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A LOW ORDER LEAST-SQUARES NONCONFORMING FINITE ELEMENT METHOD FOR STEADY MAGNETOHYDRODYNAMIC EQUATIONS

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ABSTRACT. A low order least-squares nonconforming finite element(NFE) method is proposed for magnetohydrodynamic equations with EQ_1^{rot} element and zero-order Raviart-Thomas element. Based on the above element's typical interpolations properties, the existence and uniqueness of the approximate solutions are proved and the optimal order error estimates for the corresponding variables are derived.

Key words and phrases: Magnetohydrodynamic problems; NFE; Least-squares method; Optimal order error estimates.

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1. INTRODUCTION

The magnetohydrodynamic (MHD) equations is a system of the electrically conducting fluid flow in which the electromagnetic forces can be of the same order or even greater than hydrodynamic ones. Its simplest form takes from [1]:

$$(1.1)\qquad \begin{cases} -k\triangle u + \mathbf{a}\cdot\nabla b = f & in \quad \Omega, \\ -k\triangle b + \mathbf{a}\cdot\nabla u = g & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \\ b = 0 & on \quad \partial\Gamma_D, \\ \nabla b\cdot\mathbf{n} = 0, & on \quad \partial\Gamma_N, \end{cases}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, the symbols Δ and ∇ stand for the Laplacian and gradient operators, respectively; $u = u(x, y), b = b(x, y)$are the velocity and the induced magnetic field in the z-direction, respectively; $0 < k = 1/M < 1$is the diffusivity coefficient and $M = b_0 l(\delta/\mu)^{1/2}$ is the Hartmann number, where b_0 is the intensity of the external magnetic field, l is the characteristic length of the duct, δ and μ are the electric conductivity and coefficient of viscosity of the fluid respectively, and in industrial applications one typically has $10^2 \leq M \leq 10^6$; $\mathbf{a} = (-sin\alpha, -cos\alpha)^T$, α is the angle between the externally applied magnetic field b_0 and the x-axis; $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are given source terms; $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D has a positive measure and $\Gamma_D \cap \Gamma_N = \varnothing$; $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$.

The MHD equations (1.1) are widely applied in fusion technology, novel submarine propulsion devices, the flow of liquid metals in magnetic pumps used to cool nuclear reactors. Therefore, the researches for the system of equations are of the very important and actual meaning.

The stationary MHD problems are first initially analyzed and the format of first order error estimate was derived in [2]. A different strategy to achieve convergence is realized in [3]. This observation recently motivated the works [2, 3], for example, a recent summary of known results for the MHD equations, including modeling, analysis, and numerics is [4]. However, these methods all require that the combination of finite element subspaces should satisfy LBB stability condition [5]. In order to circumvent this constraint, the stabilized finite element methods have been motivated [6]. Recently, [7] showed that the stabilized finite element method using the residual-free bubbles seems robust in MHD duct flow problems at high Hartmann numbers. However, an apparent disadvantage of this approach for convection-dominated problems is that the resulting linear system is not symmetric. Thus, some efficient and robust solvers for linear systems such as the conjugate gradient method can not be applied directly.

As we know, the least-squares methods can circumvent above two constrains. In addition, least-squares formulation satisfies a priori coercivity inequality and generates positive definite algebraic system matrices, which can be solved using standard and robust iterative methods such as conjugate gradient methods [8].

Thus the least-squares methods have become more and more frequently used to approximate MHD equations [1, 8, 9]. However, all of the analysis in [1, 8, 9] are about the conforming FEMs.

Recently, [10] proposed least-squares methods of NFEs for the second-order elliptic problem on different meshes in a unified way and gave the convergence analysis and error estimates, [11] studied the least-square Galerkin-Petrov method of NFE for the stationary conduction-convection problem and obtained the corresponding optimal order error estimates, [12, 13] proposed a family of low order nonconforming mixed FEMs to approximate MHD equations and obtained the corresponding optimal order error estimates based on H^1-conforming elements and $H(curl)$-conforming (edge) elements to approximate the magnetic field, respectively.

