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# A KIND OF FUNCTION SERIES AND ITS APPLICATIONS

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ABSTRACT. A kind of new function series is obtained in this paper. Their theorems and proofs are shown, and some applications are given. We give the expansion form of general integral and the series expansion form of function and the general expansion form of derivative. Using them in the mathematics,we get some unexpected result.

*Key words and phrases:* Function series; Integral X series; Remainder term; Expansion form.

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## 1. INTRODUCTION

When we learn the method of integration by parts, we usually just apply it to integrate, and we seldom think of whether there are other applications, and what else can we find from this? If we use it without limits, what will it lead to?

So based on above idea, we obtain the expansion form of general integral and the series expansion form of function and the general expansion form of derivative in this paper. We give some theorems and examples and prove them. Using these, we show how to make the applications of the method of the integration by parts go further...

## 2. A KIND OF FUNCTION SERIES

**Definition 2.1.** i. The function series

$$f(x)x-f'(x)\frac{x^2}{2!}+f''(x)\frac{x^3}{3!}-f^{(3)}(x)\frac{x^4}{4!}+\cdots+(-1)^{n-1}f^{(n-1)}(x)\frac{x^n}{n!}+(-1)^n\int f^{(n)}(x)\frac{x^n}{n!}dx$$

is called $X_0$ series.

The $(-1)^n\int f^{(n)}(x)\frac{x^n}{n!}dx$ is called the remainder term of $X_0$ series.

ii. The function series

$$f'(x)-f''(x)\frac{x^2}{2!}+f^{(3)}(x)\frac{x^3}{3!}-\cdots+(-1)^{n-1}f^{(n)}(x)\frac{x^n}{n!}+(-1)^n\int f^{(n+1)}(x)\frac{x^n}{n!}dx$$

is called $X_1$ series, where $(-1)^n\int f^{(n+1)}(x)\frac{x^n}{n!}dx$ is called the remainder term of $X_1$ series.

iii. The function series

$$f''(x)-f^{(3)}(x)\frac{x^2}{2!}+f^{(4)}(x)\frac{x^3}{3!}-\cdots+(-1)^{n-1}f^{(n+1)}f(x)\frac{x^n}{n!}+(-1)^n\int f^{(n+2)}(x)\frac{x^n}{n!}dx$$

is called $X_2$ series, where $(-1)^n\int f^{(n+2)}(x)\frac{x^n}{n!}dx$ is called the remainder term of $X_2$ series.

iv. All $X_0, X_1, X_2$ series are called $X$ series.

**Theorem 2.1.** *$X_0$ series has form*

$$\begin{aligned}\int f(x)dx &= f(x)x-f'(x)\frac{x^2}{2!}+f''(x)\frac{x^3}{3!}-\cdots \\ &\quad +(-1)^{n-1}f^{(n-1)}(x)+(-1)^n\int f^{(n)}(x)\frac{x^n}{n!}dx,\end{aligned} \tag{2.1}$$

*where $f^{(i)}(x), i=1,2,\cdots,n$ exists.*

*Proof.* Since $f^{(i)}(x), i = 1, 2, \cdots, n$ exists, when the formula $\int u'v dx = uv - \int uv' dx$ is applied over and over again, then we have

$$\begin{aligned}\int f(x)dx &= \int 1 \cdot f(x)dx = xf(x) - \int xf'(x)dx \\ &= xf(x) - \left(\frac{x^2}{2!}f'(x) - \int \frac{x^2}{2!}f''(x)dx\right) \\ &= f(x)x - f'(x)\frac{x^2}{2!} + \int \frac{x^2}{2!}f''(x)dx \\ &= f(x)x - f'(x)\frac{x^2}{2!} + \frac{x^3}{3!}f''(x) - \int \frac{x^3}{3!}f^{(3)}(x)dx \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= f(x)x - f'(x)\frac{x^2}{2!} + f''(x)\frac{x^3}{3!} - \cdots + (-1)^{n-1}f^{(n-1)}(x)\frac{x^n}{n!} \\ &+ (-1)^n \int f^{(n)}(x)\frac{x^n}{n!}dx.\end{aligned}$$

∎

Using Theorem 2.1, we obtain the following theorem.

