The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 1, pp. 1-10, 2019

ON A SUBSET OF BAZILEVIČ FUNCTIONS

MARJONO AND D. K. THOMAS

Received 27 August, 2018; accepted 9 January, 2019; published 31 July, 2019.

MARJONO, DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND NATURAL SCIENCES, BRAWAJAYA UNIVERSITY, MALANG, JAWA TIMUR 65145, INDONESIA.
marjono@ub.ac.id

D. K. THOMAS, DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA, SA2 8PP, UNITED KINGDOM.
d.k.thomas@swansea.ac.uk

ABSTRACT. Let $\mathcal{S}$ denote the class of analytic and univalent functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $\alpha \geq 0$, the subclass $B_1(\alpha)$ of $\mathcal{S}$ of Bazilevič functions has been extensively studied. In this paper we determine various properties of a subclass of $B_1(\alpha)$, for $\alpha \geq 0$, which extends early results of a class of starlike functions studied by Ram Singh.

Key words and phrases: Univalent functions, Starlike functions, Bazilevič functions, Coefficients.

2010 *Mathematics Subject Classification.* Primary 30C45, 30C55.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND DEFINITIONS

Denote by $\mathcal{A}$, the set of functions f, which are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and normalized so that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and by $\mathcal{S}$, the subset of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{D}$.

In recent years, a great deal of attention (see e.g. [2], [4], [9], [10]), has been given to the set $B_1(\alpha)$ of Bazilevič functions in $\mathcal{S}$, defined for $\alpha \geq 0$, as follows.

Definition 1.1. Let $f \in \mathcal{A}$ and be given by (1.1). Then for $\alpha \geq 0$, $f \in B_1(\alpha)$ if, and only if, for $z \in \mathbb{D}$

$$\operatorname{Re} f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} > 0. \tag{1.2}$$

Clearly $B_1(0)$ consists of the well-known class $\mathcal{S}^*$ of starlike functions, and $B_1(1)$ the class $\mathcal{R}$ whose elements satisfy $\operatorname{Re} f'(z) > 0$, for $z \in \mathbb{D}$.

Finding sharp bounds for $|a_n|$ for all $n \geq 2$ when $f \in B_1(\alpha)$ remains an open problem, with best possible bounds only known when $2 \leq n \leq 6$, [3], [8], and even then, only partial answers have been given when $n = 5$ and 6.

When $\alpha = -1$ in (1.2), functions defined by the following are also members of $\mathcal{S}$, [5], and provide an interesting subset of $\mathcal{S}$ which is known as the class $\mathcal{U}(\lambda)$. The class $\mathcal{U}(\lambda)$ defined below, has also been extensively studied in recent years (see e.g. [5], [6], and the references in these papers).

Definition 1.2. Let $f \in \mathcal{A}$ and be given by (1.1). Then $f \in \mathcal{U}(\lambda)$ if, and only if, for $z \in \mathbb{D}$

$$\left| f'(z)\Big(\frac{z}{f(z)}\Big)^2 - 1 \right| < \lambda.$$

It is clear from the definition that since $f'(z)/[z/f(z)]^2 \neq 0$, functions in $\mathcal{U}(\lambda)$ are non-vanishing in $\mathbb{D}\backslash\{0\}$, and locally univalent.

Finding sharp bounds for the coefficients of functions in $\mathcal{U}(\lambda)$ appears to be a difficult problem, with best possible bounds only known when $2 \leq n \leq 4$, [6]. On the other hand when $\lambda = 1$, sharp bound have been found for all $n \geq 2$ (see e.g. [6]).

In this paper we study a subset of $B_1(\alpha)$, whose definition mimics that of $\mathcal{U}(\lambda)$ in the case $\lambda = 1$, and show that it is possible to obtain sharp bounds for the first five coefficients of $f(z)$, together with the first four coefficients of the inverse function. We also give other properties of this subclass, which we define as follows.

Definition 1.3. Let $f \in \mathcal{A}$ and be given by (1.1). Then for $\alpha \geq 0$, $f \in B_1(\alpha, 1)$ if, and only if, for $z \in \mathbb{D}$,

$$\Big| f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} - 1\Big| < 1. \tag{1.3}$$

We note that when $\alpha = 0$, (1.3) reduces to

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < 1,$$

considered in [8]. Since the analysis for $\alpha = 0$ and $\alpha > 0$ can differ, we will specify this when appropriate.

