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EXISTENCE AND ESTIMATE OF THE SOLUTION FOR THE APPROXIMATE STOCHASTIC EQUATION TO THE VISCOUS BAROTROPIC GAS

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ABSTRACT. A stochastic equation of a viscous barotropic gas is considered. The application of Itô formula to a specific functional in an infinite dimensional space allows us to obtain an estimate which is useful to analyse the behavior of the solution. As it is difficult to exploit this estimate, we study an approximate problem. More precisely, we consider the equation of a barotropic viscous gas in Lagrangian coordinates and we add a diffusion of the density. An estimate of energy is obtained to analyse the behavior of the solution for this approximate problem and Galerkin method is used to prove the existence and uniqueness of the solution.

Key words and phrases: Stochastic equation; Viscous barotropic gas, Galerkin approximate.

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1. INTRODUCTION

The equations describing the motion of a viscous gas are presented in many books like [5]. They are established using the laws of conservation of the mass, of motion's amount and energy. Following [5], they are given by the system of equations

$$\partial_t \varrho + \nabla \cdot (\varrho v) = 0, \tag{1.1}$$

$$\varrho \partial_t v_j + \varrho \sum_{k=1}^{3} v_k \frac{\partial}{\partial x_k} v_j + R_1 \frac{\partial}{\partial x_j}(\varrho T) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\eta(\frac{\partial}{\partial x_k} v_j + \frac{\partial}{\partial x_j} v_k - \frac{2}{3}\delta_{j,k}\nabla \cdot v)) + \frac{\partial}{\partial x_j}(\zeta \nabla \cdot v) + \varrho f_j, \qquad j = 1,2,3, \tag{1.2}$$

$$\varrho c_v(\partial_t T + v \cdot \nabla T) + R_1 \varrho T \nabla \cdot v = \kappa \Delta v + \eta \sum_{j,k=1}^{3} (\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{j,k}\nabla \cdot v)\frac{\partial v_j}{\partial x_k} + \zeta(\nabla \cdot v)^2, \tag{1.3}$$

where $v = (v_1, v_2, v_3)$ and ϱ are the speed and the density of the gas respectively (here ϱ is obviously positive), T is the temperature, f is an external force, while η and ζ are the viscosity coefficient of the flow and the volumetric viscosity coefficient respectively, c_v the specific heat, κ the coefficient of the thermal conduction and $R_1 = \frac{R}{\mu}$, where R is the universal gas constant and μ is the molar mass of the gas. For the viscous gas, the pressure p is given by

$$p = R_1 \varrho T. \tag{1.4}$$

The barotropic model to the system of equations (1.1)-(1.3) is given by the system of equations

$$\partial_t \varrho + \nabla \cdot (\varrho v) = 0, \tag{1.5}$$

$$\varrho \partial_t v + \varrho(v \cdot \nabla)v + h\nabla \varrho^\gamma = \eta \Delta v + (\frac{\eta}{3} + \zeta)\nabla(\nabla \cdot v) + \varrho f_j. \tag{1.6}$$

This model is obtained by considering the pressure p as a function of density, i.e.,

$$p = h\varrho^\gamma, \tag{1.7}$$

where h is a positive constant and $\gamma = \frac{c_v + R_1}{c_v}$ is the adiabatic exponent. Also, by using some approximations on the equation (1.3) and by considering η and ζ as constants. For more details see [1] and [5].

Furthermore, the equations of the barotropic gas (1.5)-(1.6) in one spatial dimension in the domain $0 < x < 1$ is given by the system of equations

$$\partial_t \varrho + \partial_x(\varrho v) = 0, \tag{1.8}$$

$$\varrho \partial_t v + \varrho v \partial_x v - \eta \partial_x^2 v + h \partial_x \varrho^\gamma = \varrho f. \tag{1.9}$$

We denote that we have the conservation of the mass, i.e.,

$$\int_0^1 \varrho(t,x)dx = 1, \quad t \geq 0. \tag{1.10}$$

The existence and uniqueness of the solution to the problem (1.8), (1.9) with boundary and initial conditions are proved by Kazhikhov [4].

