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A GENERALIZATION OF OSTROWSKI'S INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION VIA A PARAMETER

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ABSTRACT. In this paper, we provide a generalization of the Ostrowski's inequality for functions of bounded variation for k points via a parameter $\lambda \in [0, 1]$. As a by product, we consider some particular cases to obtained some interesting inequalities in these directions. Our results generalizes some of the results by Dragomir in [S. S. DRAGOMIR, The Ostrowski inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60** (1999), pp. 495–508.]

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1. INTRODUCTION

We start by first recalling some basic properties of functions of bounded variations.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$ be a function. We define the total variation of f on $[a,b]$, denoted by $V_a^b(f)$, as follows;

$$V_a^b(f) = \sup\left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|; \{x_i\}_{i=0}^{n} \text{ is a partition of } [a,b] \right\}$$

We say that f is of bounded variation on $[a,b]$ if $V_a^b(f) < \infty$.

It is well known that if $f : [a,b] \to \mathbb{R}$ is a Lipschitz function, then f is of bounded variation. The following are some important properties of functions of bounded variation.

Proposition 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. The following holds;*

(1) $V_a^b(f) = V_a^x(f) + V_x^b(f)$, *for any $x \in [a,b]$.*
(2) *If $[c,d]$ is a subinterval of $[a,b]$ then $V_c^d(f) \le V_a^b(f)$.*

For more information on functions of bounded variation, we refer the interested reader to [8] and [18].

In 1938, Ostrowski [17] obtained the following inequality which is known in the literature as Ostrowski inequality.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b). If $M := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)M$$

for all $x \in [a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Many authors have studied and generalized this inequality in several different ways. For more information about the Ostrowski inequality and its associates, we refer the interested reader to the papers [1, 2, 4, 3, 9, 10, 11, 12, 15, 16].

Dragomir [2] provided the following extension of Theorem 1.2 for functions of bounded variation.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] V_a^b(f)$$

holds for all $x \in [a,b]$. The constant 1/2 is the best possible.

In [1], Dragomir obtained the following generalization of Theorem 1.3 as follows;

Theorem 1.4. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and $\alpha_i (i = 0, 1, \cdots, k+1)$ be $k+2$ points so that $\alpha_0 = a,\ \alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then we have the inequality*

$$\left| \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) - \int_a^b f(t)dt \right|$$

$$\leq \left[\frac{1}{2}\nu(h) + \max\left\{\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|; i = 0, \cdots, k-1\right\}\right] V_a^b(f)$$
$$\leq \nu(h) V_a^b(f)$$

where $\nu(h) := \max\{h_i | i = 0, 1, \cdots, k-1\}$, $h_i = x_{i+1} - x_i (i = 0, \cdots, k-1)$.

The goal of this paper is to provide a generalization of the Ostrowski inequality for functions of bounded variation with multiple points via a parameter $\lambda \in [0, 1]$. Our results generalizes Theorem 1.4

2. MAIN RESULTS

To prove our main result, we need the following lemmas.

Lemma 2.1 (see [4]). *Let* $p, g : [a, b] \to \mathbb{R}$ *be two functions. If* p *is continuous on* $[a, b]$ *and* g *is of bounded variation on* $[a, b]$*, then*

$$\left|\int_a^b p(t) dg(t)\right| \leq \int_a^b |p(t)| d\Big(V_a^t(g)\Big) \leq \sup_{t \in [a,b]} |p(t)| V_a^b(g).$$