2. REPRESENTATION EXPRESSION AND DISTORTION THEOREMS

We begin by giving a representation formula for $f \in B_1(\alpha, 1)$ when $\alpha > 0$, analogous to that given in [8] in the case $\alpha = 0$.

Theorem 2.1. *For $\alpha > 0$, $f \in B_1(\alpha, 1)$ if, and only if,*

$$f(z) = \Big(\alpha \int_0^z t^{\alpha-1}(1 + \omega(t))dt\Big)^{1/\alpha}, \tag{2.1}$$

where ω is analytic in $\mathbb{D}$ satisfying $|\omega(z)| \leq 1$, and $\omega(0) = 0$.

Proof. From (1.3), we can write

$$f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} = 1 + \omega(z). \tag{2.2}$$

Let $\phi(z) = \Big(\frac{f(z)}{z}\Big)^{\alpha}$. Then differentiation gives

$$\phi'(z) + \frac{\alpha}{z}\phi(z) = \frac{\alpha}{z}(1 + \omega(z)).$$

Multiplying by z^α and integrating gives (2.1). ∎

Theorem 2.2. *For $\alpha > 0$, let $f \in B_1(\alpha, 1)$, $z = re^{i\theta} \in \mathbb{D}$, and*

$$\beta_1(\alpha, r) = \Big(\frac{1 + \alpha + \alpha r}{1 + \alpha}\Big), \;\; \beta_2(\alpha, r) = \Big(\frac{1 + \alpha - \alpha r}{1 + \alpha}\Big).$$

Then

$$r\beta_2(\alpha, r)^{1/\alpha} \leq |f(z)| \leq r\beta_1(\alpha, r)^{1/\alpha} \tag{2.3}$$

$$(1 - r)\beta_2(\alpha, r)^{(1-\alpha)/\alpha} \leq |f'(z)| \leq (1 + r)\beta_1(\alpha, r)^{(1-\alpha)/\alpha} \tag{2.4}$$

$$\frac{1 - r}{\beta_1(\alpha, r)} \leq \Big|\frac{zf'(z)}{f(z)}\Big| \leq \frac{(1 + r)}{\beta_2(\alpha, r)}. \tag{2.5}$$

Equality holds in all cases when $f(z) = z\Big(\dfrac{1+\alpha+\alpha z}{1+\alpha}\Big)^{1/\alpha}$ for $\theta = 0$, or $\pi/2$.

Proof. It follows from the Schwarz Lemma that $|\omega(z)| \le |z|$. Using this in (2.1) and integrating easy establishes the right-hand inequality in (2.3). The left-hand inequality follows from the minimum principle for harmonic functions. Differentiating (2.1) and using (2.2) gives (2.4), from which (2.5) follows on noting that $|1+\omega(z)| \ge 1-|\omega(z)| \ge 1-|z|$. ∎

From (2.3), we at once deduce the following.

Corollary 2.1. *Let $f \in B_1(\alpha, 1)$ for $\alpha > 0$. Then $f(\mathbb{D})$ contains the disk $\{w : |w| < 1/(1+\alpha)^{1/\alpha}\}$.*

We note that letting $\alpha \to 0$ in the results of Theorem 2.2 and Corollary 2.1, gives those obtained in [9].

We will use the following lemmas, the first two and the fourth of which can be found in [1], and the third in [7].

3. LEMMAS

Denote by $\mathcal{P}$, the class of functions p of positive real part, i.e., functions satisfying $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, with Taylor expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3.1}$$

Lemma 3.1. *If $p \in \mathcal{P}$, then*

$$\Big|p_2 - \frac{\mu}{2}p_1^2\Big| \le \max\{2,\ 2|\mu-1|\} = \begin{cases} 2, & 0 \le \mu \le 2, \\ 2|\mu-1|, & \textit{elsewhere.} \end{cases}$$

Lemma 3.2. *Let $p \in \mathcal{P}$. If $0 \le B \le 1$ and $B(2B-1) \le D \le B$, then*

$$\big|p_3 - 2Bp_1p_2 + Dp_1^3\big| \le 2.$$

Lemma 3.3. *If $p \in \mathcal{P}$, and $\alpha_1, \alpha_2, \beta$ and γ satisfy $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, and*

$$8\alpha_1(1-\alpha_1)((\alpha_2\beta - 2\gamma)^2 + (\alpha_2(\alpha_1+\alpha_2) - \beta)^2) + \alpha_2(1-\alpha_2)(\beta - 2\alpha_1\alpha_2)^2 \\ \le 4\alpha_2^2(1-\alpha_2)^2\alpha_1(1-\alpha_1),$$

then

$$|\gamma p_1^4 + \alpha_1 p_2^2 + 2\alpha_2 p_1 p_3 - (3/2)\beta p_1^2 p_2 - p_4| \le 2.$$

