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ACTION OF DIFFERENTIAL OPERATORS ON CHIRPSCONSTRUCT ON L^∞

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ABSTRACT. We will study in this work the action of differential operators on L^∞ chirps and we will give a new definition of logarithmic chirp. Finally we will study the action of singular integral operators on chirps by wavelet characterization and Kernel method.

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1. INTRODUCTION

Consider a function f of the real variable which is written

$f(x) = |x|^\alpha g_\pm\left(|x|^{-\beta}\right)$, where f is an integrable function at the neighborhood of $0, \alpha > -1, \beta > 0, g_+$ and g_- are functions of global regularity r and are indefinitely oscillating in $L^\infty([T,\infty)$. This last point means that the suite $g_{\pm 1}, g_{\pm 2}, \ldots$ of iterated primitives of $g_\pm$ defined by $\frac{dg_{\pm i+1}}{dx} = g_{\pm i}$ and $g_{\pm 0}(x) = g_\pm(x)$ satisfies that $g_{\pm i}$ is in L^∞ for all i in $\mathbb{N}$.

To simplify the problem that arises, we use the following result: if f is a chirp in 0, it will be the same for the product f with the indicator function of $[0,\infty)$.

Thanks to this remark we restrict ourselves to the study of $f(x) = x^\alpha g\left(x^{-\beta}\right)$ for x in $]0,1]$, where α, β and g check the conditions mentioned above. Successive primitives $f_1, f_2, \ldots$ of f are normalized by $f_n(0) = 0$ for all $n \geq 0$.

Let f a generalized or trigonometric chirp ([12],[13]), of type (α, β, r) in 0. We want to give $D^z(f)$ where z is a complexe number without drastic assumptions whether on the regularity r of the function g where the exposant α(if $z = \gamma_1 + i\gamma_2$ and g is in de Hölder space C^r then $r - \gamma_2 > 0$ and $\alpha - \gamma_2(\beta+1) + 1 > 0$).

This requires expanding the definition of chirps in L^∞ encountered ([12],[13]) to include the possibility that g be a distribution.

Let g be a distribution of order m in $\mathbb{N}$. Define then f by $f(x) = |x|^\alpha g_\pm\left(|x|^{-\beta}\right)$ for x in $]0,\eta]$. By definition of g, there exists a function G in C^1 as $g = \frac{d^m G}{dx^m}$. After m integration, we found the term $F(x) = x^{\alpha+m(\beta+1)} G\left(x^{-\beta}\right)$ which is locally integrable function in the neighborhood of 0 .

So we have the following definition:

Definition 1.1. *g is a indefinitely oscillanting distribution in L^∞ if and only if g is a distribution of finite ordre m and if the primitive of ordre m of g is a function indefinitely oscillanting in L^∞.*

Similary α can be taken under -1. Indeed we can always find an integer n such that $\alpha + n(\beta+1) > -1$ and in this case it suffices to consider the primitive of order $\sup(m,n)$ of g.

As regards the trigonometric chirps, one takes g as a distribution of $D'(\mathbb{R})2\pi - periodic$. That is to say

$$< g, \tau_{2\pi}\varphi > = < g, \varphi >$$
$$[\tau_a \varphi](x) = \varphi(x-a)$$

Therefore for a distribution $g 2\pi - periodic$, we define the average of g, denoted $I(g)$, by

$$I(g) = < g, \varphi >$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and where $\sum_{-\infty}^{+\infty} \varphi(x + 2k\pi) = 1.$

It is immediate to verify that if φ_1 and φ_2 satisfy this last condition then $\varphi_1 - \varphi_2 = \tau_{2\pi}\varphi_3 - \varphi_3$ where $\varphi_3 \in C_0^\infty(\mathbb{R})$. So we have $< g, \varphi_1 > = < g, \varphi_2 >$ since g is $2\pi - periodic$.

Now we have to verify that g, as we defined it is indefinitely oscillating within the meaning of the Definition **(1.1)**. Or g is a distribution $2\pi - periodic$, therefore it is a distribution of finite order. We normalize the primitives $g_1, g_2, \ldots, g_n$ de g by the condition $\int_0^{2\pi} g_j(t)dt = 0 \ \forall j \in \mathbb{N}$, and all these primitives are then $2\pi - periodics$.

2. LOGARITHMIC CHIRP

Consider the function as before $f(x) = x^\alpha g\left(x^{-\beta}\right)$ for $0 < x < 1$.

It is clear that for every $\beta > 0, g(x^{-\beta})$ oscillates more violently that the function $g(\log(x))$.

Take the example of a trigonometric chirp : $f(x) = x^\alpha e^{ix^{-\beta}}$ for $0 < x < 1$, then the function h defined by :

$$\begin{aligned} h(x) &= f(x)e^{i\gamma \log x} \\ &= x^\alpha e^{i\left(x^{-\beta}+\gamma \log x\right)} \\ &= x^\alpha e^{ix^{-\beta}\left(1+x^\beta \gamma \log x\right)} \end{aligned}$$

with $\beta > 0$ and γ is any real.

The oscillations of $e^{ix^{-\beta}}$ outweigh that of $e^{i\gamma \log x}$ for any value of γ.