**Theorem 2.2.** *We have*

*i.*

$$\begin{aligned}\int_a^x f(x)dx &= f(x)x - f'(x)\frac{x^2}{2!} + f''(x)\frac{x^3}{3!} + \cdots \\ &+ (-1)^{n-1}f^{(n-1)}(x)\frac{x^n}{n!} + (-1)^n \int f^{(n)}(x)\frac{x^n}{n!}dx\Big|_a^x.\end{aligned} \tag{2.2}$$

*ii.If* $\lim_{n\to+\infty} \int f^{(n)}(x)\frac{x^n}{n!}dx = c(x)$ $(c(x)$ *can be constant*$)$, *then*

$$\int f(x)dx = \sum_{n=1}^{\infty}(-1)^{n-1}f^{(n-1)}(x)\frac{x^n}{n!} + c(x) \tag{2.3}$$

*iii.*

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty}(-1)^{n-1}f^{(n-1)}(x)\frac{x^n}{n!} + c(x)\Big|_a^b. \tag{2.4}$$

**Theorem 2.3.** *If* $f^{(i)}(x), i = 1, 2, \cdots, n$ *exists, then* $X_1$ *series has form*

$$\begin{aligned}f(x) &= f'(x)x - f''(x)\frac{x^2}{2!} + f^{(3)}(x)\frac{x^3}{3!} - \cdots \\ &+ (-1)^{n-1}f^{(n)}(x)\frac{x^n}{n!} + (-1)^n \int f^{(n+1)}(x)\frac{x^n}{n!}dx.\end{aligned} \tag{2.5}$$

*Proof.* $\int f(x)dx$ becomes $f(x)$, then 2.1 becomes 2.5. ∎

We get the next theorem similar to the proof of Theorem 2.3.

**Theorem 2.4.** *If $f^{(i)}(x), i = 1, 2, \cdots, n$ exists,then $X_2$ series has the form*

$$f'(x) = f''(x)x - f^{(3)}(x)\frac{x^2}{2!} + \cdots + (-1)^{n-1}f^{(n+1)}(x)\frac{x^n}{n!} + (-1)^n \int f^{(n+2)}(x)\frac{x^n}{n!}dx. \tag{2.6}$$

By Theorems 2.1 - 2.4, we have

**Definition 2.2.** i. $X_0$ series is called the indefinite integral expansion form.
ii. $X_1$ series is called the function expansion form.
iii. $X_2$ series is called the derivative expansion form.

Formulas 2.1 - 2.6 are very useful.

## 3. Some Applied Examples

**Example 1.** *Prove that*

$$1 \cdot \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^{n-1}\frac{1}{n}\binom{n}{n-1} + (-1)^n\frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}. \tag{3.1}$$

*Proof.* It is obvious that $\frac{1}{n+1} = \int_0^1 x^n dx$. By 2.2, we have

$$\frac{1}{n+1} = \int_0^1 x^n dx = x^n \cdot x - nx^{n-1} \cdot \frac{x^2}{2!} + n(n-1)x^{n-2} \cdot \frac{x^3}{3!}$$

$$-n(n-1)(n-2)x^{n-3} \cdot \frac{x^4}{4!} + \cdots + (-1)^{n-1}(x^n)^{n-1} \cdot \frac{x^n}{n!} + (-1)^n \int (x^n)^{(n)}\frac{x^n}{n!}dx\Big|_0^1$$

$$\Rightarrow \frac{1}{n+1} = 1 - \frac{n}{2!} + \frac{n(n-1)}{3!} - \frac{n(n-1)(n-2)}{4!} - \cdots + (-1)^{n-1}\frac{n!}{n!} + (-1)^n\frac{1}{n+1}$$

