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NUMERICAL RADIUS ISOMETRIES BETWEEN HERMITIAN BANACH ALGEBRAS

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ABSTRACT. In the case of C^*-algebras, the author in [2] showed that any linear unital and surjective numerical radius isometry is a Jordan $*$-isomorphism. In this paper, we generalize this result to the case of Hermitian Banach algebras.

Key words and phrases: Numerical Radius, Hermitian Banach Algebras, Jordan $*$-isomorphism, Preserving the numerical radius, Numerical radius isometry.

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1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be a unital complex Banach Algebras. We always denote by **1** the unit both $\mathcal{A}$ and $\mathcal{B}$. Define the set of normalized states

$$\mathcal{S}(\mathcal{A}) = \{f \in \mathcal{A}' : f(\mathbf{1}) = \|f\| = 1\},$$

where $\mathcal{A}'$ denotes the dual space of $\mathcal{A}$. It is well known that $\mathcal{S}(\mathcal{A})$ is a compact and convex in the weak*-topology of $\mathcal{A}'$. For any element $a \in \mathcal{A}$, the algebraic numerical range $V_{\mathcal{A}}(a)$ and numerical radius $\upsilon_{\mathcal{A}}(a)$ of a are defined by

$$V_{\mathcal{A}}(a) = \{f(a) : f \in \mathcal{S}(\mathcal{A})\} \text{ and } \upsilon_{\mathcal{A}}(a) = \sup_{z \in V_{\mathcal{A}}(a)} |z|.$$

The numerical radius $\upsilon_{\mathcal{A}}(.)$ is a norm on $\mathcal{A}$. In the case of C^*-algebras, this norm is equivalent to the given norm:

$$\frac{1}{2}\|a\| \leq \upsilon_{\mathcal{A}}(a) \leq \|a\|,$$

for all $a \in \mathcal{A}$. A linear map $T : \mathcal{A} \to \mathcal{B}$ is said to be numerical range preserving if $V_{\mathcal{B}}(T(a)) = V_{\mathcal{A}}(a)$, numerical radius preserving or numerical radius isometry if $\upsilon_{\mathcal{B}}(T(a)) = \upsilon_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. We say that T compresses the numerical range if $V_{\mathcal{B}}(T(a)) \subset V_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. A unital linear map $T : \mathcal{A} \to \mathcal{B}$ compresses numerical range if and only if $\upsilon_{\mathcal{B}}(T(a)) \leq \upsilon_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.*

Theorem 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras. Suppose that $T : \mathcal{A} \to \mathcal{B}$ is a linear numerical radius preserving map. Then we have that*

(1) *T is injective;*
(2) *if T surjective, then T^{-1} is the numerical radius preserving.*

Theorem 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $T : \mathcal{A} \to \mathcal{B}$ be a unital surjective linear map. Then the following are equivalent:*

(1) *$V_{\mathcal{B}}(T(a)) = V_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.*
(2) *$\upsilon_{\mathcal{B}}(T(a)) = \upsilon_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$.*

Proof. (1) ⇒ (2) is trivial. For the converse, suppose $\upsilon_{\mathcal{B}}(T(a)) = \upsilon_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. By 1.1, $V_{\mathcal{B}}(T(a)) \subset V_{\mathcal{A}}(a)$. Since T is invertible, this implies, by 1.2, that T^{-1} is the numerical radius preserving. That is,

$$\upsilon_{\mathcal{A}}(T^{-1}(b)) = \upsilon_{\mathcal{B}}(b).$$

for all $b \in \mathcal{B}$. Thus, $V_{\mathcal{A}}(T^{-1}(b)) \subset V_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$ by 1.1. Hence, $V_{\mathcal{A}}(T^{-1}(T(a)) \subset V_{\mathcal{B}}(T(a))$ for all $a \in \mathcal{A}$, i.e., $V_{\mathcal{A}}(a) \subset V_{\mathcal{B}}(T(a))$. It follows that $V_{\mathcal{B}}(T(a)) = V_{\mathcal{A}}(a)$. ∎

A linear map $T : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be $*$-homomorphism if, for all $a, b \in \mathcal{A}$, $T(ab) = T(a)T(b)$ and $T(a^*) = T(a)^*$, Jordan $*$-isomorphism (or C^*-isomorphism) if it is a linear bijective map and satisfies $T(a^*) = T(a)^*$ and $T(a^2) = T(a)^2$ for all $a \in \mathcal{A}$.

