ON A SUBSET OF BAZILEVIĆ FUNCTIONS

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ABSTRACT. Let $S$ denote the class of analytic and univalent functions in $D := \{ z \in \mathbb{C} : |z| < 1 \}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $\alpha \geq 0$, the subclass $B_1(\alpha)$ of $S$ of Bazilević functions has been extensively studied. In this paper we determine various properties of a subclass of $B_1(\alpha)$, for $\alpha \geq 0$, which extends early results of a class of starlike functions studied by Ram Singh.

Key words and phrases: Univalent functions, Starlike functions, Bazilević functions, Coefficients.

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1. INTRODUCTION AND DEFINITIONS

Denote by $A$, the set of functions $f$, which are analytic in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$, and normalized so that

\[(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\]

and by $S$, the subset of $A$ consisting of functions $f$ which are univalent in $D$.

In recent years, a great deal of attention (see e.g. [2], [4], [9], [10]), has been given to the set $B_1(\alpha)$ of Bazilevič functions in $S$, defined for $\alpha \geq 0$, as follows.

**Definition 1.1.** Let $f \in A$ and be given by (1.1). Then for $\alpha \geq 0$, $f \in B_1(\alpha)$ if, and only if, for $z \in D$

\[(1.2) \quad \Re f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} > 0.\]

Clearly $B_1(0)$ consists of the well-known class $S^*$ of starlike functions, and $B_1(1)$ the class $\mathcal{R}$ whose elements satisfy $\Re f'(z) > 0$, for $z \in D$.

Finding sharp bounds for $|a_n|$ for all $n \geq 2$ when $f \in B_1(\alpha)$ remains an open problem, with best possible bounds only known when $2 \leq n \leq 6$, [3], [8], and even then, only partial answers have been given when $n = 5$ and 6.

When $\alpha = -1$ in (1.2), functions defined by the following are also members of $S$, [5], and provide an interesting subset of $S$ which is known as the class $\mathcal{U}(\lambda)$. The class $\mathcal{U}(\lambda)$ defined below, has also been extensively studied in recent years (see e.g. [5], [6], and the references in these papers).

**Definition 1.2.** Let $f \in A$ and be given by (1.1). Then $f \in \mathcal{U}(\lambda)$ if, and only if, for $z \in D$

\[\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| < \lambda.\]

It is clear from the definition that since $f'(z)/[z/f(z)]^2 \neq 0$, functions in $\mathcal{U}(\lambda)$ are non-vanishing in $D \setminus \{0\}$, and locally univalent.

Finding sharp bounds for the coefficients of functions in $\mathcal{U}(\lambda)$ appears to be a difficult problem, with best possible bounds only known when $2 \leq n \leq 4$, [6]. On the other hand when $\lambda = 1$, sharp bound have been found for all $n \geq 2$ (see e.g. [6]).

In this paper we study a subset of $B_1(\alpha)$, whose definition mimics that of $\mathcal{U}(\lambda)$ in the case $\lambda = 1$, and show that it is possible to obtain sharp bounds for the first five coefficients of $f(z)$, together with the first four coefficients of the inverse function. We also give other properties of this subclass, which we define as follows.
Definition 1.3. Let \( f \in \mathcal{A} \) and be given by \((1.1)\). Then for \( \alpha \geq 0 \), \( f \in B_1(\alpha, 1) \) if, and only if, for \( z \in \mathbb{D} \),

\[
|f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} - 1| < 1.
\]

We note that when \( \alpha = 0 \), \((1.3)\) reduces to

\[
\left|\frac{zf'(z)}{f(z)} - 1\right| < 1,
\]

considered in [8]. Since the analysis for \( \alpha = 0 \) and \( \alpha > 0 \) can differ, we will specify this when appropriate.

2. Representation Expression and Distortion Theorems

We begin by giving a representation formula for \( f \in B_1(\alpha, 1) \) when \( \alpha > 0 \), analogous to that given in [8] in the case \( \alpha = 0 \).

Theorem 2.1. For \( \alpha > 0 \), \( f \in B_1(\alpha, 1) \) if, and only if,

\[
f(z) = \left(\alpha \int_0^z t^{\alpha-1}(1 + \omega(t))dt\right)^{1/\alpha},
\]

where \( \omega \) is analytic in \( \mathbb{D} \) satisfying \( |\omega(z)| \leq 1 \), and \( \omega(0) = 0 \).