$$= 1 - \frac{1}{2}\cdot\frac{n}{1!} + \frac{1}{3}\cdot\frac{n(n-1)}{2!} - \frac{1}{4}\cdot\frac{n(n-1)(n-2)}{3!} + \cdots + (-1)^{n-1}\frac{1}{n}\cdot\frac{n(n-1)\cdots 3\cdot 2}{(n-1)!} + (-1)^n\frac{1}{n+1}.$$

get

$$\frac{1}{n+1} = 1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^{n-1}\frac{1}{n}\binom{n}{n-1} + (-1)^n\frac{1}{n+1}\binom{n}{n} \Rightarrow 3.1.$$

∎

By example 1, we have:
i. Using 2.2, we have found 3.1.
ii. If $2 \mid n$, then $(-1)^n\frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}$, then 3.1 $\Rightarrow$

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots - \frac{1}{n}\binom{n}{n-1} = 0, \quad 2 \mid n. \tag{3.2}$$

So, we have found 3.2.
iii. We also have found two diophantine equations

$$x_1 - \frac{1}{2}x_2 + \frac{1}{3}x_3 - \frac{1}{4}x_4 + \cdots + (-1)^{n-1}\frac{1}{n}x_n + (-1)^n\frac{1}{n+1}x_{n+1} = \frac{1}{n+1}. \tag{3.3}$$

$$x_1 - \frac{1}{2}x_2 + \frac{1}{3}x_3 - \frac{1}{4}x_4 + \cdots - \frac{1}{n}x_n = 0. \tag{3.4}$$

For 3.3, 3.4, we have following conclusions:

3.3 has positive integer solutions.

$$x_i = \binom{n}{i-1},\ i = 1, 2, 3, \cdots, n+1.$$

3.4 has positive integer solutions.

$$x_i = \binom{n}{i-1}t,\ i = 1, 2, 3, \cdots, n, 2 \mid n, t\, is\, any\, positive\, integer.$$

3.4 also have positive integer solutions.

$x_i = it, i = 1, 2, \cdots, n, 2 \mid n, t$ is any positive integer.

**Example 2.** *Prove that*

$$\lim_{n\to+\infty} \int_0^x e^x \frac{x^n}{n!} dx = 0. \tag{3.5}$$

*Proof.* By 2.2, we have

$$e^x - 1 = \int_0^x e^x dx = e^x \cdot x - e^x \frac{x^2}{2!} + e^x \frac{x^3}{3!} - e^x \frac{x^4}{4!} + \cdots + (-1)^{n-1} e^x \frac{x^n}{n!} + (-1)^n \int e^x \frac{x^n}{n!} dx \Big|_0^x,$$

that is

$$\frac{e^x - 1}{e^x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1} \frac{x^n}{n!}\Big|_0^x + \frac{(-1)^n \int_0^x e^x \frac{x^n}{n!} dx}{e^x} \Rightarrow$$

$$\begin{aligned}
\lim_{n\to+\infty} \frac{(-1)^n \int_0^x e^x \frac{x^n}{n!} dx}{e^x} &= \lim_{n\to+\infty} (1 - e^{-x} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - (-1)^{n-1} \frac{x^n}{n!}) \\
&= \lim_{n\to+\infty} (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - (-1)^{n-1} \frac{x^n}{n!} - e^{-x}) \\
&= \lim_{n\to+\infty} (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots - e^{-x}) \\
&= \lim_{n\to+\infty} (e^{-x} - e^{-x}) = 0 \\
&\Rightarrow \lim_{n\to+\infty} \int_0^x e^x \frac{x^n}{n!} dx = 0
\end{aligned}$$

∎

**Example 3.** [1] $\int e^x x^n dx = ?$

*Solution.* By 2.5, we get

$$(-1)^n \int f^{(n+1)}(x) \frac{x^n}{n!} dx = f(x) - f'(x)x + f''(x)\frac{x^2}{2!} - \cdots - (-1)^{n-1} f^{(n)}(x) \frac{x^n}{n!}$$

