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HANKEL OPERATORS ON COPSON'S SPACES

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ABSTRACT. We give a characterization of boundedness of a Hankel matrix, generated by a pozitive decreasing sequence, acting on Copson's space $cop(2)$.

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It is well-known that $f \in L^1([0,2\pi])$, respectively $f \in L^\infty([0,2\pi])$, for $f(\theta) = \sum_{n=0}^\infty a_n \sin n\theta$, $a_n \downarrow 0,\ \theta \in [0,2\pi]$, if and only if

$$\sum_{n=0}^\infty \frac{a_n}{n+1} < \infty,$$

respectively

$$\sup (n+1) a_n < \infty.$$

(See [2]-vol. 1 Thm 7.3.3, and Thm 7.2.2 (4).)

The analogon of these results in the analytic case, that is whenever $f(z) = \sum_{n=0}^\infty a_n z^n,\ |z| < 1$, belong to H^1, repective in $BMOA$, was given by Pavlovic [3], respectively by Xiao [6].

Motivated by the previous papers we introduced in [4] the Banach lattices generated by the cone

$$\mathcal{M}_d^+ = \{f(z) = \sum_{k=0}^\infty a_k z^k;\ |z| < 1 \text{ and } a_k \downarrow_k 0\},$$

of all analytic functions from the Hardy spaces $H^p,\ 1 \le p < \infty$.

We denote this Banach lattice by H_d^p and proved that it is actually

$$H_d^p = \{f(z) = \sum_{k=0}^\infty a_k z^k;\ |z| < 1 \text{ and } (a_k) \in bv_0\},$$

equipped with the norm

$$||f||_{H_d^p} = (\sum_{k=0}^\infty (n+1)^{p-2} (|a|_{bv})_n^p)^{1/p} < \infty,$$

where by bv we mean the Banach space of sequences of real numbers $a = (a_n)_{n\ge 0}$ with bounded variation $||a||_{bv} := |\alpha| + \sum_{n=0}^\infty |a_n - a_{n+1}|$, where $\lim_n a_n = \alpha$. It is well-known and easy to prove that bv is a Banach lattice for the order induced by the cone $C := \{a = (a_n)_{n\ge 0}, a_n \downarrow_n \alpha \ge 0\}$.

We recall that the modulus of $a \in bv$, denoted by $|a|_{bv}$, is defined by

$$(|a|_{bv})_n := |\alpha| + \sum_{k=n}^\infty |a_k - a_{k+1}|,\ \forall n \ge 0.$$

If $\alpha = 0$, for all sequences from bv, the corresponding space is denoted by bv_0, and the latter is a Banach lattice with the norm given by $|a|_{bv_0} := \sum_{k=0}^\infty |a_k - a_{k+1}|$.

Let $(a_n)_{n\ge 0}$ be a sequence of positive real numbers with $\sum_n a_n^2 < \infty$. The infinite matrix

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & \dots & \dots \\ a_3 & a_4 & \dots & \dots & \dots \\ a_4 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

having the constant entries on each skew-diagonal, is called a *Hankel matrix.*

Denote by ℓ_d^2 the subspace $C - C$ of ℓ^2, where $C := \{a = (a_n)_{n\ge 0} \in \ell^2;\ a_n \downarrow_n 0\}$. ℓ_d^2 is equipped with the norm

$$||a|| := \inf_{a = a^1 - a^2,\, a^1, a^2 \in C} \left((\sum_{n=0}^\infty (a_n^1)^2)^{1/2} + (\sum_{n=0}^\infty (a_n^2)^2)^{1/2} \right) \sim \left(\sum_{n=0}^\infty (|a|_{bv_0})_n^2 \right)^{1/2}.$$

Of course, so equipped, ℓ_d^2 is a Banach lattice isomorphic to H_d^2. (See [4].)In [4], [5] it was stated and proved the following result:

Theorem 1.1. *Let A be the Hankel matrix defined as above, where the sequence $(a_n)_{n\geq 0}$ is, moreover, monotone decreasing $a_n \downarrow_n 0$. Then A determine a bounded operator from ℓ_d^2 into ℓ_d^2 if and only if* $\sup_{n\geq 0}(n+1)a_n < \infty$,

We call this operator *a Hankel operator on ℓ_d^2.*

ℓ_d^2 is isomorphic as a Banach lattice with the classical *Copson's sequence space* $cop(2)$. See [5] - Corollary 1.7.