Definition 2.1. Let α, β, γ real numbers as $\alpha > -1$ and $\beta > 0$. We call logarithmic chirp at 0 of type (α, β, r) all integrable function on a neighborhood $[-\eta, \eta]$ of 0 in the form $f(x) = |x|^\alpha g\ \left(|x|^{-\beta}\right) e^{i\gamma \log|x|}$, γ is a un any real , η being a strictly positive real (g satisfies the properties listed above).

We have the following assertion:

Theorem 2.1. *The successive primitives* $f_1, f_2, \ldots, f_n$ *of* f*,normalized by* $f_n(0) = 0$ *for any* $n \geq 0$ *verify :*

$$f_n(x) = |x|^{\alpha+n(\beta+1)} g^{(n)}(|x|^{-\beta}) e^{i\gamma \log|x|}$$

where $g^{(n)}$*is indefinitely oscillating and of the global regularity* $r + n$*.*

Proof. It will be limited to $f\chi_{[0,\infty)}$. Consider then the primitive of f defined by :

$$\int_0^x f(u)du = \int_0^x u^\alpha g\ \left(u^{-\beta}\right) e^{i\gamma \log u} du$$

Integration by parts gives :

$$\int_0^x f(u)du = -\frac{1}{\beta} x^{\alpha+\beta+1} g_1(x^{-\beta}) e^{i\gamma \log x} + C_{\alpha,\beta} \int_0^x u^{\alpha+\beta} g_1\ \left(u^{-\beta}\right) e^{i\gamma \log u} du$$

The first term has announced the structure. It remains to consider the second term. For this we put $u = sx$, then we have :

$$I_1 = x^{\alpha+\beta+1} e^{i\gamma \log x} \int_0^1 g_1(x^{-\beta} s^{-\beta}) s^{\alpha+\beta} e^{i\gamma \log s} ds$$

by taking $s = \frac{1}{t}$ we have

$$\begin{aligned} I_1 &= x^{\alpha+\beta+1} e^{i\gamma \log x} \int_1^\infty g_1(x^{-\beta} t^\beta) \frac{e^{-i\gamma \log t}}{t^{2+\alpha+\beta}} dt \\ &= x^{\alpha+\beta+1} e^{i\gamma \log x} h(x^{-\beta}) \end{aligned}$$

$$h(x) = \int_1^\infty g_1(xt^\beta) \frac{e^{-i\gamma \log t}}{t^{2+\alpha+\beta}} dt \tag{2.1}$$

So we have $\|h\|_\infty \leq C\|g_1\|_\infty$ since $2 + \alpha + \beta > 1$, and all primitive of h are deduced from the de g_1 by the equation 2.1. Convergence is reinforced by the fact that the amount $2 + \alpha + \beta$ is replaced with $2 + \alpha + m\beta$. ∎

We have the two following lemmas:

Lemma 2.2. *A logarithmic chirp is a generalized chirp.*

Lemma 2.3. *If* g *is an indefinitely oscillating function then* $g(x)e^{i\gamma \log x}$ *is an indefinitely oscillating fuction for any* γ *real*

Both lemmas are immediate consequences of Theorem 2.1 of the circular Y.Meyer ([12],[13]).

According to lemmas **(2.2)** et **(2.3)**, the term $e^{i\gamma \log|x|}$ may disappear in g. Therefore the definition **(2.1)** has no interest in the case where g is not periodic.

3. STUDY OF THE ACTION OF SINGULAR INTEGRAL OPERATORS ON CHIRPS

In a first, define the differential operator M_z, where z is a complex number, by :

$$\widehat{M_z f}(\xi) = sign\xi \; |\xi|^{iz} \; \widehat{f}(\xi)$$

remark :

- si $z = 0$, M_0 is then the operator Hilbert transform .
- M_i, M_{-i} are respectively integration and derivative operator.

3.1 Wavelet characterization.

M_z is the singular integral operator defined as above. Note that its adjoint operator is written $M_z^* = M_{-\overline{z}}$.

Soit ψ a function of the real variable having the following properties :

- ψ belongs to the class of Schwartz $S(\mathbb{R})$.
- $\int x^q \psi(x)dx = 0 \quad \forall q \in \mathbb{N}$.
- ψ is not degenerate sense that its Fourier transform is not identically zero over $]-\infty, 0]$ or on the other $[0, \infty[$.

We have then for every chirp as defined above, the following identity:

$$\begin{aligned} < M_z(f), \psi(a,b) > &= < f, M_z^* \left(\psi(a,b)\right) > \\ &= < f, M_{-\overline{z}}(\psi(a,b)) > \end{aligned}$$

By definition of the operator M_z we have :

$$\begin{aligned} M_{-\overline{z}}(\widehat{\psi}(a,b))(\xi) &= sign\xi \; |\xi|^{-i\overline{z}} \widehat{\psi}_{(a,b)}(\xi) \\ &= sign\xi |\xi|^{-i\overline{z}} \widehat{\psi}(a\xi) e^{-2ib\pi\xi} \\ &= a^{-i\overline{z}} \widehat{M_{-\overline{z}}(\psi)}(a\xi) e^{-2ib\pi\xi} \end{aligned}$$

We deduce the following equality:

$$M_{-\overline{z}}(\psi(a,b)) = (M_{-\overline{z}}(\psi))_{(a,b)} a^{i\overline{z}}$$

Then we will have

$$< M_z(f), \psi(a,b) > = a^{-iz} < f, \psi^*_{(a,b)} >$$

where $\psi^* = M_{-\overline{z}}(\psi)$, which means $\widehat{\psi}^*(\xi) = sign\xi \; |\xi|^{-i\overline{z}} \widehat{\psi}(\xi)$.