2. MAIN RESULT 1

A Banach $*$-algebra is said to be Hermitian if the spectrum of any self-adjoint $a = a^*$ element in $\mathcal{A}$ is a subset of $\mathbb{R}$. The class of Hermitian Banach algebras incorporates a wide class of Banach $*$-algebras and includes C^*-algebras as a very special case. One more interesting example is the group algebra $L^1(G)$, when G is commutative. Let $\mathcal{A}$ be a Hermitian Banach algebra. We denote the set of positive elements by $\mathcal{A}^+$. Hence,

$$\mathcal{A}^+ := \left\{ \sum_{k=1}^{n} a_k a_k^* \,:\, a_k \in \mathcal{A},\, n \in \mathbb{N} \right\}.$$

For a Banach $*$-algebra, the following inclusion holds:

$$\mathcal{A}_s := \left\{h^2 \,:\, h = h^* \in \mathcal{A}\right\} \subset \mathcal{A}^+.$$

In general, the above inclusion is strict, but if $\mathcal{A}$ is Hermitian, then $\mathcal{A}_s = \mathcal{A}^+$.

A linear function p is said to be *positive* if $p(aa^*) \geq 0$ for all $a \in \mathcal{A}$ (denoted by $p \geq 0$). Let us define the set

$$\mathcal{S}_*(\mathcal{A}) := \{p \in \mathcal{A}' \,:\, p \geq 0,\, p(\mathbf{1}) = 1\}.$$

It is obvious that all $p \in \mathcal{S}_*(\mathcal{A})$ are Hermitian, that is, $p(a^*) = \overline{p(a)}$ for all $a \in \mathcal{A}$. We now introduce the definition of the $*$-*numerical range* and the $*$-*numerical radius*:

$$V^*_{\mathcal{A}}(a) := \{p(a) \,:\, p \in \mathcal{S}_*(\mathcal{A})\} \text{ and } v^*_{\mathcal{A}}(a) := \sup_{z \in V^*_{\mathcal{A}}(a)} |z|.$$

In the sequel, $\mathcal{A}$ and $\mathcal{B}$ are two Hermitian semi-simple Banach algebras. Then, by [3, Corollary 33.13, p. 149], there exists an auxiliary norm $|.|$ on $\mathcal{A}$ which satisfies the C^*-condition (i.e., $|xx^*| = |x|^2$ for all $x \in \mathcal{A}$ and $|x| \leq \|x\|$ for any $x \in \mathcal{A}$). We shall denote by $\hat{\mathcal{A}}$ the completion of $\mathcal{A}$ with respect to the norm $|.|$. Observe that $\hat{\mathcal{A}}$ is a unital C^*-algebra.

We begin with the following theorem, which shows the relationship between $V^*_{\mathcal{A}}$ and $V_{\hat{\mathcal{A}}}$ and that every algebra $*$-homomorphism compresses the numerical radius $v^*_{\mathcal{A}}$.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Hermitian semi-simple Banach $*$-algebras. Then:*

(1) *For all $a \in \mathcal{A}$, we have*

$$V^*_{\mathcal{A}}(a) = V_{\hat{\mathcal{A}}}(a) \textit{ and } V^*_{\mathcal{A}}(a) \subset V_{\mathcal{A}}(a).$$