Here by $cop(2)$ we mean:

$$cop(2) := \left\{ x = (x_k)_k : \sum_{k=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k+1} \right)^2 < \infty \right\}, \text{ with the quasi-norm}$$

$$||x||_{cop(p)} = \left(\sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k+1} \right)^2 \right)^{1/2}.$$

See [1] for more details on Copson's space.

The isomorphism $T : \ell_d^2 \to cop(2)$ is given by:

$$T(x) = u, \text{ where } u_k = (k+1)[x_k - x_{k+1}],\ k \geq 0, \text{ and,}$$

$$T^{-1}(u) = x,\ x_n = \sum_{k=n}^{\infty} \frac{u_k}{k+1},\ n \geq 0.$$

See [5].

Motivated by Theorem 1 we ask ourselves whenever the Hankel matrix

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & \dots & \dots \\ a_3 & a_4 & \dots & \dots & \dots \\ a_4 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $a_k \downarrow_k 0$, is bounded on $cop(2)$.

The answer is given by the following:

Theorem 1.2. *Let $a_n \downarrow 0$, and*

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & \dots & \dots \\ a_3 & a_4 & \dots & \dots & \dots \\ a_4 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then A maps boundedly $cop(2)$ into $cop(2)$ if and only if

$$\sup_{n\geq 1} na_n < \infty.$$

Proof. Let $a_n \downarrow 0$, A as previously and $(x_n)_n \in cop(2),\ x_n \geq 0,\ \forall n.$

Denote by $(y_n)_{n\geq 1}$ $(Ax) \in cop(2)$.

Then

$$||A||^2 := \sup_{||x||_{cop(2)}\le 1}\left((\sum_{n=1}^{\infty}\frac{|y_n|}{n})^2+(\sum_{n=2}^{\infty}\frac{|y_n|}{n})^2+(\sum_{n=3}^{\infty}\frac{|y_n|}{n})^2+\cdots\right)$$

$$= \sup_{||x||_{cop(2)}\le 1}\left\{\left[(\sum_{k=1}^{\infty}\frac{a_k}{k})x_1+(\sum_{k=2}^{\infty}\frac{a_k}{k-1})x_2+(\sum_{k=3}^{\infty}\frac{a_k}{k-2})x_3+\cdots\right]^2 + \left[(\sum_{k=2}^{\infty}\frac{a_k}{k})x_1+(\sum_{k=3}^{\infty}\frac{a_k}{k-1})x_2+\cdots\right]^2+\cdots\right\}$$

$$= \sup_{||x||_{cop(2)}\le 1}\left\{\left\langle\begin{pmatrix} a_1 & a_2 & a_3 & \dots\\ a_2 & a_3 & \dots & \dots\\ a_3 & \dots & \dots & \dots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\1/2\\1/3\\ \vdots\end{pmatrix},\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\end{pmatrix}\right\rangle^2 \left\langle\begin{pmatrix} a_2 & a_3 & a_4 & \dots\\ a_3 & a_4 & \dots & \dots\\ a_4 & \dots & \dots & \dots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\1/2\\1/3\\ \vdots\end{pmatrix},\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\end{pmatrix}\right\rangle^2 +\cdots\right\}$$

$$= \sup_{||x||_{cop(2)}\le 1}\ \sup_{\alpha_i\ge 0,\sum_i\alpha_i^2=1}\left[\left\langle\alpha_1\begin{pmatrix} a_1 & a_2 & a_3 & \dots\\ a_2 & a_3 & \dots & \dots\\ a_3 & \dots & \dots & \dots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\1/2\\1/3\\ \vdots\end{pmatrix} +\alpha_2\begin{pmatrix} a_2 & a_3 & a_4 & \dots\\ a_3 & a_4 & \dots & \dots\\ a_4 & \dots & \dots & \dots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\1/2\\1/3\\ \vdots\end{pmatrix}+\cdots,\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\end{pmatrix}\right\rangle\right]^2$$