That is

$$\int f^{(n+1)}(x) x^n dx = (-1)^n n! \Big( f(x) - f'(x)x + f''(x)\frac{x^2}{2!} - \cdots - (-1)^{n-1} f^{(n)}(x) \frac{x^n}{n!} \Big) \tag{3.6}$$

Let $f(x) = e^x$, then $f^{(i)}(x) = (e^x)^{(i)} = e^x, i = 1, 2, 3, \cdots$ , we obtain

$$\int e^x x^n dx = (-1)^n e^x n!(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!}). \tag{3.7}$$

∎

Using the same method as example 3, let $f(x) = cosx, sinx, shx, chx$, and so on. We also have some other formulas. We omit them here. !!AUTHOR!!. Bad practice to ommit without saying why or where to find. Please specify.

**Example 4.** *We known that*

$$P(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots + \frac{x^{3n}}{(3n)!} + \cdots \tag{3.8}$$

$$Q(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots + \frac{x^{3n+1}}{(3n+1)!} + \cdots \tag{3.9}$$

$$R(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots + \frac{x^{3n+2}}{(3n+2)!} + \cdots \tag{3.10}$$

*Prove that*

$$P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x) = 1. \tag{3.11}$$

*Proof.* Naturally, we have

$$P'(x) = R(x),\ Q'(x) = P(x),\ R'(x) = Q(x).$$

Using 2.5,then

$$P^3(x)+Q^3(x)+R^3(x)-3P(x)Q(x)R(x) = \Big(P^3(x)+Q^3(x)+R^3(x)-3P(x)Q(x)R(x)\Big)'x$$
$$-\Big(P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x)\Big)''\frac{x^2}{2!} + \cdots$$
$$+(-1)^n \int \Big(P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x)\Big)^{(n+1)}\frac{x^n}{n!}dx$$

However,

$$\begin{aligned}
&\Big(P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x)\Big)' \\
&= 3P^2(x)P'(x) + 3Q^2(x)Q'(x) + 3R^2(x)R'(x) \\
&- 3P'(x)Q(x)R(x) - 3P(x)Q'(x)R(x) - 3P(x)Q(x)R'(x) \\
&= 3P^2(x)R(x) + 3Q^2(x)P(x) + 3R^2(x)Q(x) - 3Q(x)R^2(x) \\
&- 3R(x)P^2(x) - 3P(x)Q^2(x) = 0
\end{aligned}$$

get
$P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x) = 0 + (-1)^n \int 0dx.$
That is

$$P^3(x) + Q^3(x) + R^3(x) - 3P(x)Q(x)R(x) = c \tag{3.12}$$

Let $x = 0$, by 3.8–3.10, have $P(0) = 1, Q(0) = 0, R(0) = 0$, then 3.12$\Rightarrow c = 1 \Rightarrow 3.11.$ ∎

**Example 5.** [1] *The expansion form of* $log(1 + x)$.

*Solution.* Using 2.5, get

$$\begin{aligned} log(1+x) &= \big(log(1+x)\big)'x - \big(log(1+x)\big)''\frac{x^2}{2!} + \big(log(1+x)\big)^{(3)}\frac{x^3}{3!} - \cdots \\ &= (1+x)^{-1}x - \big((1+x)^{-1}\big)'\frac{x^2}{2!} + \big((1+x)^{-1}\big)''\frac{x^3}{3!} - \big((1+x)^{-1}\big)^{(3)}\frac{x^4}{4!} + \cdots \\ &= \frac{1}{1+x}\cdot x + \frac{1}{(1+x)^2}\frac{x^2}{2!} + 2!\frac{1}{(1+x)^3}\frac{x^3}{3!} + 3!\frac{1}{(1+x)^4}\frac{x^4}{4!} + \cdots \\ &\Rightarrow \end{aligned}$$

$$log(1+x) = \frac{x}{1+x} + \frac{1}{2}(\frac{x}{1+x})^2 + \frac{1}{3}(\frac{x}{1+x})^3 + \frac{1}{4}(\frac{x}{1+x})^4 + \cdots \tag{3.13}$$

When $x = 1$, then $log2 \approx \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} \approx 0.682.$ and $log2 = 0.693\cdots$.