Lemma 2.2. *Let*

(1) $a, b \in \mathbb{R}$, $\lambda \in [0, 1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ *is a partition of the interval* $[a, b]$*;*
(2) $\alpha_i \in \mathbb{R}$ $(i = 0, 1, \cdots, k+1)$ *is* $k+2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$;
(3) $f : [a, b] \to \mathbb{R}$ *is of bounded variation.*
(4) *Define the kernel function* $K(\cdot, I_k) : [a, b] \to \mathbb{R}$ *(see also* [19]*) as follows;*

$$K(t, I_k) = \begin{cases} t - \left(\alpha_1 - \lambda\frac{\alpha_1 - a}{2}\right), \ t \in [a, \alpha_1), \\ t - \left(\alpha_1 + \lambda\frac{\alpha_2 - \alpha_1}{2}\right), \ t \in [\alpha_1, x_1), \\ t - \left(\alpha_2 - \lambda\frac{\alpha_2 - \alpha_1}{2}\right), \ t \in [x_1, \alpha_2), \\ \vdots \\ t - \left(\alpha_{k-1} + \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\right), \ t \in [\alpha_{k-1}, x_{k-1}), \\ t - \left(\alpha_k - \lambda\frac{\alpha_k - \alpha_{k-1}}{2}\right), \ t \in [x_{k-1}, \alpha_k), \\ t - \left(\alpha_k + \lambda\frac{\alpha_{k+1} - \alpha_k}{2}\right), \ t \in [\alpha_k, b], \end{cases}$$

for all $t \in [a, b]$.

Then we have the identity

$$\int_a^b K(t, I_k) df(t) = (1 - \lambda)\sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) + \frac{\lambda}{2}\sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i)\Big(f(\alpha_i) + f(\alpha_{i+1})\Big) - \int_a^b f(t) dt. \tag{2.1}$$

Remark 2.1. It is worth noting that Lemma 2.2 is the special case of [19, Lemma 1] when the time scale $\mathbb{T} = \mathbb{R}$ and the function f is differentiable with some minor correction. However, we provide the proof here for completion.

Proof. We observe that

$$\int_a^b K(t,I_k)df(t) = \sum_{i=0}^{k-1}\Big[\int_{x_i}^{\alpha_{i+1}}(t-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}))df(t) + \int_{\alpha_{i+1}}^{x_{i+1}}(t-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))df(t)\Big].$$

By applying the integration by parts formula for the Riemann–Stieltjes integral on the right-hand side, we have

$$\begin{aligned}\int_a^b K(t,I_k)df(t) &= \sum_{i=0}^{k-1}\Bigg[(\alpha_{i+1}-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}))f(\alpha_{i+1})) \\ &\quad -(x_i-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}))f(x_i)-\int_{x_i}^{\alpha_{i+1}}f(t)dt \\ &\quad +(x_{i+1}-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))f(x_{i+1})-(\alpha_{i+1}) \\ &\quad -(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))f(\alpha_{i+1})-\int_{\alpha_{i+1}}^{x_{i+1}}f(t)dt\Bigg] \\ &= \sum_{i=0}^{k-1}\Bigg[\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(\alpha_{i+1})-(x_i-\alpha_{i+1})f(x_i)-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i) \\ &\quad +(x_{i+1}-\alpha_{i+1})f(x_{i+1})-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(\alpha_{i+1}) \\ &\quad -\int_{x_i}^{x_{i+1}}f(t)dt\Bigg].\end{aligned}$$

It follows that,

$$\begin{aligned}\int_a^b K(t,I_k)df(t) &= \sum_{i=0}^{k-1}\left[\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_{x_i}^{x_{i+1}}f(t)dt\right] \\ &\quad +\sum_{i=0}^{k-1}\Big[-(x_i-\alpha_{i+1})f(x_i)+(x_{i+1}-\alpha_{i+1})f(x_{i+1})\Big] \\ &\quad +\sum_{i=0}^{k-1}\Big[-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i)-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})\Big] \\ &= \sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_a^b f(t)dt \\ &\quad -x_0f(x_0)+x_kf(x_k)+\sum_{i=0}^{k-1}\alpha_{i+1}(f(x_i)-f(x_{i+1})) \\ &\quad +\sum_{i=0}^{k-1}-\frac{\lambda}{2}\Big[(\alpha_{i+1}-\alpha_i)f(x_i)+(\alpha_{i+2}-\alpha_{i+1})f(x_{i+1})\Big].\end{aligned}$$