(2) *If $\phi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then*

$$v^*_{\mathcal{B}}(\phi(a)) \leq v^*_{\mathcal{A}}(a).$$

Proof. Since for unital C^*-algebras a linear functional p is positive if and only if $|p| = p(\mathbf{1})$, by the Hahn-Banach Theorem, one can easily see that that $V_{\hat{\mathcal{A}}}(a) = V^*_{\mathcal{A}}(a)$. Consider now an element $z \in V^*_{\mathcal{A}}(a)$. Then there exists $p \in \mathcal{S}_*(\mathcal{A})$ such that $z = p(a)$. Since

$$|p(a)| \leq |a| \leq \|a\|$$

and $p(\mathbf{1}) = 1$, we infer that $\|p\| = p(\mathbf{1}) = 1$. Then $p \in \mathcal{S}(\mathcal{A})$ and $z = p(a) \in V_{\mathcal{A}}(a)$, as required.

For (2), let us consider any $*$-homomorphism ϕ and $\lambda \in V^*_{\mathcal{B}}(\phi(a))$. Then there exists a positive linear form $p \in \mathcal{S}_*(\mathcal{B})$ such that $\lambda = p(\phi(a))$. Define a linear functional p_1 on $\mathcal{A}$ by $p_1(a) = p \circ \phi(a)$. Obviously, p_1 is positive, and, hence, by [3, Theorem 27.2, p. 102] there exists a $*$-representation π_1 of $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}_1$ and a cyclic vector $\xi \in \mathcal{H}_1$ of norm 1 so that $p_1(a) = \langle \pi_1(a)\xi, \xi\rangle$ for all $a \in \mathcal{A}$. Therefore,

$$|p_1(a)| \leq \|\pi_1(a)\| \leq |a|$$

for all $a \in \mathcal{A}$. Hence, $p_1 \in \mathcal{S}_*(\mathcal{A})$ and $\lambda = p_1(a) \in V^*_{\mathcal{A}}(a)$. This proves that $V^*_{\mathcal{B}}(\phi(a)) \subset V^*_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. Accordingly, $v^*_{\mathcal{B}}(\phi(a)) \leq v^*_{\mathcal{A}}(a)$. The proof is thus complete. ∎

If $\mathcal{A}$ is a C^*-algebra, by the uniqueness of the C^*-norm, we get $\mathcal{A} = \hat{\mathcal{A}}$. Hence, according to 2.1 , we infer that $V_{\mathcal{A}}(a) = V^*_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. If $\mathcal{A} \subsetneq \hat{\mathcal{A}}$ (notice that $\hat{\mathcal{A}} = \mathcal{A}$ if and only if $\mathcal{A}$ is a C^*-algebra), then this equality valid for unital C^*-algebras, need not hold. That is, there can exist continuous linear functionals p on $\mathcal{A}$ such that $\|p\| = p(\mathbf{1}) = 1$, but which fail to be positive. This is shown in the following example.

Example 2.1. *For example, consider $\mathcal{A} = \ell^1(\mathbb{Z})$, the set of all complex valued functions f on $\mathbb{Z}$ such that*

$$\|f\|_1 = \sum_{n\in\mathbb{Z}} |f(n)|$$

is finite. For f and g in $\ell^1(\mathbb{Z})$ define the convolution product

$$f \star g(n) = \sum_{j\in\mathbb{Z}} f(j)g(n-j), \forall n \in \mathbb{Z}.$$

Note that $\mathcal{A}$ is a commutative Banach algebra with the (multiplicative) unit is the function **1** *in $\mathcal{A}$ defined by*

$$\mathbf{1}(n) := \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

Moreover, we know that $\mathcal{A}$ is a Banach algebra with an involution

$$f \mapsto f^*; f^*(n) = \overline{f(-n)}$$

for any $n \in \mathbb{Z}$.