As a continuous work, the main aim of this paper is to propose and analyze a low order least-squares NFE method for MHD equations by choosing suitable FE spaces: EQ_1^{rot} element [14]and zero-order Raviart -Thomas element. At the same time, through typical interpolations properties of the above two elements (see Lemma 2.2 and Remarks 2.3-2.4 below), we prove the existence and uniqueness of the discrete solutions and derive the corresponding variables optimal order error estimates. we mention that the method provided in this paper is also valid for some other very popular elements such as Q_1^{rot} NFE space discussed in [17]on the square meshes, the constrained Q_1^{rot} NFE and P_1 NFE spaces proposed in [18] and [19] on the rectangular meshes etc, but not valid for the Crouzeix-Raviart type linear triangular element [20], Wilson element [21] and quasi-Wilson element [22], etc. So it is not a easy thing to choose a appropriate space pair and derive optimal order error estimates. Furthermore, it remains open to extend the results obtained in this paper to arbitrary quadrilateral meshes.

2. LEAST-SQUARES METHOD OF NFES

In order to apply the least-square finite element method to approximate the solution of problem (1.1). We introduce two additional variables Φ and $\mathbf{j}$ by

$$\Phi = -k\nabla u \; on \; \bar{\Omega},$$
$$\mathbf{j} = -k\nabla b, \; on \; \bar{\Omega}.$$

Then we can rewrite (1.1) in the following first order system:

$$\begin{cases} \nabla\cdot\Phi + \mathbf{a}\cdot\nabla b = f \; in \; \Omega, \\ \nabla\cdot\mathbf{j} + \mathbf{a}\cdot\nabla u = g \; in \; \Omega, \\ \Phi + k\nabla u = 0 \; in \; \Omega, \\ \mathbf{j} + k\nabla b = 0 \; in \; \Omega, \\ u = 0 \; on \; \partial\Omega, \\ b = 0 \; on \; \partial\Gamma_D, \\ \mathbf{j}\cdot n = 0 \; on \; \partial\Gamma_N. \end{cases} \tag{2.1}$$

We define the following four function spaces:

$$\begin{cases} U = H_0^1(\Omega) = \{v \in H^1(\Omega), \; v|_{\partial\Omega} = 0\}, \\ \mathcal{C} = \{c : c \in H^1(\Omega), \;\; c|_{\partial\Gamma_D} = 0\}, \\ \mathcal{W} = \mathbf{H}(div;\Omega) = \{\Psi \in [L^2(\Omega)]^2, \nabla\cdot\Psi \in L^2(\Omega)\}, \\ \mathcal{Q} = \{\mathbf{q} : \mathbf{q} \in \mathbf{H}(div;\Omega), \; \mathbf{q}\cdot\mathbf{n} = 0 \; on \; \partial\Gamma_N\}, \end{cases}$$

and for $\Psi \in \mathbf{H}(div;\Omega)$, we define the norm $\|\Psi\|_{div} = (\|\Psi\|^2 + \|\nabla\cdot\Psi\|^2)^{\frac{1}{2}}$, where $\|\cdot\|$ is the L^2-norm.

A least-squares variational problem for (2.1) is to find $(u, b, \Phi, \mathbf{j}) \in U\times\mathcal{C}\times\mathcal{W}\times\mathcal{Q}$ such that

$$B((u,b,\Phi,\mathbf{j}),(v,c,\Psi,\mathbf{q})) = L(v,c,\Psi,\mathbf{q}), \; \forall(v,c,\Psi,\mathbf{q}) \in U\times\mathcal{C}\times\mathcal{W}\times\mathcal{Q}, \tag{2.2}$$

where the bilinear form $B(\cdot;\cdot)$ and the linear form $L(\cdot,\cdot,\cdot,\cdot)$ are respectively defined as follows:

$$\begin{aligned} B((u,b,\Phi,\mathbf{j}),(v,c,\Psi,\mathbf{q})) = {} & (\nabla\cdot\Phi + \mathbf{a}\cdot\nabla b, \nabla\cdot\Psi + \mathbf{a}\cdot\nabla c) \\ & +(\nabla\cdot\mathbf{j} + \mathbf{a}\cdot\nabla u, \nabla\cdot\mathbf{q} + \mathbf{a}\cdot\nabla v) \\ & +(\Phi + k\nabla u, \Psi + k\nabla v) \\ & +(\mathbf{j} + k\nabla b, \mathbf{q} + k\nabla c), \end{aligned} \tag{2.3}$$

$$L(v,c,\Psi,\mathbf{q}) = (f, \nabla\cdot\Psi + \mathbf{a}\cdot\nabla c) + (g, \nabla\cdot\mathbf{q} + \mathbf{a}\cdot\nabla v). \tag{2.4}$$

The following theorem can be found in [1].