That is,

$$\int_a^b K(t,I_k)df(t) = \sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_a^b f(t)dt$$

$$
\begin{aligned}
&+(\alpha_1-a)f(a)+(b-\alpha_k)f(b)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i)\\
&-\frac{\lambda}{2}\Big[(\alpha_1-a)f(a)+(b-\alpha_k)f(b)+2\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i)\Big]\\
&=\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-\int_a^b f(t)dt\\
&+(1-\lambda)\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i)+(1-\frac{\lambda}{2})\Big[(\alpha_1-a)f(a)+(b-\alpha_k)f(b)\Big]. \qquad (2.2)
\end{aligned}
$$

Now, consider the following

$$
\begin{aligned}
&\sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_i)f(\alpha_{i+1})=\sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_{i+1})f(\alpha_{i+1})+\sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1})\\
&=\sum_{i=1}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_i)+\sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1})\\
&=\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_i)-(\alpha_1-\alpha_0)f(\alpha_0)+\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1})-(\alpha_{k+1}-\alpha_k)f(\alpha_{k+1})\\
&=\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)(f(\alpha_i)+f(\alpha_{i+1}))-\Big[(\alpha_1-\alpha_0)f(\alpha_0)+(\alpha_{k+1}-\alpha_k)f(\alpha_{k+1})\Big].
\end{aligned}
$$

So,

$$
\begin{aligned}
\sum_{i=0}^{k-1}\frac{\lambda}{2}(\alpha_{i+2}-\alpha_i)f(\alpha_{i+1})&=\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)\\
&-\frac{\lambda}{2}\Big[(\alpha_1-a)f(a)+(b-\alpha_k)f(b)\Big]. \qquad (2.3)
\end{aligned}
$$

Substituting (2.3) in (2.2) gives the identity

$$
\begin{aligned}
&\int_a^b K(t,I_k)df(t)\\
&=(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof. ∎

Theorem 2.3. *Under the conditions of Lemma 2.2, we have the following inequalities*

$$
\begin{aligned}
&\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-\int_a^b f(t)dt\right|\\
&\le\Bigg[\frac{1}{2}\nu(h)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right\}\\
&\quad+\lambda\Big(\frac{1}{2}\nu(\tau)+\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1}-\frac{\alpha_i+\alpha_{i+2}}{2}\right|\right\}\Big)\Bigg]V_a^b(f)
\end{aligned}
$$

$$\leq \Big(\nu(h) + \lambda\nu(\tau)\Big) V_a^b(f), \tag{2.4}$$

and

$$\left|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) + \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big) - \int_a^b f(t)dt\right|$$
$$\leq \Theta\Big[(b-a) + \lambda\mu\Big] \tag{2.5}$$

where $h_i = x_{i+1} - x_i,\ \tau_i = \alpha_{i+2} - \alpha_i\ (i = 0, 1, \cdots, k-1)$,
$\nu(h) = \max\Big\{h_i : i = 0, 1, \cdots, k-1\Big\},\ \nu(\tau) = \max\Big\{\tau_i : i = 0, 1, \cdots, k-1\Big\}$,
$\Theta := \max\Big\{V_{x_i}^{\alpha_{i+1}}(f), V_{\alpha_{i+1}}^{x_{i+1}}(f); i = 0, 1, \cdots, k-1\Big\}$ *and* $\mu = \sum_{i=0}^{k-1}(\alpha_{i+2} - \alpha_i)$.

Proof. By applying Lemma 2.1 to the functions $p(t) := K(t, I_k)$ and $g(t) := f(t)$, we have