We verify easily that

- ψ^* belongs to the Schwartz class $S(\mathbb{R})$.
- $\int x^q \psi^*(x)dx = 0 \quad \forall q \in \mathbb{N}$
- ψ^* is not degenerate in the sens that its Fourier transform is identically zero over $]-\infty, 0]$ or on the other $[0, \infty[$.

Characterization of chirps in L^∞ by wavelet is given by an estimate of the module of thewavelet coefficients . Let us take : $z = \gamma_1 + i\gamma_2$ we have :

$$\begin{aligned} |< M_z(f), \psi(a,b) >| &= \left| a^{-iz} < f, \psi^*_{(a,b)} > \right| \\ &= a^{\gamma_2} | < f, \gamma^*_{(a,b)} > | \end{aligned}$$

The class of fuctions wich is stable is the set of the sum of chirp and functions in C^∞".This is also the only is characterized by the wavelet coefficients. We deduce the following lemma:

Lemma 3.1. *f is the sum of a chirp type (α, β, r) and an indefinitely differntiable function in the neighborhood of 0 with $\alpha > -1, \beta > 0$ and $r > 0$. $z = \gamma_1 + i\gamma_2$ is a complex number such that $r + \gamma_2 > 0$ and $\alpha + \gamma_2(1+\beta) > -1$, then the action of the operator M_z on f is* *a chirp of type* $(\alpha+\gamma_2(1+\beta),\beta,r+\gamma_2)$ *in* 0 *modulo an indefinitely differentiable function at the neighborhood of* 0 *.*

Let us now examine the case of a trigonometric chirp of type (α,β) and of smoothness r in 0 [12]. f therefore admits the following asymptotic expansion for all x in $[0,1]$ and all $q\geq 1$:

$f(x)=x^\alpha[g_0(x^{-\beta})+x^\beta g_1(x^{-\beta})+\cdots+x^{(q-1)\beta}g_{q-1}(x^{-\beta})]+x^{\alpha+q\beta}R_q(x^{-\beta})$ The function g_j are subject to the following conditions : g_j belongs to the Hölder space $C^{r+j}(\mathbb{R})$, is $2\pi-periodic$ for the variable t and $\int_0^{2\pi}g_j(t)dt=0$. The function R_q appearing in the error term , must belong to Hölder space $C^{r+q}(\mathbb{R})$ and be an indefinitely oscillating . Remember that $\alpha>-1,\beta>0$ and $r>0$.

Then there exists a number $\delta>0$ and a function ψ satisfying the above conditions, such that the wavelet transform $W(a,b)$ of f has the next property:

Once restricted to each of curves $a=\lambda b|b|^\beta$ $\lambda\in\mathbb{R}$, admits the following asymptotic expansion when $0<a<\delta$ et $|b|\leq\delta$:

$W(\lambda b|b|^\beta,b)=|b|^\alpha[m_\lambda^{(0)}(|b|^{-\beta})+|b|^\beta m_\lambda^{(1)}(|b|^{-\beta})+\cdots+|b|^{(q-1)\beta}m_\lambda^{(q-1)}(|b|^{-\beta})]+R_q^*(a,b)$

where each function $m_\lambda^{(j)}(t)$, $j\in\mathbb{N}$ is measurable in the set of variables ; $m_\lambda^{(j)}(t)$, $j\in\mathbb{N}$ and for each λ fixed, the function $2\pi-periodic$ of t and verify:

$$\|m_\lambda^{(j)}(.)\|_\infty\leq C_j\lambda^{r+j}\text{ if }0<\lambda\leq 1$$
$$\|m_\lambda^{(j)}(.)\|_\infty\leq C_{j,n}\lambda^{-n}\text{ if }\lambda\geq 1\ \ \forall n\in\mathbb{N}$$

Similarly R_q^* satisfies the following estimates :

$$|R_q^*|\leq C_q|b|^{\alpha+q\beta}\lambda^{r+q}\text{ if }0<\lambda\leq 1$$
$$|R_q^*|\leq C_m|b|^{\alpha+q\beta}\lambda^{-m}\quad\forall m\in\mathbb{N}\text{ if }\lambda\geq 1,a\leq|b|$$
$$|R_q^*|\leq C_m a^m\quad\forall m\in\mathbb{N}\text{ and si }|b|\leq a<\delta$$

We have already established that

$$<M_z(f),\psi(a,b)>=-a^{-iz}<f,\psi^*_{(a,b)}>$$

f is a trigonometric chirp, we will restrict ourselves to each of the curves $a=\lambda b|b|^\beta$, with λ any real. Then we have :

$$<M_z(f),\psi(a,b)>=-(\lambda b|b|^\beta)^{-iz}<f,\psi^*_{(a,b)}>$$
$$=-(\lambda b|b|^\beta)^{-i\gamma_1+\gamma_2}W^*(\lambda b|b|^\beta,b)$$

Therfore if $\gamma_1=0$ and if $r+\gamma_2>0$ then $M_{i\gamma_2}(f)$ is a trigonometric chirp of type $(\alpha+\gamma_2(\beta+1),\beta,r+\gamma_2)$.