We know that

$$log(1+x) = x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \cdots$$

When $x = 1$, we have

$$log2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.583.$$

∎

**Example 6.** *Prove that Newton's binomial theorem*

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n. \tag{3.14}$$

*Proof.* Notice that 2.1, then we have

$$(y-a)^n + c_1 = \int n(y-a)^{n-1}dy = n(y-a)^{n-1}y - \big(n(y-a)^{n-1}\big)'\frac{y^2}{2!} + \big(n(y-a)^{n-1}\big)''\frac{y^3}{3!}$$

$$-\big(n(y-a)^{n-1}\big)^{(3)}\frac{y^4}{4!} + \cdots + (-1)^{n-1}\big(n(y-a)^{n-1}\big)^{(n-1)}\frac{y^n}{n!} + (-1)^n\int\big(n(y-a)^{n-1}\big)^{(n)}\frac{y^n}{n!}dy \Rightarrow$$

$$\begin{aligned} (y-a)^n + c_1 &= n(y-a)^{n-1}y - n(n-1)(y-a)^{n-2}\cdot\frac{y^2}{2!} + n(n-1)(n-2)(y-a)^{n-3}\cdot\frac{y^3}{3!} \\ &\quad + \cdots + (-1)^{n-1}\big(n(y-a)^{n-1}\big)^{(n-1)}\cdot\frac{y^n}{n!} + (-1)^n\int 0\cdot\frac{y^n}{n!}dy \end{aligned}$$

$$= \binom{n}{1}(y-a)^{n-1}y - \binom{n}{2}(y-a)^{n-2}y^2 + \binom{n}{3}(y-a)^{n-3}y^3 - \cdots + (-1)^{n-1}\binom{n}{n}y^n + c_2.$$

Let $c_2 - c_1 = c$, get

$$\begin{aligned} (y-a)^n &= \binom{n}{1}(y-a)^{n-1}y - \binom{n}{2}(y-a)^{n-2}y^2 + \cdots \\ &\quad + (-1)^{n-2}\binom{n}{n-1}(y-a)y^{n-1} + (-1)^{n-1}y^n + c. \end{aligned} \tag{3.15}$$

Write $y - a = x$, then

$$y - a = x, a = y - x, y = x + a. \tag{3.16}$$

Notice that 3.15 , 3.16, get

$$x^n = \binom{n}{1}x^{n-1}y - \binom{n}{2}x^{n-2}y^2 + \cdots + (-1)^{n-2}\binom{n}{n-1}xy^{n-1} + (-1)^{n-1}y^n + c. \tag{3.17}$$

When $x = 0$, by 3.16 $\Rightarrow y = a$, then 3.17 $\Rightarrow 0 = (-1)^{n-1}a^n + c \Rightarrow c = -(-1)^{n-1}a^n = (-1)^n a^n$. But 3.16 have $a = y - x$, get $c = (-1)^n(y-x)^n = \Big(-(y-x)\Big)^n = (x-y)^n$. Hence 3.17 becomes

$$x^n = \binom{n}{1}x^{n-1}y - \binom{n}{2}x^{n-2}y^2 + \cdots + (-1)^{n-2}\binom{n}{n-1}xy^{n-1} + (-1)^{n-1}y^n + (x-y)^n \Rightarrow$$

$$(x-y)^n = x^n - \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 - \cdots - (-1)^{n-2}\binom{n}{n-1}xy^{n-1} - (-1)^{n-1}y^n.$$