Theorem 2.1. *There exist two positive constants C_1 and C_2, both independent of k such that for all $(v, c, \Psi, \mathbf{q}) \in U \times \mathcal{C} \times \mathcal{W} \times \mathcal{Q}$*

$$B((v, c, \Psi, \mathbf{q}); (v, c, \Psi, \mathbf{q})) \leq C_1(\|v\|_1^2 + \|c\|_1^2 + \|\Psi\|_{div}^2 + \|\mathbf{q}\|_{div}^2),$$
$$B((v, c, \Psi, \mathbf{q}); (v, c, \Psi, \mathbf{q})) \geq C_2 k^2(\|v\|_1^2 + \|c\|_1^2 + \|\Psi\|_{div}^2 + \|\mathbf{q}\|_{div}^2).$$

Thus, Lax-Milgram lemma guarantees that Problem (2.2) has a unique solution $(u, b, \Phi, \mathbf{j}) \in U \times \mathcal{C} \times \mathcal{W} \times \mathcal{Q}$.

Now we consider nonconforming finite element formulations of least-square method (2.2).

Let $\Gamma_h = \{K\}$ be a regular rectangular partition of Ω, $K = \{x_K - h_x, x_K + h_x\} \times \{y_K - h_y, y_K + h_y\}$, $h_K = diam\{K\}$ and $h = \max_{K \in \Gamma_h}\{h_K\}$. We take the EQ_1^{rot} NFE space U_h (see [14, 15]) and zero order R-T element space $\mathcal{W}_h$ to approximate U and $\mathcal{W}$, respectively.

Define the FE spaces $U_h, \mathcal{C}_h$ and $\mathcal{W}_h$ by

$$U_h = \{v_h \in L^2(\Omega); v_h|_K \in P, \forall K \in \Gamma_h, \int_l [v_h] ds = 0, l \subset \partial K\},$$
$$\mathcal{C}_h = \{v_h \in L^2(\Omega); v_h|_K \in P, \forall K \in \Gamma_h, \int_l [v_h] ds = 0, l \subset \partial K \cap \partial \Gamma_D\},$$
$$\mathcal{W}_h = \{w \in W; w|_K \in Q_{1,0}(K) \times Q_{0,1}(K) = (a_0 + a_1 x, b_0 + a_1 y), \forall K \in \Gamma_h\},$$
$$\mathcal{Q}_h = \{q \in \mathcal{W}_h; q \cdot n = 0, on\ \partial \Gamma_N\},$$

where $P = span\{1, x, y, x^2, y^2\}$, $[v_h]$ denotes the jump of v_h across the boundary l of K if l is an internal edge, and $[v_h] = v_h$ if $l \subset \partial\Omega$.

Let $I_h : H^1(\Omega) \to U_h$ and $\mathbf{\Pi}_h : (H^1(\Omega))^2 \to \mathcal{W}_h$ be the associated interpolation operators satisfying $I_h|_K = I_K, \mathbf{\Pi}_h|_K = \mathbf{\Pi}_K$, then we have

$$\int_K (v - I_K v) dxdy = 0, \int_{l_i} (v - I_K v) ds = 0, \int_{l_i} (\mathbf{q} - \mathbf{\Pi}_K \mathbf{q}) \cdot \mathbf{n}_i ds = 0,$$

where l_1, l_2, l_3, l_4 are four edges of ∂K, $\mathbf{n}_i$ is the unit outward normal vector to $l_i (i = 1, 2, 3, 4)$.

On the other hand, we can prove the following conclusion:

Lemma 2.2. *For $(v_h, c_h, \Psi_h, \mathbf{q}_h) \in (U_h \times \mathcal{C}_h \times \mathcal{W}_h \times \mathcal{Q}_h)$, there hold*

$$\sum_K \int_{\partial K} \mathbf{a} \cdot \mathbf{n}_K v_h c_h ds = 0, \tag{2.5}$$

$$\sum_K \int_{\partial K} \Psi_h \cdot \mathbf{n}_K v_h ds = 0, \tag{2.6}$$

$$\sum_K \int_{\partial K} \mathbf{q}_h \cdot \mathbf{n}_K c_h ds = 0, \tag{2.7}$$

where $\mathbf{n}_K = (n_1, n_2)$ is the unit outward normal vector to ∂K.