In this paper, we consider that the motion of the barotropic gas in spatial dimension is subjected to a random perturbation. More precisely, we consider in the domain $0 < x < 1$, the stochastic system

$$\varrho dv = (-\varrho v \partial_x v + \eta \partial_x^2 v - h \partial_x \varrho^\gamma)dt + \varrho dW, \tag{1.11}$$

$$\partial_t \varrho + \partial_x(\varrho v) = 0, \tag{1.12}$$

where $W(t)$ is a brownian motion of the Hilbert space $L^2(0,1)$ given by

$$W(t) = \sum_{k=1}^{+\infty} \lambda_k e_k(x) W^{(k)}(t), \tag{1.13}$$

where λ_k, $k = 1, 2, ...$ are taken in $\mathbf{R}^+$ satisfying

$$\sum_{k=1}^{+\infty} \lambda_k^2 < +\infty, \tag{1.14}$$

$\{e_k\}_{k=1}^{+\infty}$ is an orthonormal basis in $L^2(0,1)$ and $W^{(k)}(t)$, $k = 1, 2, ...$ are brownian independent canonical motions with real values defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$. For more details see [3]. Here, the basis $\{e_k\}_{k=1}^{+\infty}$ is given by

$$e_k(x) = \sqrt{2}\sin(k\pi x). \tag{1.15}$$

We add to the system of equations (1.11)-(1.12) the boundary conditions

$$v|_{x=0,1} = 0, \quad t \geq 0 \tag{1.16}$$

and the initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad v(0,x) = v_0(x) \quad \text{for} \quad x \in [0,1]. \tag{1.17}$$

2. ESTIMATE OF SOLUTION FOR THE SYSTEM (1.11)-(1.12)

To analyse the behavior of the solution for the system (1.11)-(1.12), with the conditions (1.16), (1.17), we define the functional $\varphi(t)$ by

$$\varphi(t) = \varphi(v(t), \varrho(t)) = \int_0^1 \varrho \frac{v^2}{2} dx + \frac{1}{\gamma - 1} h \int_0^1 \varrho^\gamma dx, \quad \text{for} \quad \gamma \neq 1. \tag{2.1}$$

Applying Itô formula to the functional φ (see [8]), we have the following result.

Proposition 2.1. *Let φ be defined by* (2.1) *and (v, ϱ) the solution of the system of equations* (1.11), (1.12) *with the conditions* (1.16), (1.17). *If $\sqrt{\varrho_0} v_0 \in L^2(0,1)$ and $\varrho_0 \in L^\gamma(0,1)$ then*

$$\begin{aligned} \varphi(t) - \varphi(0) & \\ &= -\eta \int_0^t \int_0^1 (\partial_x v)^2 dx ds + \int_0^t \langle \varrho v, dW(s) \rangle + \frac{1}{2} \int_0^t \sum_{k=1}^{+\infty} \lambda_k^2 \int_0^1 \varrho e_k^2(x) dx ds, \end{aligned} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0,1)$.

Proof. Let

$$u = v - W. \tag{2.3}$$

Itô formula (see [8]) gives

$$\varphi(t) - \varphi(0) = \int_0^t \frac{\partial \varphi}{\partial u} du + \int_0^t \frac{\partial \varphi}{\partial \varrho} d\varrho + \int_0^t \frac{\partial \varphi}{\partial W(s)} dW(s) + \frac{1}{2}\int_0^t \frac{\partial^2 \varphi}{\partial^2 W(s)} \langle dW(s), dW(s)\rangle \tag{2.4}$$

where

$$\frac{\partial \varphi}{\partial u}, \quad \frac{\partial \varphi}{\partial \varrho}, \quad \frac{\partial \varphi}{\partial W(s)}, \quad \frac{\partial^2 \varphi}{\partial^2 W(s)} \tag{2.5}$$

are Fréchet derivatives. We have for all positive functions f and g,

$$\frac{\partial \varphi}{\partial u}(f) = \int_0^1 \varrho v f dx, \tag{2.6}$$

$$\frac{\partial \varphi}{\partial \varrho}(f) = \int_0^1 (\frac{v^2}{2} + \frac{h\gamma}{\gamma - 1}\varrho^{\gamma-1}) f dx, \tag{2.7}$$

$$\frac{\partial \varphi}{\partial W(s)}(f) = \int_0^1 \varrho v f dx, \tag{2.8}$$

$$\frac{\partial^2 \varphi}{\partial W^2(s)}(f)(g) = \int_0^1 \varrho f g dx. \tag{2.9}$$