Proof. From \((1.3)\), we can write

\[
f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} = 1 + \omega(z).
\]

Let \( \phi(z) = \left(\frac{f(z)}{z}\right)^{\alpha} \). Then differentiation gives

\[
\phi'(z) + \frac{\alpha}{z}\phi(z) = \frac{\alpha}{z}(1 + \omega(z)).
\]

Multiplying by \( z^\alpha \) and integrating gives \((2.1)\). \( \blacksquare \)

Theorem 2.2. For \( \alpha > 0 \), let \( f \in B_1(\alpha, 1) \), \( z = re^{i\theta} \in \mathbb{D} \), and

\[
\beta_1(\alpha, r) = \left(\frac{1 + \alpha + \alpha r}{1 + \alpha}\right), \quad \beta_2(\alpha, r) = \left(\frac{1 + \alpha - \alpha r}{1 + \alpha}\right).
\]

Then

\[
r\beta_2(\alpha, r)^{1/\alpha} \leq |f(z)| \leq r\beta_1(\alpha, r)^{1/\alpha}
\]

\[
(1 - r)\beta_2(\alpha, r)^{(1-\alpha)/\alpha} \leq |f'(z)| \leq (1 + r)\beta_1(\alpha, r)^{(1-\alpha)/\alpha}
\]

\[
\frac{1 - r}{\beta_1(\alpha, r)} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1 + r}{\beta_2(\alpha, r)}.
\]
Equality holds in all cases when \( f(z) = z\left(\frac{1 + \alpha + \alpha z}{1 + \alpha}\right)^{1/\alpha} \) for \( \theta = 0 \), or \( \pi/2 \).

**Proof.** It follows from the Schwarz Lemma that \( |\omega(z)| \leq |z| \). Using this in (2.1) and integrating easy establishes the right-hand inequality in (2.3). The left-hand inequality follows from the minimum principle for harmonic functions. Differentiating (2.1) and using (2.2) gives (2.4), from which (2.5) follows on noting that \( |1 + \omega(z)| \geq 1 - |\omega(z)| \geq 1 - |z| \).

From (2.3), we at once deduce the following.

**Corollary 2.1.** Let \( f \in B_1(\alpha, 1) \) for \( \alpha > 0 \). Then \( f(D) \) contains the disk \( \{ w : |w| < 1/(1 + \alpha)^{1/\alpha} \} \).

We note that letting \( \alpha \to 0 \) in the results of Theorem 2.2 and Corollary 2.1 gives those obtained in [9].

We will use the following lemmas, the first two and the fourth of which can be found in [1], and the third in [7].

## 3. LEMMAS

Denote by \( \mathcal{P} \), the class of functions \( p \) of positive real part, i.e., functions satisfying \( \text{Re} p(z) > 0 \) for \( z \in \mathbb{D} \), with Taylor expansion

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.
\]

**Lemma 3.1.** If \( p \in \mathcal{P} \), then

\[
|p_2 - \frac{\mu}{2} p_1^2| \leq \max\{2, 2|\mu - 1|\} = \begin{cases} 
2, & 0 \leq \mu \leq 2, \\
2|\mu - 1|, & \text{elsewhere}.
\end{cases}
\]

**Lemma 3.2.** Let \( p \in \mathcal{P} \). If \( 0 \leq B \leq 1 \) and \( B(2B - 1) \leq D \leq B \), then

\[
|p_3 - 2B p_1 p_2 + D p_1^3| \leq 2.
\]

**Lemma 3.3.** If \( p \in \mathcal{P} \), and \( \alpha_1, \alpha_2, \beta \) and \( \gamma \) satisfy \( 0 < \alpha_1 < 1 \), \( 0 < \alpha_2 < 1 \), and

\[
8\alpha_1(1 - \alpha_1)((\alpha_2 \beta - 2\gamma)^2 + (\alpha_2(\alpha_1 + \alpha_2) - \beta)^2) + \alpha_2(1 - \alpha_2)(\beta - 2\alpha_1 \alpha_2)^2 \\
\leq 4\alpha_2^2(1 - \alpha_2)^2\alpha_1(1 - \alpha_1),
\]

then

\[
|\gamma p_1^4 + \alpha_1 p_2^2 + 2\alpha_2 p_1 p_3 - (3/2)\beta p_1^2 p_2 - p_4| \leq 2.
\]

