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# A NEW INTERPRETATION OF THE NUMBER $e$

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ABSTRACT. We show that *e* is the amount that $1 becomes when it is invested during an arbitrary time span of length $T$, at any continuously compounded interest rates as long as their average is equal to $1/T$ . A purely mathematical interpretation of $e$ is the amount a unit quantity becomes after any duration $T$ when the average of its instantaneous growth rates is $1/T$. This property can be shown to remain valid if $T$ tends to infinity as long as the integral of the growth rates converges to unity.

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## 1. INTRODUCTION

Traditionnally, calculus texts give the following interpretation of *e*: it is the amount that $1 becomes after one year if it is invested during one year at a rate of interest equal to 100% per year, continuously compounded over that year. The reason of that property is the following: suppose that the year is divided into *m* periods; after the first period, $1 becomes $1+100\%.\frac{1}{m}=1+\frac{1}{m}$; at the end of the last period $1 has become $(1+\frac{1}{m})^m$. Suppose now that the number of periods tends to infinity; equivalently, this implies that the rate of interest is compounded continuously. Then $1 becomes $\lim_{m\to\infty}(1+\frac{1}{m})^m=e$. (In fact, this is how Jacob Bernoulli in 1683 calculated for the first time this famous limit). However, as we will first show, the interpretation of $e$ can be made much more general than that, with an arbitrary investment horizon and highly variable interest rates throughout the investment period. Furthermore, an interpretation of $e$ does not need to be located in a financial context; the number $e$ can be given a purely mathematical interpretation, as we will also show.

We will proceed as follows. In Section 2 we recall the general concept of continuous compounding with variable interest rates during a time span (an investment horizon) of arbitrary length. In Section 3 we introduce a more general financial interpretation of *e*. This will be put to good use: it will enable us to answer the following question: what would be the interpretation of *e* if the investment horizon was of infinite length? In Section 4 we give a completely general, mathematical interpretation of *e*, that applies in any context.

## 2. CONTINUOUS COMPOUNDING WITH VARIABLE INTEREST RATE

Consider a time span $[0,T]$; its length $T$ (equivalently, the investment horizon), is entirely arbitrary as long as it is a positive, real number. Suppose that this time span is divided into $n$ intervals $\Delta z_j$, $j=1,...,n$. Each of these intervals is of *arbitrary* length; they are only related by the constraint $\sum_1^n \Delta z_j = T$. To each interval $\Delta z_j$ corresponds an interest rate liable to be compounded $m$ times over that interval. We denote this interest rate $i_j^{(m)}$; the lower subscript $j$ refers to the $\Delta z_j$ interval the interest rate belongs to; the upper subscript $(m)$ stands for the number of compoundings performed within that interval. A capital $C_0$ invested at time 0 will become, at the end of the first interval $C_0(1+i_1^{(1)}\Delta z_1)$ if the rate of interest is compounded once only over that first interval. If it is compounded $m$ times, it will become $C_0(1+i_1^{(m)}\frac{\Delta z_1}{m})^m$ at the end of $\Delta z_1$. Setting $\frac{1}{k}\equiv i_1^{(m)}\frac{\Delta z_1}{m}$, with $m=ki_1^{(m)}\Delta z_1$, this amount is equal to

$$C_0(1+\ i_1^{(m)}\frac{\Delta z_1}{m})^m = C_0(1+\frac{1}{k})^{ki_1^{(m)}\Delta z_1}\ . \tag{2.1}$$

.

If the number $m$ of compoundings over this first interval tends to infinity, $k$ tends to infinity as well because $i_1^{(m)}$and $\Delta z_1$ are finite, and we have in the limit

$$\lim_{k\to\infty} C_0(1+\frac{1}{k})^{ki_1^{(m)}\Delta z_1} = C_0 e^{i_1^{(\infty)}\Delta z_1} \equiv C_0 e^{i_1\Delta z_1} \tag{2.2}$$

where, to alleviate notation, we replaced $i_1^{(\infty)}$ by $i_1$; from now on, an interest rate without an upperscript will denote a continuously compounded rate of interest. Repeating the above process over the $n$ intervals, we get at time $T$

$$C_T = C_0 e^{i_1\Delta z_1}.e^{i_2\Delta z_2}...e^{i_n\Delta z_n} = C_0 e^{\sum_{j=1}^n i_j\Delta z_j}. \tag{2.3}$$