Now, consider the linear functional $p : \mathcal{A} \longrightarrow \mathbb{C}$ defined by

$$p(f) = f(0) + f(1)i$$

for all $f \in \mathcal{A}$. Easy computation show that

$$\|p\| = p(\mathbf{1}) = 1.$$

However, if we take the element $a \in \mathcal{A}$ defined by

$$a(n) := \begin{cases} 1, & \text{if } n \in \{0,1\} \\ 0, & \text{if } n \in \mathbb{Z}\backslash\{0,1\}, \end{cases}$$

then a^* is the function defined by

$$a^*(n) := \begin{cases} 1, & \text{if } n \in \{0,-1\} \\ 0, & \text{if } n \in \mathbb{Z}\backslash\{0,-1\}. \end{cases}$$

Then $a \in \mathcal{A}$ but $p(a \star a^*) = 2 + i$, which is not a real number.

Now, in the case of C^*-algebras, the author in [2] showed that any linear unital and surjective numerical radius isometry is a Jordan $*$-isomorphism. Our goal in the sequel is to generalize this result to the case of Hermitian algebras.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hermitian semi-simple Banach algebras and T be a surjective linear mapping such that $T(\mathbf{1}) = \mathbf{1}$ and $\upsilon^*_{\mathcal{B}}\big(T(a)\big) = \upsilon^*_{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. Then T is a Jordan $*$-isomorphism.*

Proof. Let us first prove that T is a vector space isomorphism. Let $a \in \mathcal{A}$ be such that $T(a) = 0$. Since $\upsilon^*_{\mathcal{B}}\big(T(a)\big) = \upsilon^*_{\mathcal{A}}(a) = 0$ and $\upsilon^*_{\mathcal{B}}$ is a norm, we infer that $a = 0$ and T is injective. 2.1 allows us to conclude that

$$\frac{1}{2}|a| \leq \upsilon^*_{\mathcal{A}} \leq |a|.$$

Keeping in mind that $\upsilon^*_{\mathcal{B}}\big(T(a)\big) = \upsilon^*_{\mathcal{A}}(a)$, we infer that

$$\frac{1}{2}|a| \leq |T(a)| \leq 2|a|$$

for all $a \in \mathcal{A}$. Consequently, T and T^{-1} are continuous with respect to $\upsilon^*_{\mathcal{B}}$ and to the C^*-norm $|\cdot|$. The extension $\tilde{T}$ of T is also a vector space isomorphism between the two C^*-algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. We will show that

$$\upsilon_{\hat{\mathcal{B}}}\big(\tilde{T}(a)\big) = \upsilon_{\hat{\mathcal{A}}}(a)$$

for all $a \in \hat{\mathcal{A}}$. Take any $a \in \hat{\mathcal{A}}$. There exists a sequence $a_n \in \mathcal{A}$ so that $\lim a_n = a$. By continuity of $\tilde{T}$, we infer that $\lim \tilde{T}(a_n) = \tilde{T}(a)$. Accordingly,

$$\lim \upsilon_{\hat{\mathcal{B}}}\big(\tilde{T}(a_n)\big) = \upsilon_{\hat{\mathcal{B}}}\big(\tilde{T}(a)\big).$$

Or $\tilde{T}(a_n) = T(a_n)$ and $\upsilon_{\hat{\mathcal{B}}}\big(T(a_n)\big) = \upsilon_{\hat{\mathcal{A}}}(a_n)$. Hence,

$$\upsilon_{\hat{\mathcal{B}}}\big(\tilde{T}(a)\big) = \lim \upsilon_{\hat{\mathcal{A}}}(a_n) = \upsilon_{\hat{\mathcal{A}}}(a).$$

So, we have shown $\upsilon_{\hat{\mathcal{B}}}\big(\tilde{T}(a)\big) = \upsilon_{\hat{\mathcal{A}}}(a)$ for all $a \in \hat{\mathcal{A}}$. Therefore, the results of [2] may be applied to show that $\tilde{T}$ is a Jordan $*$-isomorphism. So, T is a Jordan $*$-isomorphism, since it is the restriction to $\mathcal{A}$ of the Jordan $*$-isomorphism $\tilde{T}$. ∎

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