**Lemma 3.4.** If \( p \in \mathcal{P} \), then

\[
|p_3 - (\mu + 1)p_1 p_2 + \mu p_1^3| \leq \max\{2, 2|2\mu - 1|\} = \begin{cases} 
2, & 0 \leq \mu \leq 1, \\
2|2\mu - 1|, & \text{elsewhere}.
\end{cases}
\]
4. Coefficient Inequalities

**Theorem 4.1.** Let \( f \in B_1(\alpha, 1) \) for \( \alpha \geq 0 \), and be given by (1.1). Then for \( 2 \leq n \leq 5 \),
\[
|a_n| \leq \frac{1}{\alpha + n - 1}.
\]
The inequalities are sharp.

**Proof.** Recall from (1.3), that we can write
\[
f'(z) \left( \frac{f(z)}{z} \right)^{\alpha - 1} = 1 + \omega(z),
\]
where \( \omega(z) \) is analytic in \( \mathbb{D} \), \( |\omega(z)| \leq 1 \), and \( \omega(0) = 0 \).

Since \( p \in \mathcal{P} \), we can therefore write
\[
p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, \quad \text{or} \quad \omega(z) = \frac{p(z) - 1}{p(z) + 1}.
\]

From (2.2), (3.1), (4.1) and (4.2), equating coefficients we obtain
\[
a_2 = \frac{p_1}{2(1 + \alpha)}
\]
\[
a_3 = \frac{1}{2(2 + \alpha)} \left( p_2 - \frac{a(5 + 3\alpha)}{4(1 + \alpha)^2(2 + \alpha)} p_1^2 \right)
\]
\[
a_4 = \frac{1}{2(3 + \alpha)} \left( p_3 - \frac{1 + 8\alpha + 3\alpha^2}{2(1 + \alpha)^2(2 + \alpha)} p_1 p_2 + \frac{\alpha(5 + 64\alpha + 61\alpha^2 + 14\alpha^3)}{24(1 + \alpha)^3(2 + \alpha)} p_1^3 \right)
\]
\[
a_5 = \frac{1}{2(4 + \alpha)} \left( \frac{\alpha(8 + 544\alpha + 3557\alpha^2 + 5389\alpha^3 + 3329\alpha^4 + 907\alpha^5 + 90\alpha^6)}{192(1 + \alpha)^4(2 + \alpha)^2(3 + \alpha)} p_1^4
\]
\[
+ \frac{4 + 11\alpha + 3\alpha^2}{4(2 + \alpha)^2} p_2 + \frac{2 + 11\alpha + 3\alpha^2}{2(1 + \alpha)(3 + \alpha)} p_1 p_3
\]
\[
- \frac{8 + 76\alpha + 325\alpha^2 + 324\alpha^3 + 117\alpha^4 + 14\alpha^5}{8(1 + \alpha)^2(2 + \alpha)^2(3 + \alpha)} p_1^2 p_2 - p_4 \right).
\]

From (4.3) the inequality for \( a_2 \) is obvious.

For \( a_3 \) we apply Lemma 3.1 with \( \mu = \frac{\alpha(5 + 3\alpha)}{2(1 + \alpha)^2} \), which gives the inequality for \( |a_3| \), since \( 0 \leq \mu \leq 2 \) in this case.

For \( a_4 \) we use Lemma 3.2 with
\[
B = \frac{1 + 8\alpha + 3\alpha^2}{4(1 + \alpha)(2 + \alpha)},
\]
and
\[ D = \frac{\alpha(5 + 64\alpha + 61\alpha^2 + 14\alpha^3)}{24(1 + \alpha)^3(2 + \alpha)}. \]

It is easily verified that both \(0 \leq B \leq 1\), and \(B(2B - 1) \leq D \leq B\), when \(\alpha \geq 0\), and so applying Lemma 3.2 gives the required inequality for \(|a_4|\).

