag{3.14}$$

where g^{ij} are the components of the dual metric on the cotangent bundle $T_p^* M$.

The operator Δ_g extends to a self-adjoint operator on

$$L^2(M) \supset H^2(M) \to L^2(M)$$

with compact resolvent. This implies that there exists an orthonormal basis $\{f_k\} \in L^2(M)$ consisting of eigenfunctions such that

$$\Delta_g f_k = \lambda_k f_k$$

where the eigenvalues are listed with multiplicities

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty;$$

see for example, [22], [12], [24], [23], [10], [25] and [20]. The Laplacian Δ_g thus, has one-dimensional null space consisting precisely of constant functions.

4. DISCUSSION

A typical Riemannian manifold is the n-dimensional unit sphere defined by

$$S^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}. \tag{4.1}$$

In particular, the 0-sphere, 1-sphere and the 2-sphere are respectively a pair of points on a line segment, a circle on a plane and the ordinary sphere in 3-dimension.

Following [21, 3] and [1], we let $f \in S^n$ be any function on the n-sphere and $\tilde{f}$ be its extension to an open neighbourhood of S^n that is constant along rays from the centre of S^n. We say that $f \in C^2(S^n)$ if $\tilde{f}$ is a C^2 function of that neighbourhood. For such functions (not containing $\{0\}$) on S^n the Laplacian Δ_n equals

$$\Delta_n f = \Delta_g \tilde{f} \tag{4.2}$$

where Δ_g on the right-hand side of (4.2) is the usual Laplacian in $\mathbb{R}^{n+1}$.

In $\mathbb{R}^n$, $n \geq 2$, every point $x \neq 0$ can be represented in polar coordinates as a couple (r, θ) where $r := |x| > 0$ is the polar radius and $\theta := \frac{x}{|x|} \in S^{n-1}$ is the polar angle. Note that the metric $g_{S^{n-1}}$ is obtained by restricting the metric $g_{\mathbb{R}^n}$ to S^{n-1}. On S^{n-1}, the polar coordinate is $(\theta^1, \cdots, \theta^{n-1})$ whilst $r = 1$ and $dr = 0$. Indeed, for any

$$\xi \in T_x S^{n-1},\ \langle dr, \xi \rangle = \xi(r) = \xi(r|_{S^{n-1}}) = \xi(1) = 0.$$

Consider now the polar coordinates on $S^n : (\theta^1, \cdots, \theta^n)$. Let p be the north pole and q be the south pole of S^n, i.e ($p = (0, 0, \cdots, 0, 1)$) and $q = -p$. For any $x \in S^n \setminus \{p, q\}$, define $r \in (0, \pi)$ and $\theta \in S^{n-1}$ by $\cos r = x^{n+1}$ and $\theta = \frac{x'}{|x'|}$ where x' is the projection of x onto $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$. Clearly, the polar radius is the angle between the position vectors x and p. The point r can be regarded as the latitude of the point x measured from the pole. The polar angle θ can be regarded as the longitude of the point x; see figure (5).

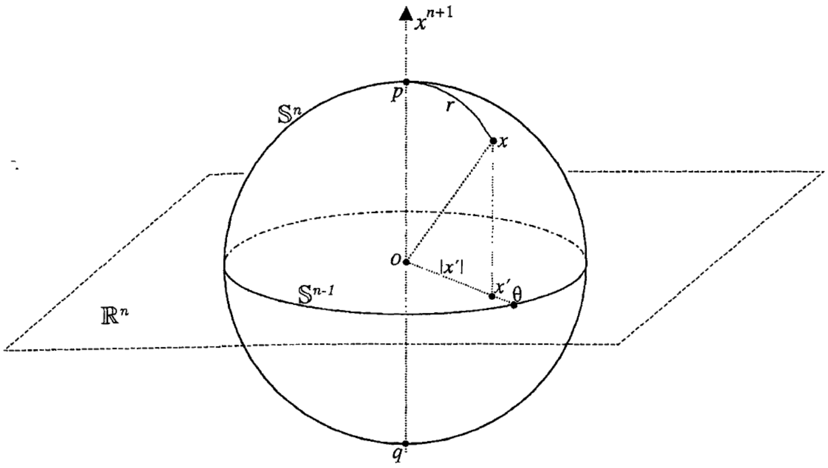


Figure 5: Polar coordinates on S^n.