Lemma 3.4. *If $p \in \mathcal{P}$, then*

$$\big|p_3 - (\mu+1)p_1p_2 + \mu p_1^3\big| \le \max\{2,\ 2|2\mu-1|\} = \begin{cases} 2, & 0 \le \mu \le 1, \\ 2|2\mu-1|, & \textit{elsewhere.} \end{cases}$$

4. COEFFICIENT INEQUALITIES

Theorem 4.1. *Let* $f \in B_1(\alpha, 1)$ *for* $\alpha \geq 0$, *and be given by* (1.1). *Then for* $2 \leq n \leq 5$,

$$|a_n| \leq \frac{1}{\alpha + n - 1}.$$

The inequalities are sharp.

Proof. Recall from (1.3), that we can write

$$f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} = 1 + \omega(z), \tag{4.1}$$

where $\omega(z)$ is analytic in $\mathbb{D}$, $|\omega(z)| \leq 1$, and $\omega(0) = 0$.

Since $p \in \mathcal{P}$, we can therefore write

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)}, \qquad \text{or} \qquad \omega(z) = \frac{p(z)-1}{p(z)+1}. \tag{4.2}$$

From (2.2), (3.1), (4.1) and (4.2), equating coefficients we obtain

$$\begin{aligned}
a_2 =& \frac{p_1}{2(1+\alpha)}\\
a_3 =& \frac{1}{2(2+\alpha)}\Big(p_2 - \frac{a(5+3\alpha)}{4(1+\alpha)^2(2+\alpha)}p_1^2\Big)\\
a_4 =& \frac{1}{2(3+\alpha)}\Big(p_3 - \frac{1+8\alpha+3\alpha^2)}{2(1+\alpha)(2+\alpha)}p_1p_2 + \frac{\alpha(5+64\alpha+61\alpha^2+14\alpha^3)}{24(1+\alpha)^3)(2+\alpha)}p_1^3\\
a_5 =& \frac{1}{2(4+\alpha)}\Big(\frac{\alpha(8+544\alpha+3557\alpha^2+5389\alpha^3+3329\alpha^4+907\alpha^5+90\alpha^6)}{192(1+\alpha)^4(2+\alpha)^2(3+\alpha)}p_1^4\\
&+ \frac{4+11\alpha+3\alpha^2}{4(2+\alpha)^2}p_2^2 + \frac{2+11\alpha+3\alpha^2}{2(1+\alpha)(3+\alpha)}p_1p_3\\
&- \frac{8+76\alpha+325\alpha^2+324\alpha^3+117\alpha^4+14\alpha^5}{8(1+\alpha)^2(2+\alpha)^2(3+\alpha)}p_1^2p_2 - p_4\Big).
\end{aligned} \tag{4.3}$$

From (4.3) the inequality for a_2 is obvious.

For a_3 we apply Lemma 3.1 with $\mu = \dfrac{\alpha(5+3\alpha)}{2(1+\alpha)^2}$, which gives the inequality for $|a_3|$, since $0 \leq \mu \leq 2$ in this case.

For a_4 we use Lemma 3.2 with

$$B = \frac{1+8\alpha+3\alpha^2}{4(1+\alpha)(2+\alpha)},$$

and

$$D = \frac{\alpha(5 + 64\alpha + 61\alpha^2 + 14\alpha^3)}{24(1+\alpha)^3(2+\alpha)}.$$

It is easily verified that both $0 \le B \le 1$, and $B(2B-1) \le D \le B$, when $\alpha \ge 0$, and so applying Lemma 3.2 gives the required inequality for $|a_4|$.

For a_5, we apply Lemma 3.3 with α_1, α_2, β and γ the respective coefficients of a_5 in (4.3), so that we need to show that

$$\begin{aligned}
&(1-\alpha)^2(4+a)^2(12544 + 427648\alpha + 5441392\alpha^2 + 33366608\alpha^3 + 117462812\alpha^4 \\
&+ 260385736\alpha^5 + 382475767\alpha^6 + 388520160\alpha^7 + 282592930\alpha^8 + 150937228\alpha^9 \\
&+ 60100454\alpha^{10} + 17921756\alpha^{11} + 3972584\alpha^{12} + 639452\alpha^{13} + 71147\alpha^{14} + 4932\alpha^{15} \\
&+ 162\alpha^{16}) \\
&\le 288(12 + 5\alpha + \alpha^2)(2 + 11\alpha + 3\alpha^2)^2(4 + 11\alpha + 3\alpha^2)(1+\alpha)^6(2+\alpha)^4(3+\alpha)^2.
\end{aligned} \tag{4.4}$$

To see that this inequality is true, write the left-hand side of the above inequality as $(1-\alpha)^2(4+\alpha)^2\phi_1(\alpha)$, and the right-hand side as $\phi_2(\alpha)$. Then clearly $(1-\alpha)^2(4+\alpha)^2\phi_1(\alpha) \le (4+\alpha)^2\phi_1(\alpha)$.