For \(a_5\), we apply Lemma 3.3 with \(\alpha_1, \alpha_2, \beta\) and \(\gamma\) the respective coefficients of \(a_5\) in (4.3), so that we need to show that

\[
(1 - \alpha)^2(4 + \alpha)^2(12544 + 427648\alpha + 5441392\alpha^2 + 33366608\alpha^3 + 117462812\alpha^4 \\
+ 260385736\alpha^5 + 382475767\alpha^6 + 388520160\alpha^7 + 282592930\alpha^8 + 150937228\alpha^9 \\
(4.4) 60100454\alpha^{10} + 17921756\alpha^{11} + 3972584\alpha^{12} + 639452\alpha^{13} + 71147\alpha^{14} + 4932\alpha^{15} \\
+ 162\alpha^{16}) \\
\leq 288(12 + 5\alpha + \alpha^2)(2 + 11\alpha + 3\alpha^2)^2(4 + 11\alpha + 3\alpha^2)(1 + \alpha)^6(2 + \alpha)^4(3 + \alpha)^2.
\]

To see that this inequality is true, write the left-hand side of the above inequality as \((1 - \alpha)^2(4 + \alpha)^2\phi_1(\alpha)\), and the right-hand side as \(\phi_2(\alpha)\). Then clearly \((1 - \alpha)^2(4 + \alpha)^2\phi_1(\alpha) \leq (4 + \alpha)^2\phi_2(\alpha)\).

Thus it enough to show that \((4 + \alpha)^2\phi_1(\alpha) \leq \phi_2(\alpha)\) when \(\alpha \geq 0\), which is easy to verify by expanding both sides and subtracting.

We note next that using Lemma 3.1 it is a simple exercise to establish the following Fekete-Szegő theorem for functions in \(B_1(\alpha, 1)\). We omit the proof.

**Theorem 4.2.** Let \(f \in B_1(\alpha, 1)\) for \(\alpha \geq 0\). Then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{2 + \alpha} & \frac{-\alpha(5 + 3\alpha)}{2(2 + \alpha)} \leq \mu \leq \frac{4 + \alpha(3 + \alpha)}{2(2 + \alpha)}, \\
\frac{\alpha - 1 + 2\mu}{2(1 + \alpha)^2}, & \text{otherwise.}
\end{cases}
\]

The inequalities are sharp.

## 5. Inverse Coefficients

We now consider the initial coefficients of the inverse function \(f^{-1}\).

For any univalent function \(f\), there exists an inverse function \(f^{-1}\) defined on some disc \(|\omega| < r_0(f)\), with Taylor expansion

\[
f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + ...
\]

Since \(f(f^{-1}(\omega)) = \omega\), comparing coefficients from (1.1) and (5.1) gives
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\[ A_2 = -a_2 \]
\[ A_3 = -2a_2^2 + a_3 \]
\[ A_4 = -5a_2^3 + 5a_2a_3 - a_4, \]

which, on substituting from (4.3), gives

\[
\begin{align*}
A_2 &= -\frac{p_1}{1 + \alpha} \\
A_3 &= -\frac{1}{2(2 + \alpha)} \left( p_2 - \frac{8 + 9\alpha + 3\alpha^2}{4(1 + \alpha)^2} p_1^2 \right) \\
A_4 &= -\frac{1}{2(3 + \alpha)} \left( p_3 - \frac{16 + 13\alpha + 3\alpha^2}{2(1 + \alpha)(2 + \alpha)} p_1 p_2 + \frac{90 + 190\alpha + 152\alpha^2 + 53\alpha^3 + 7\alpha^4}{12(1 + \alpha)^3(2 + \alpha)} p_1^2 \right).
\end{align*}
\]

We are now able to find sharp estimates for the above coefficients.

**Theorem 5.1.** Let \( f \in B_1(\alpha, 1) \) for \( \alpha \geq 0 \), with inverse coefficients given by (5.2). Then

\[
|A_2| \leq \frac{1}{1 + \alpha}, \quad |A_3| \leq \begin{cases} 
\frac{1}{2 + \alpha}, & \alpha \geq \frac{1}{2}(1 + \sqrt{17}) \\
\frac{3 + \alpha}{2(1 + \alpha)^2}, & 0 \leq \alpha \leq \frac{1}{2}(1 + \sqrt{17}), \\
\frac{1}{3 + \alpha}, & \alpha \geq \alpha_0,
\end{cases}
\]

\[
|A_4| \leq \begin{cases} 
\frac{(2 + \alpha)(4 + \alpha)}{3(1 + \alpha)^3}, & \alpha \leq \alpha_0,
\end{cases}
\]

where \( \alpha_0 \) is the positive root of the equation \( 21 + 17\alpha - 2\alpha^3 = 0 \).