The canonical metric g_{S^n} on S^n has the following representation in polar coordinates:

$$g_{S^n} = dr^2 + \sin^2 r g_{S^{n-1}};$$

see [12].

In the polar coordinates, the Riemannian measure on S^n is given by $dV = \sin^{n-1} r dr d\theta$.

For the unit n-sphere, the Laplacian (4.2) in polar coordinates reduces to

$$\Delta_n = \frac{1}{\sin^{n-1}\theta}\frac{\partial}{\partial\theta}\{\sin^{n-1}\theta\frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta}\Delta_{n-1} \tag{4.3}$$

where Δ_{n-1} is the Laplacian on S^{n-1}.

The Laplacian in polar coordinates (θ, ϕ) on S^2 endowed with the round metric

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2 = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

using equation (3.14) is

$$\Delta_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\{\sin\theta\frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta}\Delta_1$$

where $\Delta_1 = \frac{\partial^2}{\partial\theta^2}$ is the Laplacian on S^1.

On S^3, where the round metric is

$$g_{S^3} = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & \sin^2\theta\sin^2\phi \end{pmatrix}$$

using equation (3.14) is

$$\Delta_3 = \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\{\sin^2\theta\frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta}\Delta_2$$

where Δ_2 is the Laplacian on S^2. Continuing this way, one arrives at (4.3).

The Legendre equation

$$(1-x^2)P_m(x)'' - 2xP_m(x)' + (k(k+1))P_m(x) = 0. \tag{4.4}$$

is the usual equation on the S^n which is solved by the Legendre polynomial function given by

$$f(x) = P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m. \tag{4.5}$$

The Legendre polynomial function is generalised as follows.

Definition 4.1. The Gegenbauer polynomial $P_k^n(t)$ is given by

$$P_k^n(t) = k!\Gamma(\frac{n-1}{2})\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\frac{(1-t^2)^j t^{k-2j}}{4^j j!(k-2j)!\Gamma(j+\frac{n-1}{2})}, \tag{4.6}$$

or given through the extended Rodrigues formula

$$P_k^n(t) = (-1)^k R_{k,n}(1-t^2)^{\frac{3-n}{2}}\frac{d^k}{dt^k}(1-t^2)^{k+\frac{n-3}{2}} \text{ with } n \geq 2, \tag{4.7}$$

where the Rodrigues constant $R_{k,n}$ is given by

$$R_{k,n} = \frac{\Gamma(\frac{n-1}{2})}{2^k\Gamma(k+\frac{n-1}{2})}.$$

Gegenbauer polynomials are relevant to the study of the heat and zeta kernels of the laplacian on Riemannian manifolds because of the following result.

Lemma 4.1. *(Addition formula, c.f: Morimoto* [21]*).*
Let $\{\psi_{k,j} : 1 \leq j \leq d_k(n)\}$ be an orthonormal basis of the space of n-dimensional spherical harmonics $\mathcal{H}_k(S^n)$, that is,

$$\int_{S^n}\psi_{k,j}(x)\bar{\psi}_{k,l}(x)dV_g(x) = \delta_{jl};\ 1 \leq j,l \leq d_k(n). \tag{4.8}$$

Then

$$\sum_{j=1}^{d_k(n)}\psi_{k,j}(x)\bar{\psi}_{k,l}(y) = \frac{d_k(n)}{|S^n|}P_k^{\frac{(n-1)}{2}}(x\cdot y) \tag{4.9}$$

where as before, $P_k^n(t)$ are the Gegenbauer polynomials of degree k in n dimensions.