$$\begin{aligned}
\left|\int_a^b K(t,I_k)df(t)\right| &\leq \sum_{i=0}^{k-1}\left[\left|\int_{x_i}^{\alpha_{i+1}} K(t,I_k)df(t)\right| + \left|\int_{\alpha_{i+1}}^{x_{i+1}} K(t,I_k)df(t)\right|\right] \\
&\leq \sum_{i=0}^{k-1}\left[\sup_{t\in[x_i,\alpha_{i+1}]}|K(t,I_k)|V_{x_i}^{\alpha_{i+1}}(f) + \sup_{t\in[\alpha_{i+1},x_{i+1}]}|K(t,I_k)|V_{\alpha_{i+1}}^{x_{i+1}}(f)\right] \\
&= \sum_{i=0}^{k-1}\left[\sup_{t\in[x_i,\alpha_{i+1}]}|t-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2})|V_{x_i}^{\alpha_{i+1}}(f)\right. \\
&\qquad \left. + \sup_{t\in[\alpha_{i+1},x_{i+1}]}|t-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2})|V_{\alpha_{i+1}}^{x_{i+1}}(f)\right] \\
&= \sum_{i=0}^{k-1}\left[\max\left\{\left|x_i-\alpha_{i+1}+\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right|, \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right\}V_{x_i}^{\alpha_{i+1}}(f)\right. \\
&\qquad \left. + \max\left\{\left|x_{i+1}-\alpha_{i+1}-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right|, \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right\}V_{\alpha_{i+1}}^{x_{i+1}}(f)\right] \\
&\leq \sum_{i=0}^{k-1}\left[\left(\left|x_i-\alpha_{i+1}+\lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right| + \lambda\frac{\alpha_{i+1}-\alpha_i}{2}\right)V_{x_i}^{\alpha_{i+1}}(f)\right. \\
&\qquad \left. + \left(\left|x_{i+1}-\alpha_{i+1}-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right| + \lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}\right)V_{\alpha_{i+1}}^{x_{i+1}}(f)\right] \\
&\leq \sum_{i=0}^{k-1}\left[\Big(\alpha_{i+1}-x_i+\lambda(\alpha_{i+1}-\alpha_i)\Big)V_{x_i}^{\alpha_{i+1}}(f)\right. \\
&\qquad \left. + \Big(x_{i+1}-\alpha_{i+1}+\lambda(\alpha_{i+2}-\alpha_{i+1})\Big)V_{\alpha_{i+1}}^{x_{i+1}}(f)\right] \\
&=: M(\lambda, I_k).
\end{aligned}$$

That is,

$$\left|\int_a^b K(t,I_k)df(t)\right| \leq M(\lambda, I_k) \tag{2.6}$$

where

$$M(\lambda, I_k) = \sum_{i=0}^{k-1} \Big[\Big(\alpha_{i+1} - x_i + \lambda(\alpha_{i+1} - \alpha_i)\Big) V_{x_i}^{\alpha_{i+1}}(f) + \Big(x_{i+1} - \alpha_{i+1} + \lambda(\alpha_{i+2} - \alpha_{i+1})\Big) V_{\alpha_{i+1}}^{x_{i+1}}(f) \Big]$$

Now, we consider the following estimates for $M(\lambda, I_k)$. By using the fact that $\max\{A, B\} = \frac{A+B}{2} + \frac{|A-B|}{2}$, we observe that