Thus it enough to show that $(4+\alpha)^2\phi_1(\alpha) \le \phi_2(\alpha)$ when $\alpha \ge 0$, which is easy to verify by expanding both sides and subtracting.
∎

We note next that using Lemma 3.1, it is a simple exercise to establish the following Fekete-Szegő theorem for functions in $B_1(\alpha, 1)$. We omit the proof.

Theorem 4.2. *Let $f \in B_1(\alpha, 1)$ for $\alpha \ge 0$. Then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{1}{2+\alpha}, & -\dfrac{\alpha(5+3\alpha)}{2(2+\alpha)} \le \mu \le \dfrac{4+\alpha(3+\alpha)}{2(2+\alpha)}, \\ \dfrac{\alpha - 1 + 2\mu}{2(1+\alpha)^2}, & \text{otherwise.} \end{cases}$$

The inequalities are sharp.

5. INVERSE COEFFICIENTS

We now consider the initial coefficients of the inverse function f^{-1}.

For any univalent function f, there exists an inverse function f^{-1} defined on some disc $|\omega| < r_0(f)$, with Taylor expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots \tag{5.1}$$

Since $f\left(f^{-1}(\omega)\right) = \omega$, comparing coefficients from (1.1) and (5.1) gives

$$A_2 = -\,a_2$$
$$A_3 = -\,2{a_2}^2 + a_3$$
$$A_4 = -\,5{a_2}^3 + 5a_2a_3 - a_4,$$

which, on substituting from (4.3), gives

(5.2)
$$A_2 = -\,\frac{p_1}{1+\alpha}$$
$$A_3 = -\,\frac{1}{2(2+\alpha)}\Big(p_2 - \frac{8+9\alpha+3\alpha^2}{4(1+\alpha)^2}p_1^2\Big)$$
$$A_4 = -\,\frac{1}{2(3+\alpha)}\Big(p_3 - \frac{16+13\alpha+3\alpha^2}{2(1+\alpha)(2+\alpha)}p_1p_2 + \frac{90+190\alpha+152\alpha^2+53\alpha^3+7\alpha^4}{12(1+\alpha)^3(2+\alpha)}p_1^3\Big).$$

We are now able to find sharp estimates for the above coefficients.

Theorem 5.1. *Let $f \in B_1(\alpha,1)$ for $\alpha \geq 0$, with inverse coefficients given by (5.2). Then*

$$|A_2| \leq \frac{1}{1+\alpha}, \qquad |A_3| \leq \begin{cases} \dfrac{1}{2+\alpha}, & \alpha \geq \dfrac{1}{2}(1+\sqrt{17}), \\[2ex] \dfrac{3+\alpha}{2(1+\alpha)^2}, & 0 \leq \alpha \leq \dfrac{1}{2}(1+\sqrt{17}), \end{cases}$$

$$|A_4| \leq \begin{cases} \dfrac{1}{3+\alpha}, & \alpha \geq \alpha_0, \\[2ex] \dfrac{(2+\alpha)(4+\alpha)}{3(1+\alpha)^3}, & \alpha \leq \alpha_0, \end{cases}$$

where α_0 is the positive root of the equation $21 + 17\alpha - 2\alpha^3 = 0$.

All the inequalities are sharp.

Proof. The inequality for $|A_2|$ is obvious, and sharp when $p_1 = 2$.

For A_3 we apply Lemma 3.1 with $\mu = \dfrac{8+9\alpha+3\alpha^2}{2(1+\alpha)^2}$, so that $0 \leq \mu \leq 2$ when $\alpha \geq \dfrac{1}{2}(1+\sqrt{17})$. This gives the first inequality for $|A_3|$. The second inequality follows from Lemma 3.1 on noting that if μ is outside the interval [0,2], then $0 \leq \alpha \leq \dfrac{1}{2}(1+\sqrt{17})$.