If $y$ becomes $-y$, get

$$\big(x-(-y)\big)^n = x^n - \binom{n}{1}x^{n-1}(-y) + \binom{n}{2}x^{n-2}(-y)^2 - \cdots - (-1)^{n-2}\binom{n}{n-1}x(-y)^{n-1}$$
$$-(-1)^{n-1}(-y)^n.$$

that is

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n \Rightarrow 3.14.$$

∎

Note the knowledge of permutation and combination is not applied in the proof. So using the binomial theorem, we have proved the formulas as followings:

$$\binom{n}{r} = \binom{n}{n-r}, \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, etc.$$

**Theorem 3.1.** *We have*

$$\int_2^x \frac{dt}{logt} = \frac{x}{logx} - (\frac{1}{logx})'\frac{x^2}{2!} + (\frac{1}{logx})''\frac{x^3}{3!} - (\frac{1}{logx})^{(3)}\frac{x^4}{4!} + \cdots$$
$$+ (-1)^{n-1}(\frac{1}{logx})^{(n-1)}\frac{x^n}{n!} + (-1)^n\int(\frac{1}{logx})^{(n)}\frac{x^n}{n!}dx\Big|_2^x. \tag{3.18}$$

*Proof.* By 2.2, let $f(x) = \dfrac{1}{logx}$, we obtain 3.18. ∎

Since

$$(\frac{1}{logx})' = \Big((logx)^{-1}\Big)' = -(logx)^{-2}\cdot\frac{1}{x} = \frac{-1}{xlog^2x}(\frac{1}{logx})'' = \frac{2+logx}{x^2log^3x}, \cdots$$

then 3.18 becomes

$$\int_2^x \frac{dt}{logt} = \frac{x}{logx} + \frac{1}{2!}\frac{x}{log^2x} + \frac{1}{3!}\frac{(2+logx)x}{log^3x} + \cdots + (-1)^n\int(\frac{1}{logx})^{(n)}\frac{x^n}{n!}dx\Big|_2^x. \tag{3.19}$$

We know, if $\pi(x) =$ the number of primes $p$ satisfy $2 \le p \le x$, then

$$\lim_{x\to+\infty}\frac{\pi(x)}{\frac{x}{logx}} = 1 \, or \, \lim_{x\to+\infty}\frac{\frac{x}{logx}}{\pi(x)} = 1 \tag{3.20}$$

are called the prime number theorem. By 3.18 , 3.19, then

$$\lim_{x\to+\infty}\frac{\int_2^x \frac{dt}{logt}}{\frac{x}{logx}}=\lim_{x\to+\infty}\frac{\frac{x}{logx}+\frac{1}{2!}\frac{x}{log^2x}+\cdots}{\frac{x}{logx}}=\lim_{x\to+\infty}(1+\frac{1}{2!}\frac{1}{log^2x}+\cdots)=1$$

Therefore, we have

$$\lim_{x\to+\infty}\frac{\pi(x)}{\int_2^x \frac{dt}{logt}}=1\, or\, \lim_{x\to+\infty}\frac{\int_2^x \frac{dt}{logt}}{\pi(x)}=1. \tag{3.21}$$

So, $\int_2^x \frac{dt}{logt}$ is very important.

Hence,the expansion formula 3.19 is also very important.

By 3.19 , 3.20, then we have the approximate formulas.

$$\pi(x)\approx\frac{x}{logx},$$

$$\pi(x)\approx\frac{x}{logx}+\frac{1}{2}\frac{x}{log^2x},$$

$$\pi(x)\approx\frac{x}{logx}+\frac{1}{2}\frac{x}{log^2x}+\frac{1}{6}\frac{(2+logx)x}{log^3x}, \tag{3.22}$$

## 4. CONCLUSION

In conclusion, we obtain three important formulas from college calculus–the integral expansion form, the function expansion form and the derivative expansion form of function. Using these theories, we get some interesting results. For example, we find $P(x), Q(x), R(x)$ and their relationship. We also give an important series expansion form of function in number theory.

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