$$\begin{aligned} M(\lambda, I_k) &= \sum_{i=0}^{k-1} \Big[\Big(\alpha_{i+1} - x_i\Big) V_{x_i}^{\alpha_{i+1}}(f) + \Big(x_{i+1} - \alpha_{i+1}\Big) V_{\alpha_{i+1}}^{x_{i+1}}(f) \\ &\qquad + \lambda(\alpha_{i+1} - \alpha_i) V_{x_i}^{\alpha_{i+1}}(f) + \lambda(\alpha_{i+2} - \alpha_{i+1}) V_{\alpha_{i+1}}^{x_{i+1}}(f) \Big] \\ &\le \sum_{i=0}^{k-1} \Big[\max\Big\{\alpha_{i+1} - x_i, x_{i+1} - \alpha_{i+1}\Big\} V_{x_i}^{x_{i+1}}(f) \\ &\qquad + \lambda \max\Big\{\alpha_{i+1} - \alpha_i, \alpha_{i+2} - \alpha_{i+1}\Big\} V_{x_i}^{x_{i+1}}(f) \Big] \\ &= \sum_{i=0}^{k-1} \Big[\Big(\frac{1}{2}(x_{i+1} - x_i) + \Big|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\Big|\Big) V_{x_i}^{x_{i+1}}(f) \\ &\qquad + \lambda\Big(\frac{1}{2}(\alpha_{i+2} - \alpha_i) + \Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big) V_{x_i}^{x_{i+1}}(f) \Big] \\ &= \sum_{i=0}^{k-1} \Big[\Big(\frac{1}{2}h_i + \Big|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\Big|\Big) V_{x_i}^{x_{i+1}}(f) \\ &\qquad + \lambda\Big(\frac{1}{2}\tau_i + \Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big) V_{x_i}^{x_{i+1}}(f) \Big] \\ &\le \max_{i=0,1,\cdots,k-1} \Big\{\frac{1}{2}h_i + \Big|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\Big|\Big\} \sum_{i=0}^{k-1} V_{x_i}^{x_{i+1}}(f) \\ &\qquad + \lambda \max_{i=0,1,\cdots,k-1} \Big\{\frac{1}{2}\tau_i + \Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big\} \sum_{i=0}^{k-1} V_{x_i}^{x_{i+1}}(f) \\ &\le \Big[\frac{1}{2}\nu(h) + \max_{i=0,1,\cdots,k-1} \Big\{\Big|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\Big|\Big\} \\ &\qquad + \lambda\Big(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\cdots,k-1} \Big\{\Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big\}\Big) \Big] V_a^b(f). \end{aligned}$$

That is,

$$\begin{aligned} M(\lambda, I_k) &\le \Big[\frac{1}{2}\nu(h) + \max_{i=0,1,\cdots,k-1} \Big\{\Big|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\Big|\Big\} \\ &\qquad + \lambda\Big(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\cdots,k-1} \Big\{\Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big\}\Big) \Big] V_a^b(f). \end{aligned} \tag{2.7}$$

Using (2.1), (2.6) and (2.7), we obtained the first inequality of (2.4). Now, we observe that

$$\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}h_i$$

and

$$\left|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\right| \leq \frac{1}{2}\tau_i.$$

So, it follows that

$$\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|\right\} \leq \frac{1}{2}\nu(h) \tag{2.8}$$

and

$$\max_{i=0,1,\cdots,k-1}\left\{\left|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\right|\right\} \leq \frac{1}{2}\nu(\tau). \tag{2.9}$$

Hence, by using (2.7), (2.8) and (2.9), we have

$$M(\lambda, I_k) \leq \Big(\nu(h) + \lambda\nu(\tau)\Big)V_a^b(f). \tag{2.10}$$

This proves the second inequality in (2.4). Finally, to obtain the inequality in (2.5), we observe that

$$\begin{aligned} M(\lambda, I_k) &\leq \Theta\sum_{i=0}^{k-1}\Big(\alpha_{i+1} - x_i + \lambda(\alpha_{i+1} - \alpha_i) + x_{i+1} - \alpha_{i+1} + \lambda(\alpha_{i+2} - \alpha_{i+1})\Big) \\ &= \Theta\sum_{i=0}^{k-1}\Big(x_{i+1} - x_i + \lambda(\alpha_{i+2} - \alpha_i)\Big) = \Theta\Big[(b-a) + \lambda\mu\Big]. \end{aligned}$$

That is

$$M(\lambda, I_k) \leq \Theta\Big[(b-a) + \lambda\mu\Big] \tag{2.11}$$

This proves the theorem. ∎

Corollary 2.4. *If we let $\lambda = 0$ in Theorem 2.3, then the inequalities in (2.4) reduces to the inequalities in Theorem 1.4 and the inequality in (2.5) becomes*

$$\left|\sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)f(x_i) - \int_a^b f(t)dt\right| \leq \Theta(b-a) \tag{2.12}$$