Hence the following result, reasoning modulo functions C^∞ :

Lemma 3.2. *Soit f a trigonometric chirp of type (α,β) of regularity r in 0. Si $r+\gamma>0$, then $M_{i\gamma}(f)$ is a trigonometric chirp of type $(\alpha+\gamma(\beta+1),\beta)$ of regularity $r+\gamma$ in 0.*

3.2 Direct Kernel method.

In this part we try to improve results in the former, by explaining the first term of the asymptotic expansion of the action of singular integral operator on the chirp [15]. The Kernel operator of M_z, noted K_z, is given by:

$$K_z(x)=C_z\frac{|x|^{iz}}{x}$$

Therefore :

$$M_z(f)(x)=\int_{IR}f(y)\frac{|x-y|^{-iz}}{x-y}dy$$

Before stating the fundamental theorem, making the following hypotheses:

Let $\eta>0$ and let f a generalized chirp of type (α,β,r) with $\alpha>-1,\beta>0$ and $r>0$.

$$\begin{cases} f(x) = x^{\alpha} g_{+}\left(x^{-\beta}\right) & \text{if} \quad 0 < x < \eta \\ f(x) = |x|^{\alpha} g_{-}\left(|x|^{-\beta}\right) & -\eta < x < 0 \end{cases}$$

g_{+} dans g_{-} are both indefinitely oscillating on $[T, \infty[$ with $T = \eta^{-\beta}$. We will take $z = \gamma_1 + i\gamma_2$ is a complex number such that $r - \gamma_2 > 0$. So we have the following assertion:

Theorem 3.3. *For all x in $]-\eta, \eta[$ we have :*

$$M_z\left(x^{\alpha}\mathbf{g}_{\pm}\left(x^{-\beta}\right)\right) =$$
$$-\beta^{iz}x^{\alpha-iz(\beta+1)}\left(\mathbf{M}_z\left(\mathbf{g}_{\pm}\right)\right)\left(x^{-\beta}\right)+x^{\alpha-iz(\beta+1)+\beta}g_{\pm}^{(1)}(x^{-\beta})+\cdots+x^{\alpha-iz(\beta+1)+q\beta}g_{\pm}^{(q)}(x^{-\beta})+\cdots$$

where the functions $g_{\pm}^{(i)}$ are indefinitely oscillating for all i is an integer other than zero.

This theorem explicitly gives the first term of the asymptotic expansion of the action of the operator M_z on a chirp.

We write :

$$\begin{aligned} \widehat{M_z(f)}(\xi) &= sign\xi\, |\xi|^{-iz}\widehat{f}(\xi) \\ &= |\xi|^{\gamma_2} sign\xi\, |\xi|^{-i\gamma_1}\widehat{f}(\xi) \\ &= |\xi|^{\gamma_2}\widehat{M_{\gamma_1}(f)}(\xi) \end{aligned}$$

let $\Lambda = (-\Delta)^{\frac{1}{2}}$ whose symbol $|\xi|$ we will have Λ^{γ_2} whose symbol $|\xi|^{\gamma_2}$(is a fractional derivative operator).

3.2. Operator study M_γ,γ is real.

Always with the same notations and by limiting to $f\chi_{[0,\infty)}$, we set $f(x) = x^{\alpha}g(x^{-\beta})$ with $\alpha > -1, \beta > 0$ and g is an indefinitely oscillating function. And for all γ real we have :

$$M_\gamma f(x) = \int_{-1}^{1} f(t)\frac{|t-x|^{-i\gamma}}{t-x}dt$$

It is clear that outside the interval $[0, 1]$, $M_\gamma f$ is a function C^{∞}. The decomposition of our integral is given by :

$$\begin{aligned} M_\gamma f(x) &= \int_{0}^{x(1-\epsilon)} f(t)\frac{|t-x|^{-i\gamma}}{t-x}dt \\ &+ \int_{x(1-\epsilon)}^{x(1+\epsilon)} f(t)\frac{|t-x|^{-i\gamma}}{t-x}dt \\ &+ \int_{x(1+\epsilon)}^{1} f(t)\frac{|t-x|^{-i\gamma}t-x}{d}t \end{aligned}$$

ϵ is a strictly positive real such that $\epsilon << 1$ as well as $\beta\epsilon << 1$.

We have this theorem :

Theorem 3.4. $\int_{x(1-\epsilon)}^{x(1+\epsilon)} f(t)\frac{|t-x|^{-i\gamma}}{t-x}dt = -\beta^{i\gamma}x^{\alpha-i\gamma(\beta+1)}\int_{x^{-\beta}(1-\epsilon)^{-\beta}}^{x^{-\beta}(1+\epsilon)^{-\beta}} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt + x^{\alpha-i\gamma(\beta+1)+\beta}g_1(x^{-\beta})$ *where g_1 is a function indefinitely oscillating of regularity $r+1$.*