Proof. It is not difficult to check that

$$\begin{aligned} \sum_K \int_{\partial K} \mathbf{a} \cdot \mathbf{n}_K v_h c_h ds &= \sum_K \int_{\partial K} (a_1 \cdot n_1 + a_2 \cdot n_2) v_h c_h ds \\ &= \sum_K (\int_{l_3} - \int_{l_1}) a_2 (v_h - P_{0,i} v_h)(c_h - P_{0,i} c_h) dx \\ &+ \sum_K (\int_{l_2} - \int_{l_4}) a_1 (v_h - P_{0,i} v_h)(c_h - P_{0,i} c_h) dy \\ &= I_1 + I_2, \end{aligned} \tag{2.8}$$

where

$$P_{0,i}v = \frac{1}{|l_i|}\int_{l_i} vds, (i = 1, 2, 3, 4), \forall l \in L^2(L_i),$$

$$I_1 = (\int_{l_3} - \int_{l_1})a_2(v_h - P_{0,i}v_h)(c_h - P_{0,i}c_h)dx,$$

$$I_2 = (\int_{l_2} - \int_{l_4})a_1(v_h - P_{0,i}v_h)(c_h - P_{0,i}c_h)dy.$$

Since $\frac{\partial v_h}{\partial x} = \{1, x\}$, we know that

$$\frac{\partial v_h}{\partial x}(x, y_K + h_y) = \frac{\partial v_h}{\partial x}(x, y_K - h_y).$$

So with the similar argument of [15], we have

$$|I_1| \leq \frac{4h_x^2}{3}\|\frac{\partial v_h}{\partial x}\|_{0,K}\|\frac{\partial^2 c_h}{\partial x\partial y}\|_{0,K}. \tag{2.9}$$

Similarly, $\frac{\partial v_h}{\partial y} = \{1, y\}$, we get

$$|I_2| \leq \frac{4h_y^2}{3}\|\frac{\partial v_h}{\partial y}\|_{0,K}\|\frac{\partial^2 c_h}{\partial y\partial x}\|_{0,K}. \tag{2.10}$$

On the other hand, noting that $c_h \in \mathcal{C}_h = span\{1, x, y, x^2, y^2\}$, we have

$$\frac{\partial^2 c_h}{\partial y\partial x} = \frac{\partial^2 c_h}{\partial x\partial y} = 0. \tag{2.11}$$

Thus

$$\sum_K \int_{\partial K} \mathbf{a}\cdot\mathbf{n}_K v_h c_h ds = 0.$$

Analogously, we can prove (2.6) and (2.7). The proof of Lemma 2.2 is completed. □

Also, it has been shown in [16] that

Lemma 2.3. *For all $v_h \in U_h$, there exists a positive constant C_* such that*

$$\|v_h\| \leq C_*\|v_h\|_h. \tag{2.12}$$

The least-squares scheme for Problem (2.2) is to find $(u_h, b_h, \Phi_h, \mathbf{j}_h) \in (U_h \times \mathcal{C}_h \times \mathcal{W}_h \times \mathcal{Q}_h)$ such that for any $(v_h, c_h, \Psi_h, \mathbf{q}_h) \in U_h \times \mathcal{C}_h \times \mathcal{W}_h \times \mathcal{Q}_h$

$$B_h((u_h, b_h, \Phi_h, \mathbf{j}_h), (v_h, c_h, \Psi_h, \mathbf{q}_h)) = L_h(v_h, c_h, \Psi_h, \mathbf{q}_h), \tag{2.13}$$

where the bilinear form $B_h(\cdot;\cdot)$ and the linear form $L_h(\cdot)$ are respectively defined as follows:

$$\begin{aligned} &B_h((u_h, b_h, \Phi_h, \mathbf{j}_h), (v_h, c_h, \Psi_h, \mathbf{q}_h)) \\ &= (\nabla_h\cdot\Phi_h + \mathbf{a}\cdot\nabla_h b_h, \nabla_h\cdot\Psi_h + \mathbf{a}\cdot\nabla_h c_h) \\ &+(\nabla_h\cdot\mathbf{j}_h + \mathbf{a}\cdot\nabla_h u_h, \nabla_h\cdot\mathbf{q}_h + \mathbf{a}\cdot\nabla_h v_h) \\ &+(\Phi_h + k\nabla_h u_h, \Psi_h + k\nabla_h v_h) \\ &+(\mathbf{j}_h + k\nabla_h b_h, \mathbf{q}_h + k\nabla_h c_h), \end{aligned} \tag{2.14}$$