From (1.11), (1.12), we obtain

$$\frac{\partial \varphi}{\partial u} du = \int_0^1 [-\varrho v^2 \partial_x v + \eta v \partial_x^2 v - h v \partial_x \varrho^\gamma] dx ds, \tag{2.10}$$

$$\frac{\partial \varphi}{\partial \varrho} d\varrho = \int_0^1 [\frac{-v^3}{2}\partial_x \varrho - \varrho \frac{v^2}{2}\partial_x v] dx ds - \frac{\gamma}{\gamma - 1} h \int_0^1 (v\varrho^{\gamma-1}\partial_x \varrho + \varrho^{\gamma-1}\varrho \partial_x v) dx ds,$$

$$\frac{\partial \varphi}{\partial W_s} dW(s) = \langle \varrho v, dW(s)\rangle$$

and

$$\frac{\partial^2 \varphi}{\partial W^2(s)} \langle dW(s), dW(s)\rangle = \int_0^1 \varrho \sum_{k=1}^{+\infty} \lambda_k^2 e_k^2(x) dx ds.$$

Substituting these relations in Itô formula (2.4), we get

$$\begin{aligned} \varphi(t) - \varphi(0) &= \int_0^t \int_0^1 [-\varrho v^2 \partial_x v + \eta v \partial_x^2 v - h v \partial_x \varrho^\gamma] dx ds \\ &+ \int_0^t \int_0^1 [\frac{-v^3}{2}\partial_x \varrho - \varrho \frac{v^2}{2}\partial_x v - h\frac{\gamma}{\gamma - 1}(v\varrho^{\gamma-1}\partial_x \varrho + \varrho^{\gamma-1}\varrho \partial_x v)] dx ds \\ &+ \int_0^1 \langle \varrho v, dW(s)\rangle + \frac{1}{2}\int_0^t \int_0^1 \varrho \sum_{k=1}^{+\infty} \lambda_k^2 e_k^2(x) dx ds. \end{aligned} \tag{2.11}$$

From (1.16), we have

$$-\int_0^1 (\varrho v^2\partial_x v + \frac{v^3}{2}\partial_x\varrho + \varrho\frac{v^2}{2}\partial_x v)dx = -\int_0^1 \partial_x[\varrho v\frac{v^2}{2}] = 0,$$

$$\int_0^1 (v\varrho^{\gamma-1}\partial_x\varrho + \varrho^{\gamma-1}\varrho\partial_x v + \frac{\gamma-1}{\gamma}v\partial_x\varrho^\gamma)dx = \int_0^1 \partial_x[v\varrho^\gamma] = 0,$$

then, we obtain (2.2). □

Corollary 2.2. *Let (v_0, ϱ_0) be the initial data given by $\sqrt{\varrho_0}v_0 \in L^2(0,1)$ and $\varrho_0 \in L^\gamma(0,1)$. For $T \geq 1$, there exists a positive constant C such that*

$$\frac{1}{T}\mathbf{E}\int_0^T \|\partial_x v\|^2_{L^2(0,1)}\,dt \leq C. \tag{2.12}$$

Proof. Applying mathematical expectation to the formula (2.2), we obtain

$$\mathbf{E}\left(\int_0^1 \varrho\frac{v^2}{2}dx + \frac{h}{\gamma-1}\int_0^1 \varrho^\gamma dx\right) = \mathbf{E}\left(\int_0^1 \varrho_0\frac{v_0^2}{2}dx + \frac{h}{\gamma-1}\int_0^1 \varrho_0^\gamma dx\right) - \eta\mathbf{E}\int_0^1 \|\partial_x v\|^2_{L^2(0,1)}\,ds + \frac{1}{2}\mathbf{E}\int_0^t \sum_{k=1}^{+\infty}\lambda_k^2\int_0^1 \varrho e_k^2(x)dxds. \tag{2.13}$$

As

$$\mathbf{E}\left(\int_0^1 \varrho\frac{v^2}{2}dx + \frac{h}{\gamma-1}\int_0^1 \varrho^\gamma dx\right) > 0$$

and $\sqrt{\varrho_0}v_0 \in L^2(0,1)$, $\varrho_0 \in L^\gamma(0,1)$ then, (2.13) becomes

$$\eta\mathbf{E}\int_0^1 \|\partial_x v\|^2_{L^2(0,1)}\,ds \leq C_1 + \frac{1}{2}\mathbf{E}\int_0^t \sum_{k=1}^{+\infty}\lambda_k^2\int_0^1 \varrho e_k^2(x)dxds, \tag{2.14}$$

where

$$C_1 = \mathbf{E}\left(\int_0^1 \varrho_0\frac{v_0^2}{2}dx + \frac{h}{\gamma-1}\int_0^1 \varrho_0^\gamma dx\right).$$