$$= \sup_{\substack{\alpha_i\ge 0,\\ \sum_i\alpha_i^2=1}}\left\langle\begin{pmatrix} \alpha_1a_1+\alpha_2a_2+\cdots & \alpha_1a_2+\alpha_2a_3+\cdots & \alpha_1a_3+\alpha_2a_4+\cdots\\ \alpha_1a_2+\alpha_2a_3+\cdots & \alpha_1a_3+\alpha_2a_4+\cdots & \alpha_1a_4+\alpha_2a_5+\cdots\\ \alpha_1a_3+\alpha_2a_4+\cdots & \alpha_1a_4+\alpha_2a_5+\cdots & \alpha_1a_5+\alpha_2a_6+\cdots\\ \vdots & \vdots & \vdots\end{pmatrix} \times\begin{pmatrix}1\\1/2\\1/3\\ \vdots\end{pmatrix},\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\end{pmatrix}\right\rangle^2.$$

Let $d(2)$ the Banach sequence space:

$$d(2) := \{a=(a_n)_n;\ ||a||_{d(2)} = (\sum_{n=1}^{\infty}\sup_{k\ge n}|a_k|^2\})^{1/2} < \infty.$$

See [1]. It is known, by Corollary 12.17 -[1], that the Köthe dual $cop(2)^{\times}$ of $cop(2)$ coincide with $d(2)$.

Consequently, denoting by $b_i \sum_{k=1}^{\infty} \alpha_k a_{k+i},\ i \geq 1,$

$$||A||^2 = \sup_{\substack{\alpha_i \geq 0, \\ \sum_i \alpha_i^2 = 1}} \left\| \begin{pmatrix} b_1 & b_2 & b_3 & \dots \\ b_2 & b_3 & \dots & \dots \\ b_3 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \\ \vdots \end{pmatrix} \right\|_{d(2)}^2 .$$

Since the sequence in the norm is positive and monotonically decreasing we have:

$$||A||^2 = \sup_{\substack{\alpha_i \geq 0, \\ \sum_i \alpha_i^2 = 1}} \left\| \begin{pmatrix} b_1 & b_2 & b_3 & \dots \\ b_2 & b_3 & \dots & \dots \\ b_3 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \\ \vdots \end{pmatrix} \right\|_{\ell^2}^2$$

$$= \sup_{\substack{\alpha_i, \beta_i \geq 0, \\ \sum_i \alpha_i^2 = 1, \sum_i \beta_i^2 = 1}} \left[\beta_1 (b_1 + \frac{1}{2}\beta_2 + \frac{1}{3} b_3 + \cdots) + \beta_2 (b_2 + \frac{1}{2} + \frac{1}{3} b_4 + \cdots) + \cdots \right]^2$$

$$= \sup_{\substack{\alpha_i, \beta_i \geq 0, \\ \sum_i \alpha_i^2 = 1, \sum_i \beta_i^2 = 1}} \Bigg\{ \alpha_1 \left[\beta_1 \sum_{k=1}^{\infty} \frac{1}{k} a_k + \beta_2 \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} + \beta_3 \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} + \cdots \right]$$

$$+ \alpha_2 \left[\beta_1 \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} + \beta_2 \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} + \cdots + \beta_3 \sum_{k=1}^{\infty} \frac{1}{k} a_{k+3} + \cdots \right] + \cdots \Bigg\}^2$$

$$= \sup_{\substack{\beta_i \geq 0, \\ \sum_i \beta_i^2 = 1}} \left\| \begin{pmatrix} \sum_{k=1}^{\infty} \frac{1}{k} a_k & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots \\ \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots & \dots \\ \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix} \right\|_{\ell^2}^2$$

$$= \left\| \begin{pmatrix} \sum_{k=1}^{\infty} \frac{1}{k} a_k & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+1} & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots \\ \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots & \dots \\ \sum_{k=1}^{\infty} \frac{1}{k} a_{k+2} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right\|_{B(\ell^2)}^2 .$$

Using Theorem 4.1 -[4] and Corollary 3.3.1-[6] we get that

$$||A||^2 = \sup_n n \sum_{k=n}^{\infty} \frac{1}{k} a_{n+k-1} \approx sup_n n a_n .$$

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