Remark : The main component of this development is given by $M_\gamma(f)$ in the neighborhood of the singularity. Let the integral above we will note $I_1(x)$, then we have :

$$\begin{aligned} I_1(x) &= \int_{x(1-\epsilon)}^{x(1+\epsilon)} f(t)K_\gamma\left(x-t\right)dt \\ &= \int_{x(1-\epsilon)}^{x(1+\epsilon)} t^{\alpha}g\left(t^{-\beta}\right)\frac{|t-x|^{-i\gamma}}{t-x}dt \end{aligned}$$

Proceed to the next change of variable : $u = t^{-\beta}$ and $X = x^{-\beta}$, $I_1(x)$ then becomes:

$$\begin{aligned} I_1(X) &= \frac{1}{\beta}\int_{X(1-\epsilon)^{-\beta}}^{X(1+\epsilon)^{-\beta}} g(u)u^{-\left(\frac{\alpha+\beta+1}{\beta}\right)}\frac{\left|u^{-\frac{1}{\beta}}-X^{-\frac{1}{\beta}}\right|^{-i\gamma}}{u^{-\frac{1}{\beta}}-X^{-\frac{1}{\beta}}}du \\ &= \frac{1}{\beta}\int_{X(1-\epsilon)^{-\beta}}^{X(1+\epsilon)^{-\beta}} g(u)\left(\frac{u}{X}\right)^{-\frac{\alpha+\beta}{\beta}}X^{\frac{i\gamma-\alpha-\beta}{\beta}}\frac{\left|\frac{u}{X}^{-\frac{1}{\beta}}-1\right|^{-i\gamma}}{1-\frac{u}{X}^{\frac{1}{\beta}}}du \end{aligned}$$

Proceed to the change of variable :

$$\theta = \frac{u}{X} - 1, \gamma_1 = -\frac{\alpha+\beta}{\beta} \text{ et } \gamma_2 = \frac{1}{\beta}$$
$$F(\theta) = (1+\theta)^{\gamma_1} \frac{\left|(1+\theta)^{-\gamma_2} - 1\right|^{-i\gamma}}{1-(1+\theta)^{\gamma_2}}$$

A limited development of F in θ in the neighborhood of 0 gives

$$F(\theta) = \frac{1+O(\theta)}{\theta(-\gamma_2+O(\theta))} | -\gamma_2\theta + O(\theta^2)|^{-i\gamma}$$
$$= -\beta^{1+i\gamma}|\theta|^{-i\gamma-1} + |\theta|^{-i\gamma}O(1)$$

We take this formula and inject there in the integral I_1, he comes :

$$I_1(x) = -\beta^{i\gamma} X^{\frac{i\gamma-\alpha-\beta}{\beta}} \int_{X(1+\epsilon)^{-\beta}}^{X(1-\epsilon)^{-\beta}} g(u)|\theta|^{-i\gamma-1} du + O(1) X^{\frac{i\gamma-\alpha-\beta}{\beta}} \int_{X(1+\epsilon)^{-\beta}}^{X(1-\epsilon)^{-\beta}} g(u)|\theta|^{-i\gamma} du$$

θ is replaced by $\frac{u}{X} - 1$ and $x = X^{\frac{-1}{\beta}}$ and we have :

$$I_1(x) = -\beta^{i\gamma} x^{\alpha - i\gamma(\beta+1)} \int_{x^{-\beta}(1+\epsilon)^{-\beta}}^{x^{-\beta}(1-\epsilon)^{-\beta}} g(u) \frac{|u-x^{-\beta}|^{-i\gamma}}{u-x^{-\beta}} du +$$
$$x^{\alpha - i\gamma(\beta+1)+\beta} O(1) \int_{x^{-\beta}(1+\epsilon)^{-\beta}}^{x^{-\beta}(1-\epsilon)^{-\beta}} g(u)|u-x^{-\beta}|^{-i\gamma} du$$

It remains to proof that the function h defined by :

$$h(x) = \int_{x(1+\epsilon)^{-\beta}}^{x(1-\epsilon)^{-\beta}} g(u)|u-x|^{-i\gamma} du$$

is a indefinitely oscillating function of global regularity is $r+1$.

Indeed

$$h(x) = \int_{x(1+\epsilon\prime)}^{x(1-\epsilon\prime)} g(u)|u-x|^{-i\gamma} du$$
$$= \int_{-x\epsilon\prime}^{x\epsilon\prime} g(u+x)|u|^{-i\gamma} du$$

with $\epsilon' \simeq \beta\epsilon << 1$. So we have :

$$h(x) = \int_0^{x\epsilon\prime} [g(x+u) - g(x-u)]|u|^{-i\gamma} du$$
$$= \int_0^1 [g(x+u) - g(x-u)]|u|^{-i\gamma} du + \int_1^{x\epsilon\prime} [g(x+u) - g(x-u)]|u|^{-i\gamma} du$$

The first term of the second member is majorises easily by $2\|g\|_\infty$. Against by, in the second term of the second member, noted by $S(x)$, we proceed to integration by parts :

$$S(x) = \int_1^{x\epsilon\prime} [g(x+u) - g(x-u)]|u|^{-i\gamma} du$$
$$= [|u|^{-i\gamma}(g_1(x+u) - g_1(x-u))]_1^{x\epsilon\prime} + i\gamma \int_1^{x\epsilon\prime} |u|^{-i\gamma-1}[g_1(x+u) - g_1(x-u)] du$$

The first term of the second member is an indefinitely oscillating function according to Lemma **(2.3)**.

