A NOTE ON TAYLOR EXPANSIONS WITHOUT THE DIFFERENTIABILITY ASSUMPTION
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ABSTRACT. We introduce new Taylor expansions when the function is not differentiable.

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In this paper, we provide a pioneering contributions that overcome a major obstacles in mathematical sciences. In doing so, we introduce exact Taylor’s expansions even if the function is not differentiable. Needless to say, this pioneering contribution is extremely useful in many applications, such as the areas of regression analysis, optimization, integration, partial differential equations PDEs (see [1] for potential important applications). Another possible application is option valuation. It is especially useful in the estimation of the value function (see, for example, [4], among others).

Consider a continuous, bounded function \( f(x) \) that is not differentiable with respect to \( x \); it can be expressed as \( f(x + \alpha) \), where \( \alpha \) is a shift parameter with an initial value equal to zero (see [1] and [3]). In [2], Alghalith introduced the idea without a proof, using a different approach. We also define \( z = x + \alpha \), so that \( f(z) = f(x + \alpha) \). We can show that \( f \) is differentiable w.r.t. \( \alpha \) (since every function can be shifted), and consequently \( f \) is differentiable w.r.t. \( z \) (see Appendix 1 for the proof).

We obtain the following Taylor expansion around \( c \)

\[
(i) \quad f(z) = f(c) + f'(c)(z - c) + R(z, c),
\]

where \( R \) is the remainder. Evaluating \( i \) at \( z = x \), we obtain

\[
(ii) \quad f(x) = f(c) + f'(c)(x - c) + R(x, c).
\]

Multiple-variable functions:
We examine a two-variable function \( f(x, y) \), however, the extension to a multiple-variable function is straightforward. As before, we define \( v = y + \beta \), where \( \beta \) is a shift parameter with an initial value equal to zero; so that \( f(x + \alpha, y + \beta) \equiv f(z, v) \). As before, \( f \) is differentiable w.r.t. \( v \) and \( z \) (see the appendix). The Taylor expansion is given by

\[
f(z, v) = f(c_1, c_2) + f_z(c_1, c_2)(z - c_1) + f_v(c_1, c_2)(v - c_2) + R(.)\]

As before, evaluating at \( z = x \) and \( v = y \), we obtain

\[
f(x, y) = f(c_1, c_2) + f_z(c_1, c_2)(x - c_1) + f_v(c_1, c_2)(y - c_2) + R(.)\]

An application to finance:

The (non-differentiable) value function (frequently used in finance) \( v(t, x) \) can be approximated as

\[
v(t, x) \approx v(c_1, c_2) + v_z(t, x)(t - c_1) + v_h(t, x)(x - c_2).
\]

Appendix 1. Proof of the differentiability
Differentiability with respect to the shift parameter (as opposed to a variable) stems from the fact that the small change in the shift parameter is a constant (graphically, this is evidenced by a small horizontal shift of the function). Let \( d\alpha = \epsilon - 0 = \epsilon, d\beta = \delta - 0 = \delta \) (since the initial values are zero), where \( \epsilon \) and \( \delta \) are small non-zero constants. Thus, \( dz = dx + d\alpha = dx + \epsilon \).
Now,

\[ \frac{df(z)}{dz} |_{\Delta x=0} = \lim_{\Delta z \to 0} \frac{f(x + \Delta z) - f(x)}{\Delta z} |_{\Delta x=0} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x + \epsilon) - f(x)}{\Delta x + \epsilon} |_{\Delta x=0} = \frac{f(x + \epsilon) - f(x)}{\epsilon}. \]

By the continuity of \( f \) and the fact that \( \epsilon \neq 0 \), the derivative exists. For a multiple-variable function, we apply the same procedure to the partial derivatives. That is, for example, we obtain \( \frac{\partial f(z,v)}{\partial z} |_{\Delta x=0}. \) □

REFERENCES