All the inequalities are sharp.

**Proof.** The inequality for \( |A_2| \) is obvious, and sharp when \( p_1 = 2 \).

For \( A_3 \) we apply Lemma 3.1 with \( \mu = \frac{8 + 9\alpha + 3\alpha^2}{2(1 + \alpha)^2} \), so that \( 0 \leq \mu \leq 2 \) when \( \alpha \geq \frac{1}{2}(1 + \sqrt{17}) \). This gives the first inequality for \( |A_3| \). The second inequality follows from Lemma 3.1 on noting that if \( \mu \) is outside the interval \([0,2]\), then \( 0 \leq \alpha \leq \frac{1}{2}(1 + \sqrt{17}) \).

The first inequality for \( |A_3| \) is sharp on choosing \( p_1 = 0 \) and \( p_2 = 2 \). The second inequality is sharp when \( p_1 = p_2 = 2 \).
For $A_4$, we first use Lemma 3.4 with $\mu = \frac{(3 + \alpha)(4 + \alpha)}{2(1 + \alpha)(2 + \alpha)}$, so that

$$A_4 = -\frac{1}{2(3 + \alpha)} \left(p_3 - (\mu + 1)p_1p_2 + \mu p_1^3 + \frac{18 + 4\alpha - 10\alpha^2 - \alpha^3 + \alpha^4}{12(1 + \alpha)^3(2 + \alpha)} p_1^3\right).$$

Noting that $\mu > 1$, when $0 \leq \alpha < \frac{1}{2}(1 + \sqrt{33})$, we use the inequality $|p_1| \leq 1/2$, and apply Lemma 3.4 to obtain the bound for $|A_4|$ on the interval $0 \leq \alpha < \frac{1}{2}(1 + \sqrt{33})$.

We now use Lemma 3.2.

From (4.2) let

$$B = \frac{16 + 13\alpha + 3\alpha^2}{4(1 + \alpha)(2 + \alpha)}, \quad \text{and} \quad D = \frac{90 + 190\alpha + 53\alpha^2 + 7\alpha^4}{12(1 + \alpha)^3(2 + \alpha)}.$$

Then $0 \leq B \leq 1$ when $\alpha \geq \frac{1}{2}(1 + \sqrt{33})$, and $B(2B - 1) \leq D \leq B$ when $\alpha \geq \alpha_0$, where $\alpha_0$ is the unique real root of the equation $21 + 17\alpha + 2\alpha^3 = 0$. Since both these inequalities are satisfied when $\alpha \geq \alpha_0$, the first inequality for $|A_4|$ follows on this interval by applying Lemma 3.2.

Thus we are left with the interval $\frac{1}{2}(1 + \sqrt{33}) \leq \alpha \leq \alpha_0$.

Write

$$A_4 = -\frac{1}{2(3 + \alpha)} \left(p_3 - 2Bp_1p_2 + Bp_1^3 + (D - B)p_1^3\right),$$

and note that $D - B = \frac{(21 + 17\alpha - 2\alpha^3)}{12(1 + \alpha)^3} \geq 0$ when $0 \leq \alpha \leq \alpha_0$. Noting that we still require that $\alpha \geq \frac{1}{2}(1 + \sqrt{33})$ (since $0 \leq B \leq 1$), we now apply Lemma 3.2 in the case $D = B$, to obtain the second inequality for $|A_4|$ on the interval $\frac{1}{2}(1 + \sqrt{33}) \leq \alpha \leq \alpha_0$.

The first inequality for $|A_4|$ is sharp on choosing $p_1 = 0$, and $p_3 = 2$. The second inequality is sharp when $p_1 = p_2 = p_3 = 2$.

6. The Fifth Inverse Coefficient

We have seen in Theorem 4.1 that it is possible to find complete and sharp bounds of the fifth coefficient of $f(z)$. Finding sharp bounds for the fifth inverse coefficient $A_5$ seems more difficult.