## 3. AN EXTENDED, FINANCIAL INTERPRETATION OF $e$.

We recognize in the power of $e$ the weighted sum of the interest rates, and, since $\sum_{j=1}^{n} \Delta z_j = T$, we can make their weighted average, denoted $\bar{I}$, appear. It is equal to $\bar{I} = \sum_1^n i_j \Delta z_j / \sum_1^n \Delta z_j = \sum_1^n i_j \Delta z_j / T$, and we can write $\sum_1^n i_j \Delta z_j = \bar{I}T$. Therefore equation (3) can be expressed as

$$C_T = C_0 e^{\sum_{j=1}^{n} i_j \Delta z_j} = C_0 e^{\bar{I}T}. \tag{3.1}$$

It immediately appears that $e$ is what \$1 becomes if $\bar{I}$, the average of the interest rates weighted by the intervals $\Delta z_j$, is equal to the inverse of the investment period $T$.

Let now the number $n$ of intervals $\Delta z_j$ tend to infinity and the maximum length of those intervals tend to zero. If the sum $\sum_1^n i_j \Delta z_j$ tends to a limit, we may write it as

$$\lim_{n\to\infty;\ \max \Delta z_j \to 0} \sum_{j=1}^{n} i_j \Delta z_j \equiv \int_0^T i(t)dt \tag{3.2}$$

and therefore

$$\lim_{n\to\infty;\ \max \Delta z_j \to 0} C_T = C_0 e^{\int_0^T i(t)dt}. \tag{3.3}$$

Now defining $\bar{I}$ as

$$\bar{I} = \frac{\int_0^T i(t)dt}{T}, \tag{3.4}$$

we have

$$C_T = C_0 e^{\left[\{\int_0^T i(t)dt\}/T\right]T} = C_0 e^{\bar{I}T}. \tag{3.5}$$

We can immediately see that *e is what \$1 becomes when it is invested during an arbitrary time span of length $T$, at any continuously compounded interest rates as long as their average is equal to* $1/T$ .

For instance suppose that the horizon investment $T$ is equal to 25 years. The inverse of $T$ is 0.04/year, i.e. 4% per year. One dollar will become *e* dollars if it is invested at *any* continuously compounded interest rates as long as their average is 4% per year. There is of course an infinite number of evolutions of the interest rate that would share this property. Note in particular that the interest rate may very well be *negative* over a number of periods during the horizon investment.

Figures 1 and 2 give examples. We have considered possible evolutions of the interest rate corresponding to 4 horizons: 10, 20, 50 and 100 years, all leading to *e*. In Figure 1, those trajectories correspond to the family of parabolas