Corollary 2.5. *Under the conditions of Theorem 2.3, if we choose*

$$\alpha_0 = a, \alpha_{i+1} = \frac{x_i + x_{i+1}}{2}\ (i = 0, \cdots, k-1) \text{ and } \alpha_{k+1} = b,$$

then we have the inequalities;

$$\begin{aligned} \Bigg|&\frac{1-\lambda}{2}\Big[(x_1 - a)f(a) + \sum_{i=1}^{k-1}(x_{i+1} - x_{i-1})f(x_i) + (b - x_{k-1})f(b)\Big] \\ &+ \frac{\lambda}{4}\Big[(x_1 - a)\Big(f(a) + f\Big(\frac{x_1 + a}{2}\Big)\Big) + \sum_{i=1}^{k-1}(x_{i+1} - x_{i-1})\Big(f\Big(\frac{x_i + x_{i-1}}{2}\Big) \\ &+ f\Big(\frac{x_{i+1} + x_i}{2}\Big)\Big) + (b - x_{k-1})\Big(f(b) + f\Big(\frac{b + x_{k-1}}{2}\Big)\Big)\Big] - \int_a^b f(t)dt\Bigg| \end{aligned}$$

$$\leq \Big[\frac{1}{2}\nu(h) + \lambda\Big(\frac{1}{2}\nu(\tau) + \max_{i=0,1,\cdots,k-1}\Big\{\Big|\alpha_{i+1} - \frac{\alpha_i + \alpha_{i+2}}{2}\Big|\Big\}\Big)\Big]V_a^b(f)$$

$$\leq \Big(\frac{1}{2}\nu(h) + \lambda\nu(\tau)\Big)V_a^b(f) \tag{2.13}$$

and

$$\Big|\frac{1-\lambda}{2}\Big[(x_1 - a)f(a) + \sum_{i=1}^{k-1}(x_{i+1} - x_{i-1})f(x_i) + (b - x_{k-1})f(b)\Big]$$
$$+ \frac{\lambda}{4}\Big[(x_1 - a)\Big(f(a) + f\Big(\frac{x_1 + a}{2}\Big)\Big) + \sum_{i=1}^{k-1}(x_{i+1} - x_{i-1})\Big(f\Big(\frac{x_i + x_{i-1}}{2}\Big)$$
$$+ f\Big(\frac{x_{i+1} + x_i}{2}\Big)\Big) + (b - x_{k-1})\Big(f(b) + f\Big(\frac{b + x_{k-1}}{2}\Big)\Big)\Big] - \int_a^b f(t)dt\Big|$$
$$\leq \Theta\Big[(b - a) + \lambda\mu\Big]. \tag{2.14}$$

Proof. In this case, we have

$$\alpha_1 - \alpha_0 = \frac{x_1 - a}{2},\ \alpha_{i+1} - \alpha_i = \frac{x_{i+1} - x_{i-1}}{2}\ (i = 1, \cdots, k-1),$$

$$\alpha_{k+1} - \alpha_k = \frac{b - x_{k-1}}{2} \text{ and } \alpha_{i+1} - \frac{x_i + x_{i-1}}{2} = 0\ (i = 0, \cdots, k-1).$$

∎

Now, if we choose I_k to be the equidistant partition of $[a, b]$, then we have the following corollary.

Corollary 2.6. *Let*

$$I_k : x_i = a + (b - a)\frac{i}{k}\ (i = 0, 1, \cdots, k)$$

be the equidistant partitioning of $[a, b]$ and the $\alpha_i's$ be as in Corollary 2.5. Then the following inequality holds;

$$\Big|\frac{1-\lambda}{2}\Big[\frac{b-a}{k}\Big(f(a) + f(b)\Big) + \frac{2(b-a)}{k}\sum_{i=1}^{k-1} f\Big(\frac{(k-1)a + bi}{k}\Big)\Big]$$
$$+ \frac{\lambda}{4}\Big[\frac{b-a}{k}\Big(f(a) + f(b) + f\Big(\frac{(2k-1)a + b}{2k}\Big) + f\Big(\frac{a + (2k-1)b}{2k}\Big)$$
$$+ \frac{2(b-a)}{k}\sum_{i=1}^{k-1}\Big(f\Big(\frac{(2k-2i+1)a + (2i-1)b}{2k}\Big) + f\Big(\frac{(2k-2i-1)a + (2i+1)b}{2k}\Big)\Big)\Big]$$
$$- \int_a^b f(t)dt\Big| \leq \frac{b-a}{2k}\Big(1 + 4\lambda\Big)V_a^b(f). \tag{2.15}$$