The first inequality for $|A_3|$ is sharp on choosing $p_1 = 0$ and $p_2 = 2$. The second inequality is sharp when $p_1 = p_2 = 2$.

For A_4, we first use Lemma 3.4 with $\mu = \dfrac{(3+\alpha)(4+\alpha)}{2(1+\alpha)(2+\alpha)}$, so that

$$A_4 = -\frac{1}{2(3+\alpha)}\Big(p_3 - (\mu+1)p_1p_2 + \mu p_1^3 + \frac{(18+4\alpha-10\alpha^2-\alpha^3+\alpha^4)}{12(1+\alpha)^3(2+\alpha)}p_1^3\Big).$$

Noting that $\mu > 1$, when $0 \le \alpha < \frac{1}{2}(1+\sqrt{33})$, we use the inequality $|p_1| \le 2$, and apply Lemma 3.4 to obtain the bound for $|A_4|$ on the interval $0 \le \alpha < \frac{1}{2}(1+\sqrt{33})$.

We now use Lemma 3.2.

From (4.2) let

$$B = \frac{16+13\alpha+3\alpha^2}{4(1+\alpha)(2+\alpha),} \qquad \text{and} \qquad D = \frac{90+190\alpha+152\alpha^2+53\alpha^3+7\alpha^4}{12(1+\alpha)^3(2+\alpha)}.$$

Then $0 \le B \le 1$ when $\alpha \ge \frac{1}{2}(1+\sqrt{33})$, and $B(2B-1) \le D \le B$ when $\alpha \ge \alpha_0$, where α_0 is the unique real root of the equation $21 + 17\alpha + 2\alpha^3 = 0$. Since both these inequalities are satisfied when $\alpha \ge \alpha_0$, the first inequality for $|A_4|$ follows on this interval by applying Lemma 3.2.

Thus we are left with the interval $\frac{1}{2}(1+\sqrt{33}) \le \alpha \le \alpha_0$.

Write

$$A_4 = -\frac{1}{2(3+a)}\Big(p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3\Big),$$

and note that $D - B = \dfrac{(21+17\alpha-2\alpha^3)}{12(1+\alpha)^3} \ge 0$ when $0 \le \alpha \le \alpha_0$. Noting that we still require that $\alpha \ge \frac{1}{2}(1+\sqrt{33})$ (since $0 \le B \le 1$), we now apply Lemma 3.2 in the case $D = B$, to obtain the second inequality for $|A_4|$ on the interval $\frac{1}{2}(1+\sqrt{33}) \le \alpha \le \alpha_0$.

The first inequality for $|A_4|$ is sharp on choosing $p_1 = 0$, and $p_3 = 2$. The second inequality is sharp when $p_1 = p_2 = p_3 = 2$.

∎

6. The Fifth Inverse Coefficient

We have seen in Theorem 4.1 that it is possible to find complete and sharp bounds of the fifth coefficient of $f(z)$. Finding sharp bounds for the fifth inverse coefficient A_5 seems more difficult.

It is easy to see that $A_5 = 14a_2^4 - 21a_2^2a_3 + 3a_3^2 + 6a_2a_4 - a_5$, and then expressing A_5 in terms of the coefficients p_1, p_2, p_3 and p_4, obtain an expression similar to that found for a_5 in (4.3). Applying Lemma 3.3 to the resulting expression gives the sharp bound $|A_5| \le 1/(4+\alpha)$, provided $\alpha > 6.029\ldots$. This leaves open the problem of finding sharp bounds for $|A_5|$ on the interval $0 \le \alpha \le 6.029\ldots$.

We next give a subordination property for functions in $B_1(\alpha, 1)$ for $\alpha \geq 0$, similar to that proved by Marjono [3], noting that the result is valid for all functions in $\mathcal{A}$.

7. SUBORDINATION

Theorem 7.1. *Let $f \in B_1(\alpha, 1)$ for $\alpha \geq 0$, and $\gamma > 0$. Then*

$$f'(z)\Big(\frac{f(z)}{z}\Big)^{\alpha-1} \prec (1+z)^{\beta(\gamma)}$$

implies

$$\Big(\frac{f(z)}{z}\Big)^{\alpha} \prec (1+z)^{\gamma},$$

where

$$\beta(\gamma) = \gamma + \frac{4}{\pi}\arctan\Big(\frac{\gamma}{\gamma+2\alpha}\Big).$$