We proceed to a new integration by parts for the second term of the second member, noted by $L(x)$:

$$L(x) = \int_1^{x\epsilon\prime} |u|^{-i\gamma-1}[g_1(x+u) - g_1(x-u)] du$$
$$= [|u|^{-i\gamma-1} g_2(x+u) - g_2(x-u)]_1^{x\epsilon'} + (i\gamma+1) \int_1^{x\epsilon\prime} |u|^{-i\gamma} \frac{g_2(x+u) - g_2(x-u)}{u^2} du$$

Similarly, the first term of the second member is an indefinitely oscillating function while the second term is indeed a function in L^∞. We conclude that h is in L^∞. So we have

$$h(x) = \int_{x(1-\epsilon\prime)}^{x(1+\epsilon\prime)} g(u)|u-x|^{-i\gamma} du$$
$$= [g_1(u)|u-x|^{-i\gamma}]_{x(1-\epsilon\prime)}^{x(1+\epsilon\prime)} + i\gamma \int_{x(1-\epsilon\prime)}^{x(1+\epsilon\prime)} g_1(u)|u-x|^{-i\gamma-1} du$$

$$= G_1^*(x) x^{-i\gamma} + i\gamma \int_{x(1-\epsilon\prime)}^{x(1+\epsilon\prime)} g_1(u)|u-x|^{-i\gamma-1} du \tag{3.1}$$

with $G_1^*(x) = \epsilon'^{-i\gamma}[g_1(x(1+\epsilon')) - g_1(x(1-\epsilon'))]$ which is indefinitely oscillating of regularity $r+1$. We noted g_i the ith primitive of g. The second term of the second member of equality becomes after the change of variable $u = sx$

$$\int_{x(1-\epsilon\prime)}^{x(1+\epsilon\prime)} g_1(u)|u-x|^{-i\gamma-1} du = x^{-i\gamma} \int_{1-\epsilon'}^{1+\epsilon'} g_1(sx)|s-1|^{-i\gamma-1} ds$$

Therefore we have:

$$h(x) = x^{-i\gamma} G_1^*(x) + i\gamma \int_{1-\epsilon'}^{1+\epsilon'} g_1(sx)|s-1|^{-i\gamma-1} ds$$

We deduce a first information that h has a regularity equal to $r+1$.

G_1^* is an indefinitely oscillating function, and h is in L^∞, therefore

$$\int_{1-\epsilon\prime}^{1+\epsilon\prime} g_1(sx)|s-1|^{-i\gamma-1} ds$$

is in L^∞, and all the primitives de plus toutes les primitives of that amount can be deduced directly from the primitive ofg_1 ; it is therefore an indefinitely oscillating function. And thanks to the Lemma **(2.3)**, we can deduce that h is an indefinitely oscillating function.

With the same notation as before, we have the following lemma:

Lemma 3.5. $\int_0^{x(1-\epsilon)} t^\alpha g(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt = x^{\alpha - i\gamma(\beta+1)+\beta} \widetilde{G}_1(x^{-\beta})$ *where* $\widetilde{G}_1$*is an indefinitely oscillating function of the global regularity is* $r+1$.

An integration by parts gives:

$$I(x) = \int_0^{x(1-\epsilon)} t^\alpha g(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt$$

$$= x^{\alpha+\beta-i\gamma} \widetilde{g}_1(x^{-\beta}) - \frac{1}{\beta} \int_0^{x(1-\epsilon)} t^{\alpha+\beta} g_1(t^{-\beta}) \left((1+i\gamma) \frac{t}{t-x} - (\alpha+\beta+1) \right) \frac{|t-x|^{-i\gamma}}{t-x} dt$$

where $\widetilde{g}$ is an indefinitely oscillating function of the global regularity is $r+1$. In the integral of the secon term , noted I_2, we proceed to the classical change of variable $t = sx$ and we have :

$$I_2(x) = x^{\alpha+\beta-i\gamma} \int_0^{1-\epsilon} g_1(x^{-\beta} s^{-\beta}) s^{\alpha+\beta} ((1+i\gamma) \frac{s}{s-1} - (\alpha+\beta+1) \frac{|s-1|^{-i\gamma}}{s-1} ds$$

I_2 is written :

$$I_2(x) = x^{\alpha+\beta-i\gamma} h(x^{-\beta})$$

where h is analogously, an indefinitely oscillating function of a global regularity $r+1$. By Lemma **(2.3)**, we can always write $x^{\alpha+\beta-i\gamma} h(x^{-\beta}) = x^{\alpha+\beta-i\gamma+(\beta+1)} H(x^{-\beta})$, with h an indefinitely oscillating function with the same smoothness. With the notation previously encountered, we have the following result:

Lemma 3.6. $\int_{x(1+\epsilon)}^1 t^\alpha g(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt = x^{\alpha - i\gamma(\beta+1)+\beta} F(x^{-\beta}) + r(x)$

where F *is an indefinitely oscillating function of the global regularity* $r+1$ *et* $r(x)$ *is an indefinitely differentiable function in the neighborhood of* 0.