$$i(t) = \left(6/T^3\right)\left(-t^2 + tT\right). \tag{3.6}$$

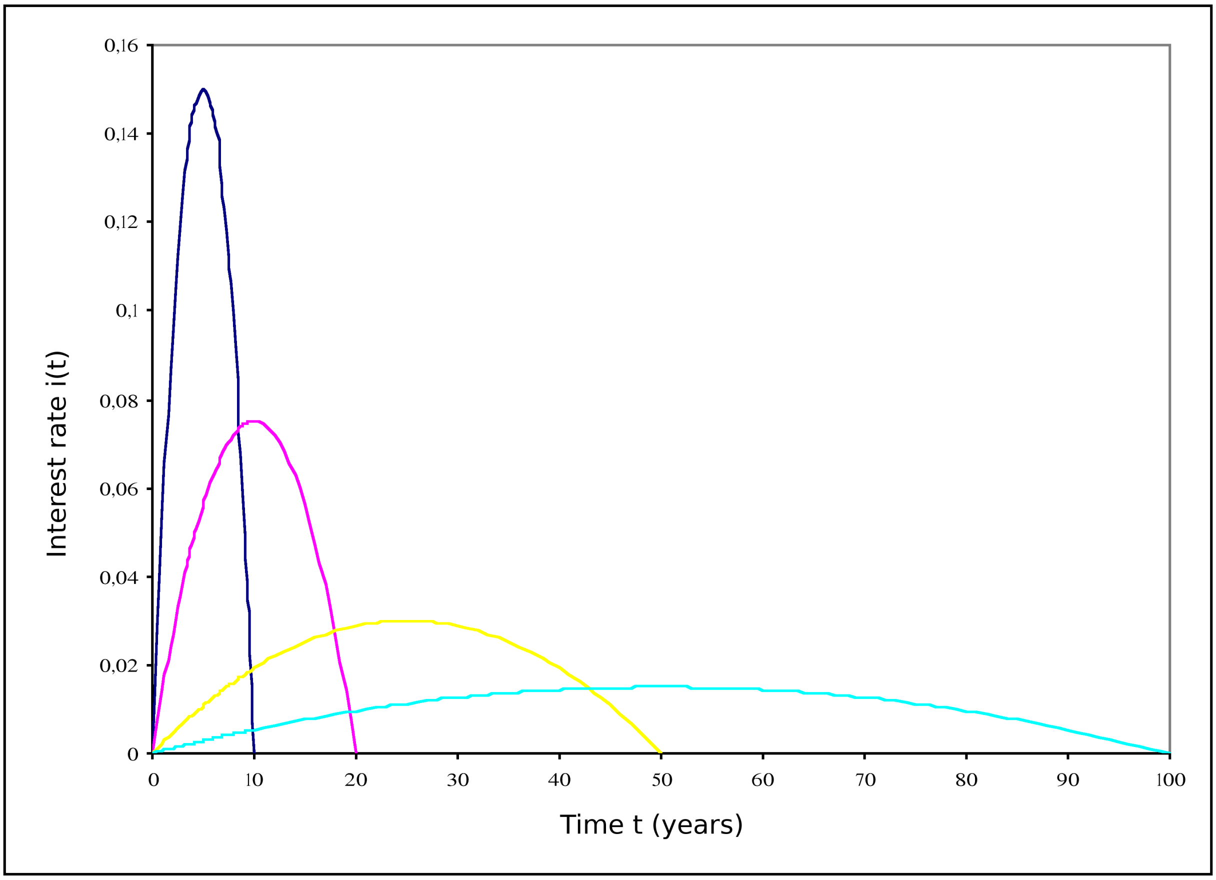


FIGURE 1. Four members of the family of parabolas $i(t) = (6/T^3)(-t^2 + tT)$ yielding $e$ after $T = 10, 20, 50$ and $100$ years, respectively. Their averages are $\bar{I} = 0.1, 0.05, 0.02$, and $0.01$ per year.

Figure 2 presents examples where the interest rate may become negative (this can be the case if we consider the so-called "real" interest rate, defined by the nominal interest diminished by the inflation rate); they belong to the family of third order polynomials

$$i(t) = \frac{12}{T^4}\left(\frac{0.2}{3}T - 1\right)\left(t^3 - Tt^2\right) - \frac{0.1}{T^2}t^2 + 0.1. \tag{3.7}$$

Needless to say, any function $i(t)$ having the property $\int_0^T i(t)dt = 1$ can receive an infinite number of variations such that the property would be preserved.

Finally, we can immediately see that the above mentionned interpretation of *e* holds even for horizons of infinite length, as long as the average of the interest rates converges to zero and $\lim_{T\to\infty} \int_0^T i(t)dt = 1$. An infinite number of functions share this property.