$$L_h(v_h, c_h, \Psi_h, \mathbf{q}_h) = (f, \nabla_h\cdot\Psi_h + \mathbf{a}\cdot\nabla_h c_h) + (g, \nabla_h\cdot\mathbf{q}_h + \mathbf{a}\cdot\nabla_h v_h). \tag{2.15}$$

3. SOLVABILITY OF THE DISCRETE PROBLEM AND ERROR ESTIMATES

In this section, we will prove the solvability of Problem (2.13) and give error estimates. The following theorem guarantees that Problem (2.13) has a unique solution.

Theorem 3.1. *For $(v_h, c_h, \Psi_h, \mathbf{q}_h) \in U_h \times \mathcal{C}_h \times \mathcal{W}_h \times \mathcal{Q}_h$, there exist two positive constants C_3 and C_4, both independent of k and h such that*

$$B_h((v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h)) \le C_3(\|v_h\|_h^2 + \|c_h\|_h^2 + \|\Psi_h\|_*^2 + \|\mathbf{q}_h\|_*^2), \tag{3.1}$$

$$B_h((v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h)) \ge C_4 k^2(\|v_h\|_h^2 + \|c_h\|_h^2 + \|\Psi_h\|_*^2 + \|\mathbf{q}_h\|_*^2), \tag{3.2}$$

where $\|\cdot\|_*^2 = \|\cdot\|_{0,h}^2 + \|\nabla_h \cdot\|_{0,h}^2$, ∇_h is the gradient operator defined element by element, $\|\cdot\|_h = \sqrt{\sum_K |\cdot|_{1,K}^2}$ is a norm over U_h and $\mathcal{C}_h$.

Proof. The first inequality (3.1) is obvious. We proceed to show the second inequality (3.2).

Let α be a positive constant that will be determined later, utilizing (2.5)-(2.7) and Lemma 2.3, then, we have

$$\begin{aligned}
&B_h((v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h))\\
&= \|\nabla \cdot \Psi_h + \mathbf{a} \cdot \nabla_h c_h - \alpha v_h\|_{0,h}^2 + 2\alpha(\nabla_h \cdot \Psi_h + \mathbf{a} \cdot \nabla_h c_h, v_h) - \alpha^2\|v_h\|_{0,h}^2\\
&+\|\nabla_h \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h - \alpha c_h\|_{0,h}^2 + 2\alpha(\nabla_h \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h, c_h) - \alpha^2\|c_h\|_{0,h}^2\\
&+\|\Psi_h + k\nabla_h v_h - \alpha\nabla_h v_h\|_{0,h}^2 + 2\alpha(\Psi_h + k\nabla_h v_h, \nabla_h v_h) - \alpha^2\|\nabla_h v_h\|_{0,h}^2\\
&+\|\mathbf{q}_h + k\nabla_h c_h - \alpha\nabla_h c_h\|_{0,h}^2 + 2\alpha(\mathbf{q}_h + k\nabla_h c_h, \nabla_h c_h) - \alpha^2\|\nabla_h c_h\|_{0,h}^2\\
&\ge 2\alpha(\nabla_h \cdot \Psi_h + \mathbf{a} \cdot \nabla_h c_h, v_h) - \alpha^2\|v_h\|_{0,h}^2\\
&+2\alpha(\nabla_h \cdot \mathbf{q}_h + \mathbf{a} \cdot \nabla_h v_h, c_h) - \alpha^2\|c_h\|_{0,h}^2\\
&+2\alpha(\Psi_h + k\nabla_h v_h, \nabla_h v_h) - \alpha^2\|\nabla_h v_h\|_{0,h}^2\\
&+2\alpha(\mathbf{q}_h + k\nabla_h c_h, \nabla_h c_h) - \alpha^2\|\nabla_h c_h\|_{0,h}^2\\
&\ge -\alpha^2\|v_h\|_{0,h}^2 - \alpha^2\|c_h\|_{0,h}^2 + 2\alpha k\|\nabla v_h\|_{0,h}^2 - \alpha^2\|\nabla_h v_h\|_{0,h}^2\\
&+2\alpha k\|\nabla_h c_h\|_{0,h}^2 - \alpha^2\|\nabla_h c_h\|_{0,h}^2\\
&\ge \alpha(2k - \alpha(1 + C_*^2))(\|\nabla_h v_h\|_{0,h}^2 + \|\nabla_h c_h\|_{0,h}^2).
\end{aligned} \tag{3.3}$$