For proof, one may see Morimoto [21]. Note in particular, this means that $P_k^{(n-1)/2}(x\cdot y)$ is a harmonic function on S^n with eigenvalue $\lambda_k = k(k+n-1)$.

The Gegenbauer polynomials enable one to write the heat kernel on S^n explicitly, namely, for all $t > 0$, and $x, y \in S^n$:

$$K(t,x,y) := \frac{1}{V}\sum_{k=0}^{\infty}\sum_{j=1}^{d_k(n)}e^{-k(k+n-1)t}\psi_{k,j}(x)\bar{\psi}_{k,j}(y) \tag{4.10}$$

$$= \frac{1}{V}\sum_{k=0}^{\infty}e^{-k(k+n-1)t}\frac{d_k(n)}{P_k^{\frac{(n-1)}{2}}(1)}P_k^{\frac{(n-1)}{2}}(x\cdot y). \tag{4.11}$$

where V is the volume of S^n, and $d_k(n)$ is the dimension of the λ_k eigenspace. It is also known that the zeta kernel $\zeta_{S^n}(s,x,y)$ on S^n is explicitly given by

$$\zeta_{S^n}(s,x,y) = \frac{1}{V}\sum_{k=1}^{\infty}\frac{d_k(n)}{(k(k+n-1))^s}\cdot\frac{1}{P_k^{\frac{(n-1)}{2}}(1)}P_k^{\frac{(n-1)}{2}}(x\cdot y) \tag{4.12}$$

(see e.g Wogu [30], Camporesi [9], Buser [8] and Morimoto [21]).

Lemma 4.2. *The multiplicities $d_k(n)$ of the eigenspace of the spectrum $\{\lambda_k\}$ of the Laplacian on S^n can be expressed as*

$$d_k(n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \tag{4.13}$$

where $k \in \mathbb{N}_0$ *and* $n \geq 1$ is the dimension of the manifold S^n.

Proof. It is clear that

$$\begin{aligned} d_k(n) &= \binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(k+n)!}{k!n!} - \frac{(k+n-2)!}{(k-2)!n!} \\ &= \frac{(k+n-2)!}{n!}\Big[\frac{(k+n)(k+n-1)}{k!} - \frac{1}{(k-2)!}\Big] \end{aligned}$$

which simplifies as

$$\begin{aligned} & \frac{(k+n-2)!}{n!}\Big[\frac{(k+n)(k+n-1)}{k!} - \frac{1}{(k-2)!}\Big] \\ &= \frac{(k+n-2)!}{n!(k-2)!}\Big[\frac{(k+n)(k+n-1)}{k(k-1)} - 1\Big] \\ &= \frac{(k+n-2)!}{k!}\frac{n}{n!}(2k+n-1) \\ &= \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}. \end{aligned}$$

∎

We further illustrate the results with the following examples.

Example 4.1. *Consider the systems of ordinary differential equations*

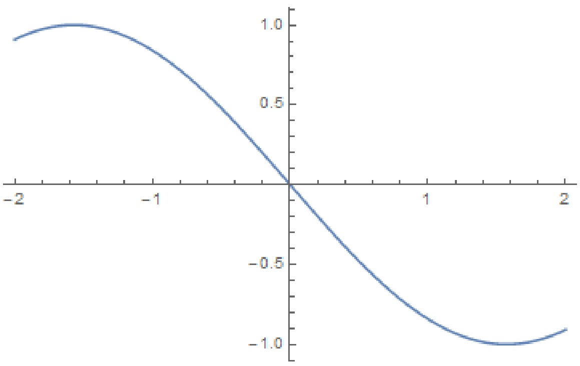
$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x \end{cases}$$

with $x(0) = r \in M$ *and* $y(0) = 0 \in M$.