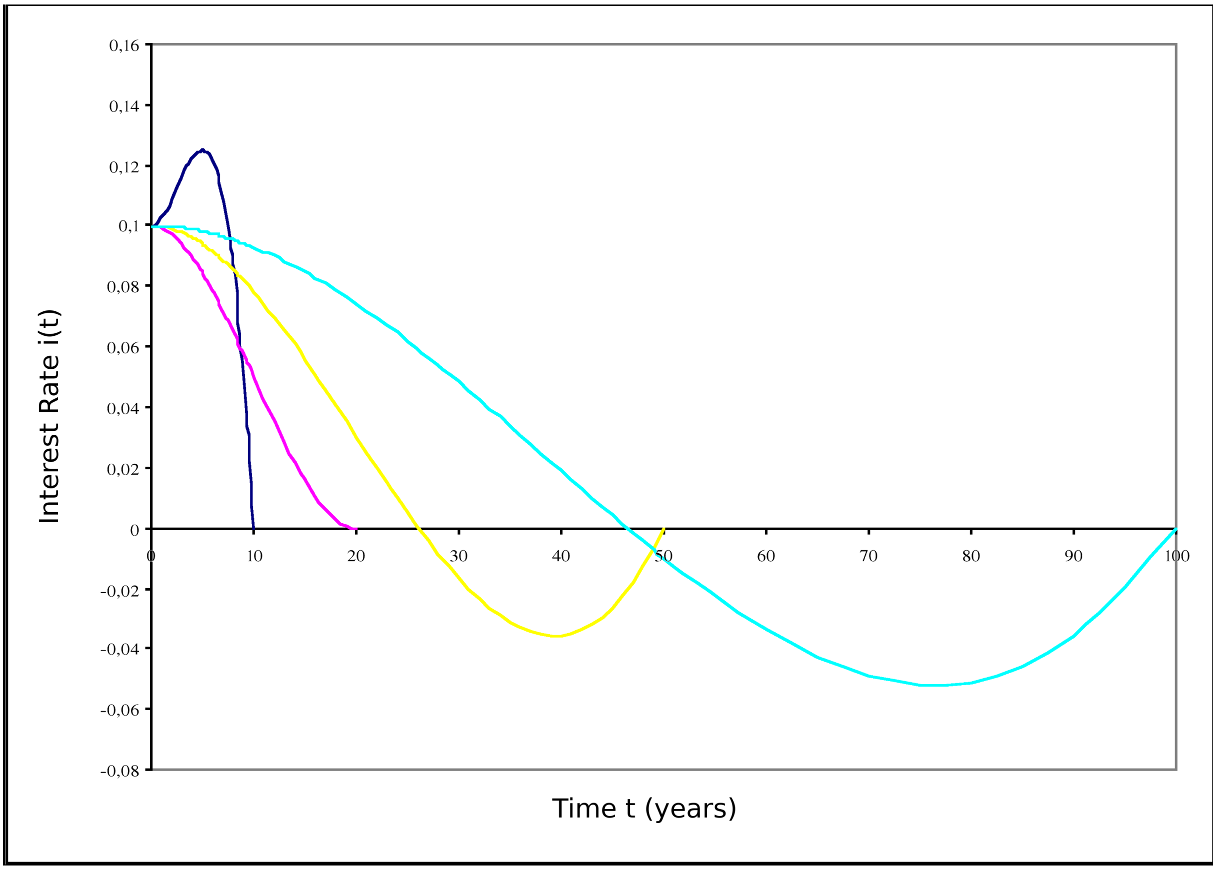


FIGURE 2. Four members of the family of the third order polynomials $i(t) = \frac{12}{T^4}\left(\frac{0.2}{3}T-1\right)(t^3-Tt^2)-\frac{0.1}{T^2}t^2+0.1$ yielding $e$ after $T = 10, 20, 50$ and $100$ years, respectively. Their averages are $= 0.1, 0.05, 0.02$, and $0.01$ per year.

## 4. A MATHEMATICAL INTERPRETATION OF $e$

An interpretation of *e* does not need to belong to a financial context; it can be given a completely general form, applying in any field. We can immediately prove the following: *e is the amount a unit quantity becomes after any duration $T$ when the average of its instantaneous growth rates is* $1/T$.

This can be seen by considering the differential equation

$$\frac{1}{y}\frac{dy}{dt} = g(t) \tag{4.1}$$

where $g(t)$ is the instantaneous growth rate of $y(t)$.With the initial condition $y(0) = 1$, we have

$$y(T) = e^{\int_0^T g(t)dt}. \tag{4.2}$$

Defining the average of the growth rates over $[0, T]$ as $\bar{g} \equiv [\int_0^T g(t)dt]/T$ we get, if $\bar{g} = 1/T$, $y(T) = e^{\bar{g}T} = e$, which completes the proof.

Equivalently, define over period $[0, T]$ the *growth factor* of $y$ as the ratio $y_T/y_0$. Then *e* is the growth factor of $y(t)$ over $[0, T]$ if the average of its instantaneous growth rates $\dot{y}(t)/y(t)$ is $1/T$.

Finally, we observe that this property of $e$ remains valid if $T \to \infty$; the only condition is that the average of $g(t)$ tends to zero while $\int_0^\infty g(t)dt = 1$.

## 5. CONCLUDING REMARK

The generality of the interpretation of $e$ given in Section 4 above comes from the fact that for *any* value of $T$ $(T > 0)$ there is are infinite number of functions $g(t)$ sharing the property that the integral $\int_0^T g(t)dt$ equals unity.

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