Taking $\alpha = k/(1 + C_\star^2)$, we get

$$B_h((v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h)) \ge (k^2/(1 + C_\star^2))(\|v_h\|_h^2 + \|c_h\|_h^2). \tag{3.4}$$

On the other hand, it is not difficult to check that

$$\begin{aligned}
&\|\Psi_h\|_*^2 = \|\Psi_h\|_{0,h}^2 + \|\nabla_h \cdot \Psi_h\|_{0,h}^2\\
&= \|\Psi + k\nabla_h v_h - k\nabla_h v_h\|_{0,h}^2 + \|\nabla_h \cdot \Psi_h + \mathbf{a} \cdot \nabla_h c_h - \mathbf{a} \cdot \nabla_h c_h\|_{0,h}^2\\
&\le 2(\|\Psi_h + k\nabla_h v_h\|_{0,h}^2 + k^2\|\nabla_h v_h\|_{0,h}^2\\
&+ \|\nabla_h \cdot \Psi_h + \mathbf{a} \cdot \nabla_h c_h\|_{0,h}^2 + \|\mathbf{a} \cdot \nabla_h c_h\|_{0,h}^2)\\
&\le 2(B_h(v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h)\\
&+ (1 + C_\star^2)B_h(v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h))\\
&= 2(2 + C_\star^2)B_h(v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h),
\end{aligned} \tag{3.5}$$

$$\|\mathbf{q_h}\|_*^2 \le (C/k^2)B_h(v_h, c_h, \Psi_h, \mathbf{q}_h); (v_h, c_h, \Psi_h, \mathbf{q}_h). \tag{3.6}$$

Combining (3.3)-(3.6) yields the desired result(3.2). The proof is completed. □

Remark 3.1. We can see that the following typical features are essential in the proof of Lemma 2.2, i.e.,

(I)For all $v_h \in U_h$, $\frac{\partial v_h}{\partial x}$ and $\frac{\partial v_h}{\partial y}$ should be independent of y and x respectively;

(II)For all $v_h \in U_h$, $\int_F [v_h] ds = 0, F \subset \partial K$;

Remark 3.2. We can check that Lemma 2.2 is also valid if U_h and $\mathcal{C}_h$ are taken as Q_1^{rot} NFE space discussed in [17] on the square meshes, the constrained Q_1^{rot} NFE and P_1 NFE spaces proposed in [18] and [19] on the rectangular meshes, or $\mathcal{C}_h$ is Q_1 FE space, $\mathcal{W}_h$ and $\mathcal{Q}_h$ are the zero order R-T element space, respectively, however, not valid for other popular nonconforming FEs, such as Crouzeix-Raviart type linear triangular element [20], Wilson element [21] and quasi-Wilson element [22], etc.

Remark 3.3. It can be checked that if U_h is not changed in this paper but we replace the FE space $\mathcal{C}_h$ with the spaces used in [17, 18, 19] and $\mathcal{W}_h = \mathcal{Q}_h$ with that of nonconforming elements used in [23, 24], then the above results of Lemma 2.2 are still valid.

Theorem 3.2. *Let* $(u, b, \Phi, \mathbf{j}) \in (U \cap H^2(\Omega)) \times (\mathcal{C} \cap H^2(\Omega)) \times (\mathcal{W} \cap H^1(div; \Omega)) \times (\mathcal{Q} \cap H^1(div; \Omega))$ *and* $(u_h, b_h, \Phi_h, \mathbf{j}_h) \in U_h \times \mathcal{C}_h \times \mathcal{W}_h \times \mathcal{Q}_h$ *be the solutions of Problems (2.2) and (2.13), respectively. Then*

$$\begin{aligned} &\|u - u_h\|_h + \|b - b_h\|_h + \|\Phi - \Phi_h\|_* + \|\mathbf{j} - \mathbf{j}_h\|_* \\ &\leq Ch\{|u|_2 + |b|_2 + |\Phi|_1 + |\nabla \cdot \Phi|_1 + |\mathbf{j}|_1 + |\nabla \cdot \mathbf{j}|_1\}. \end{aligned} \tag{3.7}$$