Proof. Write

$$P(z) = \Big(\frac{f(z)}{z}\Big)^{\alpha},$$

so that P is analytic in $\mathbb{D}$, $P(0) = 1$ and

$$P(z) + \frac{zP'(z)}{\alpha} = \Big(\frac{f(z)}{z}\Big)^{\alpha-1} f'(z).$$

We therefore need to show that

$$P(z) + \frac{zP'(z)}{\alpha} \prec (1+z)^{\beta(\gamma)}$$

implies

$$P(z) \prec (1+z)^{\gamma}.$$

For $z \in \mathbb{D}$, let $h(z) = (1+z)^{\beta(\gamma)}$ and $q(z) = (1+z)^{\gamma}$, so that $|\arg h(z)| < \dfrac{\pi\beta(\gamma)}{4}$ and $|\arg q(z)| < \dfrac{\pi\gamma}{4}$.

Suppose that $p(z) \not\prec q(z)$. Then from the Clunie-Jack Lemma, there exits $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}$, such that $P(z_0) = q(\zeta_0)$, $(p(|z| < |z_0|) \subset q(\mathbb{D})$ and $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$ for $k \geq 1$.

Thus we can write

$$\begin{aligned} P(z_0) + \frac{z_0 P'(z_0)}{\alpha} &= q(\zeta_0) + \frac{\zeta_0 q'(\zeta_0)}{\alpha} \\ &= (1+\zeta_0)^{\gamma}\Big[1 + \frac{k\gamma\zeta_0}{\alpha(1+\zeta_0)}\Big]. \end{aligned} \tag{7.1}$$

Now write $\zeta_0 = e^{i\theta}$, so that (7.1) becomes

$$P(z_0) + \frac{z_0 P'(z_0)}{\alpha} = (1+e^{i\theta})^{\gamma}\Big[\frac{1}{2} + i\frac{k\gamma}{2\alpha}\frac{\sin\theta}{1+\cos\theta}\Big].$$

Writing $\sin\theta = t$, and taking arguments, we obtain

$$\arg\left(P(z_0) + \frac{z_0 P'(z_0)}{\alpha}\right) = \gamma \arctan\left[\frac{t}{1+\sqrt{1-t^2}}\right] + \arctan\left[\frac{k\gamma t}{(2\alpha + k\gamma)\sqrt{1-t^2}}\right].$$

Noting that the above expression is minimum when $t = -1$, taking the modulus and using the fact that $k \geq 1$, we deduce that

$$\left|\arg\left(P(z_0) + \frac{z_0 P'(z_0)}{\alpha}\right)\right| \geq \frac{\gamma\pi}{4} + \arctan\left[\frac{\gamma}{2\alpha+\gamma}\right] = \frac{\beta(\gamma)\pi}{4},$$

which is a contradiction. ∎

REFERENCES

[1] R. M. ALI, Coefficients of the inverse of strongly starlike functions, *Bull. Malaysian Math. Soc.* , **26** (2003), pp. 63–71.

[2] MARJONO & D. K.THOMAS, A note on the powers of Bazilevič functions, *International Journal of Mathematical Analysis*, **9(42)** (2015) pp. 2061–2067.

[3] MARJONO, Subordination of analytic functions, *Australian Journal of Mathematical Analysis and Applications*, **14(1)** (2017), pp. 1–5.

[4] MARJONO, J. SOKÓL & D. K. THOMAS, The fifth and sixth coefficients for Bazilevič functions $B_1(\alpha)$, *Mediterranean J. Math.*, **14(4)**: 158 (2017).

[5] M. OBRADOVIČ & S. PONNUSAMY, On the class $\mathcal{U}$. *Proceedings 21st Annual Conference of the Jammu. Math. Soc.* February 25–27, (2011).

[6] M. OBRADOVIČ, S. PONNUSAMY & K-J. WIRTHS, Logarithmic coefficients and a coefficient conjecture for univalent functions, *arXiv:1701.05413v1[math.CV] 19 Jan 2017*.

[7] V. RAVICHANDRAN & S. VERMA, Bound for the fifth coefficient of certain starlike functions, *Comptes Rendus Mathematique*, **353**, 6, (2015), pp. 505–510.

[8] R. SINGH, On a Class of star-like functions, *Compositio Mathematica*, **19** no 1, (1968), pp. 78–82.

[9] R. SINGH, On Bazilevič functions, *Proc. Amer. Math. Soc.*, **38**, (1973), pp. 261–271.

[10] D. K. THOMAS, On the coefficients of Bazilevič functions with logarithmic growth, *Indian J. Math.*, **57**. no 3, (2015), pp. 403–418.