and

$$\Big|\frac{1-\lambda}{2}\Big[\frac{b-a}{k}\Big(f(a) + f(b)\Big) + \frac{2(b-a)}{k}\sum_{i=1}^{k-1} f\Big(\frac{(k-1)a + bi}{k}\Big)\Big]$$
$$+ \frac{\lambda}{4}\Big[\frac{b-a}{k}\Big(f(a) + f(b) + f\Big(\frac{(2k-1)a + b}{2k}\Big) + f\Big(\frac{a + (2k-1)b}{2k}\Big)$$

$$+\frac{2(b-a)}{k}\sum_{i=1}^{k-1}\left(f\Big(\frac{(2k-2i+1)a+(2i-1)b}{2k}\Big)+f\Big(\frac{(2k-2i-1)a+(2i+1)b}{2k}\Big)\right)\Big]$$

$$-\int_a^b f(t)dt\Big| \le \frac{b-a}{k}\Big[(1+2\lambda)k-\lambda\Big]\Theta. \tag{2.16}$$

Proof. We consider the following computations.

$$x_1-a=\frac{b-a}{k},\ x_1+a=\frac{(2k-1)a+b}{k},\ x_{i+1}-x_{i-1}=\frac{2(b-a)}{k}\ (i=1,\cdots,k-1),$$
$$\frac{x_i+x_{i-1}}{2}=\frac{(2k-2i+1)a+(2i-1)b}{k},\ \frac{x_i+x_{i+1}}{2}=\frac{(2k-2i-1)a+(2i+1)b}{k},$$
$$(i=1,\cdots,k-1),\ b-x_{k-1}=\frac{b-a}{k},\ b+x_{k-1}=\frac{a+(2k-1)b}{k},\ h_i=\frac{b-a}{k}$$
$$(i=0,\cdots,k-1),\ \tau_0=\frac{3(b-a)}{2k},\ \tau_i=\frac{2(b-a)}{k}\ (i=1,\cdots,k-2),$$
$$\tau_{k-1}=\frac{3(b-a)}{2k},\ \text{and}\ \mu=\frac{b-a}{k}\Big(2k-1\Big).$$

Using these computations and the last inequality in (2.13), we have the inequality in (2.15). Similarly, the inequality in (2.16) follows from the inequality in (2.14). ∎

3. SOME PARTICULAR CASES

In this section, we consider some particular cases for the inequality (2.4) in Theorem 2.3. Similar results can be obtained from the inequality (2.5) as well.

Corollary 3.1. *Let $a, b \in \mathbb{R}, a < b, \lambda \in [0,1], a \le \alpha_1 \le x_1 \le \alpha_2 \le b$ and $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function. Then, we have the following inequalities;*

$$\Big|(1-\lambda)\Big[(\alpha_1-a)f(a)+(\alpha_2-\alpha_1)f(x_1)+(b-\alpha_2)f(b)\Big]$$
$$+\frac{\lambda}{2}\Big[(\alpha_1-a)(f(a)+f(\alpha_1))+(\alpha_2-\alpha_1)(f(\alpha_1)+f(\alpha_2))$$
$$+(b-\alpha_2)(f(\alpha_2)+f(b))\Big]-\int_a^b f(t)dt\Big|$$
$$\le\Big[\frac{1}{2}\max\Big\{x_1-a,b-x_1\Big\}+\max\Big\{\Big|\alpha_1-\frac{a+x_1}{2}\Big|,\Big|\alpha_2-\frac{x_1+b}{2}\Big|\Big\}$$
$$+\lambda\Big(\frac{1}{2}\max\Big\{\alpha_2-a,b-\alpha_1\Big\}+\max\Big\{\Big|\alpha_1-\frac{a+\alpha_2}{2}\Big|,\Big|\alpha_2-\frac{\alpha_1+b}{2}\Big|\Big\}\Big)\Big]V_a^b(f)$$
$$\le\Big(\max\Big\{x_1-a,b-x_1\Big\}+\lambda\max\Big\{\alpha_2-a,b-\alpha_1\Big\}\Big)V_a^b(f). \tag{3.1}$$