It is easy to see that $A_5 = 14a_4^3 - 21a_3^2a_2 + 3a_3^2 + 6a_2a_4 - a_5$, and then expressing $A_5$ in terms of the coefficients $p_1, p_2, p_3$ and $p_4$, obtain an expression similar to that found for $a_5$ in (4.3). Applying Lemma 3.3 to the resulting expression gives the sharp bound $|A_5| \leq 1/(4 + \alpha)$, provided $\alpha > 6.029\ldots$. This leaves open the problem of finding sharp bounds for $|A_5|$ on the interval $0 \leq \alpha \leq 6.029\ldots$. 

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We next give a subordination property for functions in $B_1(\alpha, 1)$ for $\alpha \geq 0$, similar to that proved by Marjono [3], noting that the result is valid for all functions in $A$.

7. SUBORDINATION

**Theorem 7.1.** Let $f \in B_1(\alpha, 1)$ for $\alpha \geq 0$, and $\gamma > 0$. Then

$$f'(z) \left( \frac{f(z)}{z} \right)^{\alpha - 1} \prec (1 + z)^{\beta(\gamma)}$$

implies

$$\left( \frac{f(z)}{z} \right)^{\alpha} \prec (1 + z)^{\gamma},$$

where

$$\beta(\gamma) = \gamma + \frac{4}{\pi} \arctan \left( \frac{\gamma}{\gamma + 2\alpha} \right).$$

**Proof.** Write

$$P(z) = \left( \frac{f(z)}{z} \right)^{\alpha},$$

so that $P$ is analytic in $D$, $P(0) = 1$ and

$$P(z) + \frac{zP'(z)}{\alpha} = \left( \frac{f(z)}{z} \right)^{\alpha - 1} f'(z).$$

We therefore need to show that

$$P(z) + \frac{zP'(z)}{\alpha} \prec (1 + z)^{\beta(\gamma)}$$

implies

$$P(z) \prec (1 + z)^{\gamma}.$$ 

For $z \in D$, let $h(z) = (1 + z)^{\beta(\gamma)}$ and $q(z) = (1 + z)^{\gamma}$, so that $|\arg h(z)| < \frac{\pi \beta(\gamma)}{4}$ and $|\arg q(z)| < \frac{\pi \gamma}{4}$.

Suppose that $p(z) \not\prec q(z)$. Then from the Clunie-Jack Lemma, there exits $z_0 \in D$ and $\zeta_0 \in \partial D$, such that $P(z_0) = q(\zeta_0)$, $(p(|z| < |z_0|) \subset q(D)$ and $z_0 p'(z_0) = k \zeta_0 q'(\zeta_0)$ for $k \geq 1$.

Thus we can write

$$P(z_0) + \frac{z_0 P'(z_0)}{\alpha} = q(\zeta_0) + \frac{\zeta_0 q'(\zeta_0)}{\alpha} = (1 + \zeta_0)^{\gamma} \left[ 1 + \frac{k\gamma \zeta_0}{\alpha (1 + \zeta_0)} \right].$$

(7.1)

Now write $\zeta_0 = e^{i\theta}$, so that (7.1) becomes

$$P(z_0) + \frac{z_0 P'(z_0)}{\alpha} = (1 + e^{i\theta})^{\gamma} \left[ \frac{1}{2} + i \frac{k\gamma}{2\alpha} \frac{\sin \theta}{1 + \cos \theta} \right].$$
Writing $\sin \theta = t$, and taking arguments, we obtain
\[
\arg \left( P(z_0) + \frac{z_0 P'(z_0)}{\alpha} \right) = \gamma \arctan \left[ \frac{t}{1 + \sqrt{1 - t^2}} \right] + \arctan \left[ \frac{k\gamma t}{(2\alpha + k\gamma) \sqrt{1 - t^2}} \right].
\]

Noting that the above expression is minimum when $t = -1$, taking the modulus and using the fact that $k \geq 1$, we deduce that
\[
\left| \arg \left( P(z_0) + \frac{z_0 P'(z_0)}{\alpha} \right) \right| \geq \frac{\gamma \pi}{4} + \arctan \left[ \frac{\gamma}{2\alpha + \gamma} \right] = \frac{\beta(\gamma) \pi}{4},
\]
which is a contradiction.

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