An integration by parts gives:

$$\int_{x(1+\epsilon)}^1 t^\alpha g(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt = \left[-\frac{1}{\beta} g_1(t^{-\beta}) t^{\alpha+\beta+1} \frac{|t-x|^{-i\gamma}}{t-x} \right]_{x(1+\epsilon)}^1$$

$$-\frac{1}{\beta} \int_{x(1+\epsilon)}^1 t^{\alpha+\beta} g_1(t^{-\beta}) \left((1+i\gamma) \frac{t}{t-x} - (\alpha+\beta+1) \right) \frac{|t-x|^{-i\gamma}}{t-x} dt$$

The first term of the second member is written in the form of a function indefinitely differentiable in the neighborhood of 0 and of $x^{\alpha+\beta-i\gamma} \widetilde{g}_1(x^{-\beta})$, where $\widetilde{g}_1$ is an indefinitely oscillating function of the global regularity $r+1$. The second term of the second member consists of two terms:

$$I_3(x) = \int_{x(1+\epsilon)}^1 t^{\alpha+\beta+1} g_1(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{(t-x)^2} dt$$

$$I_4(x) = \int_{x(1+\epsilon)}^1 t^{\alpha+\beta} g_1(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt$$

The integral I_4, poses no problems since it is the same type that $I(x)$.

As regards the other term, we begin by posing

$$I_5(x) = \int_{x(1+\epsilon)}^1 t^{\alpha+\beta+1} g_1(t^{-\beta}) \frac{|t-x|^{-i\gamma}}{t-x} dt$$

Differentiating I_5, we get :

$$I_5'(x) = x^{\alpha+\beta-i\gamma}\widetilde{h}(x^{-\beta}) + C_\gamma I_3(x)$$

$\widetilde{h}$ is an indefinitely oscillating function of the global regularity $r+1$, and $I_5(x)$ is the same type $I(x)$, wich is a chirp in 0 and it is written :

$$I(x) = x^{\alpha+\beta-i\gamma}\widetilde{h}(x^{-\beta}) + r(x)$$

where $\widetilde{h}$ is an indefinitely oscillating function of the global regularity $r+1$, $r(x)$ is an indefinitely differentiable function in the neighborhood of 0.

So we just proved the following result :

$$M_\gamma(f)(x) = -\beta^{i\gamma}x^{\alpha-i\gamma(\beta+1)}\int_{x^{-\beta}(1-\epsilon)^{-\beta}}^{x^{-\beta}(1+\epsilon)^{-\beta}} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt + x^{\alpha-i\gamma(\beta+1)+\beta}\widetilde{g}_1(x^{-\beta})$$

where $\widetilde{g}$ is an indefinitely oscillating function of the global regularity $r+1$.

Analyze conclusion, the two remaining integrals and with the same previous notation we have:

Lemma 3.7. $x^{\alpha-i\gamma(\beta+1)}\int_1^{x^{-\beta}(1+\epsilon)^{-\beta}} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt = x^{\alpha-i\gamma(\beta+1)+\beta}h_0(x) + r(x)$

$x^{\alpha-i\gamma(\beta+1)}\int_{x^{-\beta}(1-\epsilon)^{-\beta}}^{+\infty} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt = x^{\alpha-i\gamma(\beta+1)+\beta}h_1(x)$ *where the functions* h_0 *et* h_1 *are indefinitely oscillating function of the global regularity* $r+1$, *and* $r(x)$ *is an indefinitely differentiable function in the neighborhood of* 0.

Indeed an integration by parts gives:

$$\int_{x^{-\beta}(1+\epsilon\prime)}^{+\infty} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt = \left[g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}\right]_{x^{-\beta}(1-\epsilon\prime)}^{+\infty} + C_\gamma\int_{x^{-\beta}(1+\epsilon\prime)}^{+\infty} g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{\left(t-x^{-\beta}\right)^2}dt$$

The first term is written:

$$x^\beta u(x^{-\beta}) \text{ where } u(x) = -\epsilon'^{-i\gamma-1}g_1(x(1+\epsilon'))x^{-i\gamma}$$

Therefore, u is an indefinitely oscillating function of the global regularity $r+1$.

In the other integral, we proceed to the change of variable $t = sx^{-\beta}$. This will provide :

$$\int_{x^{-\beta}(1+\epsilon\prime)}^{+\infty} g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{\left(t-x^{-\beta}\right)^2}dt = x^{(i\gamma+1)\beta}\int_{1+\epsilon\prime}^{+\infty} g_1(sx^{-\beta})\frac{|s-1|^{-i\gamma}}{(s-1)^2}ds$$
$$= x^\beta H(x^{-\beta})$$

where H is an indefinitely oscillating function of the global regularity $r+1$. It is given by the following formula:

$$H(x) = x^{-i\gamma}\int_{1+\epsilon\prime}^{+\infty} g_1(sx)\frac{|s-1|^{-i\gamma}}{(s-1)^2}ds$$

Indeed H is in L^∞ and all the primitives of $\int_{1+\epsilon\prime}^{+\infty} g_1(sx)\frac{|s-1|^{-i\gamma}}{(s-1)^2}ds$ are deduced directly from the derivatives of g_1. Come in to ultimate integral:

$$J(x) = \int_1^{x^{-\beta}(1+\epsilon)^{-\beta}} g(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt$$

From what precedes, and already we know that $J(x)$ is a chirp in 0. An integration by parts gives :

$$J(x) = x^{\alpha+\beta-i\gamma(\beta+1)}\widetilde{h}(x^{-\beta}) + C_\gamma x^{\alpha-i\gamma(\beta+1)}\int_1^{x^{-\beta}(1+\epsilon)^{-\beta}} g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{\left(t-x^{-\beta}\right)^2}dt + r(x)$$

where $\widetilde{h}(x) = \epsilon'^{-i\gamma-1}g_1(x(1-\epsilon'))x^{-i\gamma}$ wich is an indefinitely oscillating function of global regularity $r+1$, and $r(x)$ is an indefinitely differentiable function in the neighborhood of 0.