Proof. The proof follows directly from Theorem 2.3 by choosing $k = 2$. ∎

Corollary 3.2. (1) *If we choose $\alpha_1 = a, x_1 = x$ and $\alpha_2 = b$ in Corollary 3.1, then we have the inequalities*

$$\Big|(b-a)\Big[(1-\lambda)f(x)+\frac{\lambda}{2}\Big(f(a)+f(b)\Big)\Big]-\int_a^b f(t)dt\Big|$$

$$\leq \Big(\max\Big\{x-a, b-x\Big\} + \lambda(b-a)\Big)V_a^b(f)$$

$$= \Big(\frac{1}{2}(b-a) + \Big|x - \frac{a+b}{2}\Big| + \lambda(b-a)\Big)V_a^b(f) \tag{3.2}$$

for all $x \in [a,b]$.

(2) *If we choose* $x = \dfrac{a+b}{2}$ *in (3.2), then we have the following "midpoint inequality";*

$$\Big|(b-a)\Big[(1-\lambda)f(\frac{a+b}{2}) + \frac{\lambda}{2}\Big(f(a)+f(b)\Big)\Big] - \int_a^b f(t)dt\Big|$$

$$\leq \frac{1}{2}(b-a)\Big(1+2\lambda\Big)V_a^b(f) \tag{3.3}$$

(3) *If we choose* $\alpha_1 = \dfrac{5a+b}{6}, \alpha_2 = \dfrac{a+5b}{6}$ *and* $x_1 = x$ *in Corollary 3.1, then we have*

$$\Big|(1-\lambda)\frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f(x)\Big] + \frac{\lambda(b-a)}{3}\Big[\frac{f(a)+f(b)}{2} + \frac{5}{2}f\Big(\frac{5a+b}{6}\Big)$$

$$+ \frac{5}{2}f\Big(\frac{a+5b}{6}\Big)\Big] - \int_a^b f(t)dt\Big|$$

$$\leq \Big[\frac{1}{2}\max\Big\{x-a, b-x\Big\} + \frac{1}{2}\max\Big\{\Big|x - \frac{2a+b}{3}\Big|, \Big|x - \frac{a+2b}{3}\Big|\Big\}$$

$$+ \frac{2\lambda}{3}(b-a)\Big]V_a^b(f)$$

$$= \Big[\frac{1}{2}\Big(\frac{b-a}{2} + \Big|x - \frac{a+b}{2}\Big| + \max\Big\{\Big|x - \frac{2a+b}{3}\Big|, \Big|x - \frac{a+2b}{3}\Big|\Big\}\Big)$$

$$+ \frac{2\lambda}{3}(b-a)\Big]V_a^b(f). \tag{3.4}$$

(4) *In particular, if we choose* $x = \dfrac{a+b}{2}$ *in (3.4), then we have the following perturbed "Simpson's inequality";*

$$\Big|(1-\lambda)\frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big] + \frac{\lambda(b-a)}{3}\Big[\frac{f(a)+f(b)}{2} + \frac{5}{2}f\Big(\frac{5a+b}{6}\Big)$$

$$+ \frac{5}{2}f\Big(\frac{a+5b}{6}\Big)\Big] - \int_a^b f(t)dt\Big|$$

$$\leq \frac{1}{3}(b-a)\Big(1+2\lambda\Big)V_a^b(f). \tag{3.5}$$

4. CONCLUSION

A new Ostrowski-type inequality for functions of bounded variation for k points via a parameter has been established. Some particular cases have be considered as examples. By considering different partitions, different points and/or different values of the parameter we will obtain several interesting inequalities.

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