Proof. For $(v_h, c_h, \Psi_h, \mathbf{q}_h) \in U_h \times C_h \times \mathcal{W}_h \times \mathcal{Q}_h$, we have

$$\begin{aligned} &\|v_h - u_h\|_h^2 + \|c_h - b_h\|_h^2 + \|\Psi_h - \Phi_h\|_*^2 + \|\mathbf{q}_h - \mathbf{j}_h\|_*^2 \\ &\leq B_h((v_h - u_h, c_h - b_h, \Psi_h - \Phi_h, \mathbf{q}_h - \mathbf{j}_h); (v_h - u_h, c_h - b_h, \Psi_h - \Phi_h, \mathbf{q}_h - \mathbf{j}_h)) \\ &\leq B_h((v_h - u, c_h - b, \Psi_h - \Phi, \mathbf{q}_h - \mathbf{j}); (v_h - u_h, c_h - b_h, \Psi_h - \Phi_h, \mathbf{q}_h - \mathbf{j}_h)) \\ &+ B_h((u - u_h, b - b_h, \Phi - \Phi_h, \mathbf{j} - \mathbf{j}_h); (v_h - u_h, c_h - b_h, \Psi_h - \Phi_h, \mathbf{q}_h - \mathbf{j}_h)). \end{aligned} \tag{3.8}$$

From (2.2)-(2.4) and (2.13)-(2.15), we find

$$B_h((u - u_h, b - b_h, \Phi - \Phi_h, \mathbf{j} - \mathbf{j}_h); (v_h - u_h, c_h - b_h, \Psi_h - \Phi_h, \mathbf{q}_h - \mathbf{j}_h)) = 0. \tag{3.9}$$

Therefore,

$$\|u - u_h\|_h + \|\Phi - \Phi_h\|_* \leq C\{\|u - I_h u\|_h + \|\Phi - \mathbf{\Pi}_h \Phi\|_*\}. \tag{3.10}$$

On the other hand, for each $K \in \Gamma_h$

$$\begin{aligned} \nabla \cdot \mathbf{\Pi}_h \Phi|_K &= \frac{1}{|K|} \int_K \nabla \cdot \mathbf{\Pi}_h \Phi dxdy = \frac{1}{|K|} \int_{\partial K} \mathbf{\Pi}_h \Phi \cdot \mathbf{n}_K ds \\ &= \frac{1}{|K|} \int_{\partial K} \Phi \cdot \mathbf{n}_K ds = \frac{1}{|K|} \int_K \nabla \cdot \Phi dxdy. \end{aligned} \tag{3.11}$$

Thus we have $\nabla \cdot \mathbf{\Pi}_h|_K = P_0 \nabla \cdot |_K$, where P_0 is the local L^2 projection satisfying

$$\|\nabla \cdot (\Phi - \mathbf{\Pi}_h \Phi)\|_{0,K} = \|\nabla \cdot \Phi - P_0 \nabla \cdot \Phi\|_{0,K} \leq Ch|\nabla \cdot \Phi|_{1,K}. \tag{3.12}$$

Substituting (3.12) into (3.10) and applying the interpolation theory, we have

$$\|u - u_h\|_h + \|\Phi - \Phi_h\|_* \leq Ch\{|u|_2 + |\Phi|_1 + |\nabla \cdot \Phi|_1\}. \tag{3.13}$$

Similarly, we obtain

$$\|b - b_h\|_h + \|\mathbf{j} - \mathbf{j}_h\|_* \leq Ch\{|b|_2 + |\mathbf{j}|_1 + |\nabla \cdot \mathbf{j}|_1\}. \tag{3.14}$$

The proof is completed. □

Remark 3.4. We point out that (2.5)-(2.7) are the key conditions leading to the optimal order error estimates in this present work. So it is not a easy thing for one to choose a appropriate space pair to derive Lemma 2.2. On the other hand, the investigation of this paper is carried out for rectangular meshes, how to extend it to arbitrary quadrilateral cases [25, 26] still remains open.

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