Note the integral $K(x)$:

$$K(x) = \int_1^{x^{-\beta}(1+\epsilon)^{-\beta}} g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{t-x^{-\beta}}dt + r(x)$$
$$K'(x) = x^\beta g^*(x^{-\beta}) + C_\gamma\int_1^{x^{-\beta}(1+\epsilon)^{-\beta}} g_1(t)\frac{|t-x^{-\beta}|^{-i\gamma}}{\left(t-x^{-\beta}\right)^2}dt$$

where $g^* = x^{-i\gamma}\epsilon'^{-i\gamma-1}g_1(x(1-\epsilon'))$ is an indefinitely oscillating function of global regularity $r+1$. So we have the last lemma. We had seen that the action of an operator $M_{i\gamma}$ on a trigonometric chirp of type $(\alpha,\beta,r,2\pi)$ in 0 as we have $r-\gamma>0$ et $\alpha-\gamma(\beta+1)>-1$, is a trigonometric chirp of type $(\alpha-\gamma(\beta+1),\beta,r-\gamma,2\pi)$ in 0.

Let g_+ et g_- are two functions in C^r and $2\pi-periodic$ of the variable t. And we have $\int_0^{2\pi} g_+(t)dt = 0, \alpha > -1$ and $\beta > 0$.

Definition 3.1. f is a trigonometric pure chirp of type $(\alpha,\beta,r,2\pi)$ in 0 if and only if we have :

$$f(x) = x^\alpha g_+(x^{-\beta}) \text{ pour } 0 < x < \eta$$
$$f(x) = |x|^\alpha g_-(|x|^{-\beta}) \text{ pour } -\eta < x < 0$$

where $g_\pm$ are subject to the above properties (frequency, regularity and vanishing integral).

Precisely this definition was not adequate in the sense that the action of an operator $M_{i\gamma}$ (which is other than an operator of derivation or integration in the case where $\gamma\neq 0$) on a pure trigonometric chirp does not give a pure trigonometric chirp

Now we will show a more accurate result in the case of the Hilbert transform: $(\gamma = 0)$. With the same notations as we have:

Lemma 3.8. *$H(f)(x) = -|x|^\alpha H(g_\pm)(|x|^{-\beta}) + r(x)$ where $r(x)$ is an indefinitely differentiable in the neighborhood of 0.*

The above results allow us to write that:

$$H(f)(x) = -x^\alpha H(g)(x^{-\beta}) + x^{\alpha+\beta}g^*(x^{-\beta}) + r(x)$$

where $r(x)$ is a function in C^∞ , and where g^* is an indefinitely oscillating function of global regularity $r+1$. Suppose that f is a sum of two pure trigonometric chirp :

$$f(x) = x^\alpha g(x^{-\beta}) + x^{\alpha+\beta}g_1(x^{-\beta})$$

g, g_1 are $2\pi-periodic$ of class respectively C^r and C^{r+1} and such that

$$\int_0^{2\pi} g(t)dt = \int_0^{2\pi} g_1(t)dt = 0$$

Then we have (modulo a function C^∞) :

$$\begin{aligned} H(f)(x) &= -x^\alpha H(g)(x^{-\beta}) + x^{\alpha+\beta}g^*(x^{-\beta}) - x^{\alpha+\beta}H(g_1)(x^{-\beta}) + x^{\alpha+2\beta}g_1^*(x^{-\beta}) \\ &= -x^\alpha H(g)(x^{-\beta}) + x^{\alpha+\beta}[g^* - H(g_1)](x^{-\beta}) + x^{\alpha+2\beta}g_1^*(x^{-\beta}) \end{aligned}$$

g^*et $g_1^*$are two indefinitely oscillating functions of global regularity respectively $r+1$ and $r+2$.

we know that for any given row, the asymptotic expansion of a trigonometric chirp is unique.

f is a trigonometric chirp, $H(f)$ is too. Then necessarily we have $g^* - H(g_1)$ is a function $2\pi-periodic$. We conclude that g^* and therefore g_1^* are $2\pi-periodic$ average to zero. Or

$$g^*(x) = xH(g)(x) + x^{1+\frac{\alpha}{\beta}}H(f)(x^{-\frac{1}{\beta}})$$

We verifie that g^* is not periodic since g^* is written in the following form :

$$g^*(x) = x\int \frac{g(sx)+s^\alpha g(xs^{-\frac{1}{\beta}})}{s-1}ds$$

The function $xg(sx) + s^\alpha g(xs^{-\frac{1}{\beta}})$ is not periodic if g is not periodic. Therefore the function $x^{\alpha+\beta}g^*(x^{-\beta})$ is an indefinitely differentiable.

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