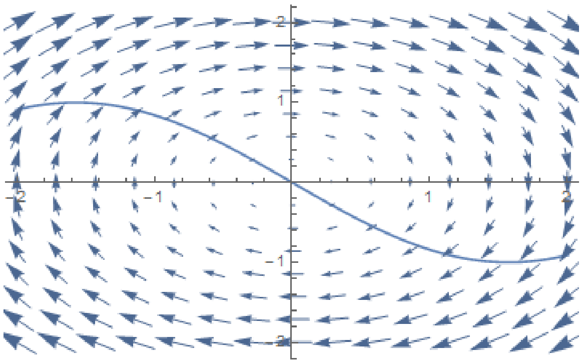
Clearly, the system is solved by

$$\begin{cases} x(t) &= r\cos t, \\ y(t) &= -r\sin t. \end{cases}$$

Take $r = 1$ *for example, to have the integral curve on* $I = [-2, 2]$ *as*



The integral curve plotted on the associated vector field on the same interval is



Example 4.2. *Similarly, consider another systems of ordinary differential equations*

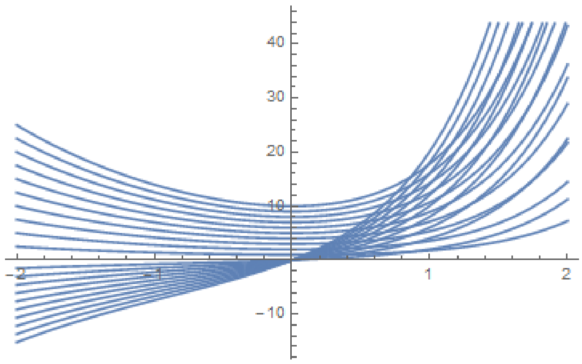
$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= x + y \end{cases}$$

with $x(0) = 0 \in M$.

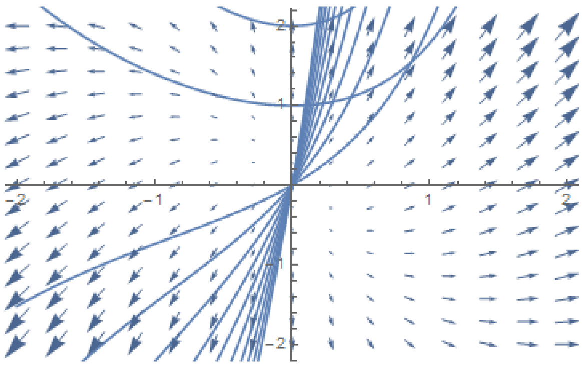
Solving, we have

$$\begin{cases} x(t) &= \frac{2}{\sqrt{5}} c e^{\frac{t}{2}} \sinh \frac{\sqrt{5}}{2} t \text{ and} \\ y(t) &= -\frac{1}{\sqrt{5}} c e^{\frac{t}{2}} \Big[-5 \cosh \frac{\sqrt{5}}{2} t + \sqrt{5} \sinh \frac{\sqrt{5}}{2} t \Big] \end{cases}$$

for some arbitrary constant c. *For integer values of* $c \in [0, 10]$ *for example, we obtain a family of the integral curves on* $I = [-2, 2]$ *as*



The family of the integral curves plotted on the associated vector field on the same interval is



So in both cases, the vector fields determine the direction of the flow of the integral curves.

5. CONCLUSION

Global analysis has been systematically explained in this work as a bunch of theories on geometrical and topological properties of differential equations on manifolds. We dwelt specifically on the actions of vector fields on systems of ordinary differential equations on Riemannian manifolds. A number of results emerged especially theorems relating what it means to defined a differential equation on Riemannian manifolds and we proved that unique solutions of the equations exist for the manifolds. The Laplace equation on the unit n-dimensional sphere is also illustrated as a prototype case of partial differential equations on Riemannian manifolds.

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