As

$$\sum_{k=1}^{+\infty}\lambda_k^2 < +\infty,$$

and using the relations

$$\sup_{0\leq x\leq 1} e_k^2(x) = \sup_{0\leq x\leq 1}(\sqrt{2}\sin(k\pi x))^2 = 2,$$

$$\int_0^1 \varrho e_k^2 dx \leq \sup_{0\leq x\leq 1} e_k^2(x)\int_0^1 \varrho dx = 2,$$

(2.14) becomes

$$\eta\mathbf{E}\int_0^1 \|\partial_x v\|^2_{L^2(0,1)}\,ds \leq C_1 + C_2, \tag{2.15}$$

where

$$C_2 = t\sum_{k=1}^{+\infty}\lambda_k^2.$$

Dividing (2.15) by $T > 0$, we obtain the sought after estimate, where $C = \frac{C_1+C_2}{T}$. □

Remark 2.1. From Prokhorov theorem [3], the inequality (2.12) allows us to extract a subsequence of invariant measures

$$\mu_{T_k} = \frac{1}{T_k}\int_0^{T_k} \nu_t dt,$$

converging weakly to a measure μ in the space $L^p(0,1)$, $p \in [0,+\infty]$. It is clear that this result is valid only for v. To find an invariant measure for the system of equations (1.11)-(1.12) (using Krylov-Bogoliubov theorem [3]), we need to have also an estimate for ϱ. As it is difficult to find an adequate estimate for ϱ, we propose to study an approximate problem.

3. POSITION OF THE PROBLEM

We consider the system of equations (1.11)-(1.12), with the conditions (1.16), (1.17), in lagrangian coordinates in the domain $[0,1]$ (for the passage of eulerian coordinates to lagrangian coordinates see [1]). the system of equations is written as

$$dv = (\eta\partial_\xi(\varrho\partial_\xi v) - h(\partial_\xi \varrho^\gamma))dt + dG, \quad 0<\xi<1, \tag{3.1}$$

$$\partial_t \varrho = -\varrho^2 \partial_\xi v, \tag{3.2}$$

with

$$v|_{\xi=0,1} = 0, \quad t\in[0,T] \tag{3.3}$$

and

$$\varrho(0,\xi) = \varrho_0(\xi), \quad v(0,\xi) = v_0(\xi) \quad \text{for} \quad \xi\in[0,1]. \tag{3.4}$$

where G is a Brownian motion defined in the probability space $(\Omega, F, \mathbf{P})$ such that

$$G(t) = \sum_{k=1}^{+\infty} \lambda_k e_k G^{(k)}(t),$$

where $\lambda_k \in \mathbf{R}$, $k=1,2,..$, $\{e_k\}_{k=1}^{+\infty}$ is an orthonormal basis in $L^2(0,1)$ and $G^{(k)}$, $k=1,2,..$, are independent real Brownian motions.
The existence and uniqueness of the solution to the problem (3.1)-(3.2) with the conditions (3.3) and (3.4) are proved by Tornatore-Fujita Yashima [9].

We add a diffusion of density to the equation (3.2), given by

$$\varepsilon\varrho\partial_\xi((\varrho^{-(\gamma-2)} + 2\varrho^{-\gamma} + \varrho^{-(\gamma+2)})\partial_\xi\varrho), \ \varepsilon>0,$$

i.e., we consider the system

$$dv = (\eta\partial_\xi v(\varrho\partial_\xi v) - h(\partial_\xi\varrho^\gamma))dt + dG, \tag{3.5}$$

$$\partial_t\varrho = -\varrho^2\partial_\xi v + \varepsilon\varrho\partial_\xi((\varrho^{-(\gamma-2)} + 2\varrho^{-\gamma} + \varrho^{-(\gamma+2)})\partial_\xi\varrho), \tag{3.6}$$

with

$$v|_{\xi=0,1} = 0, \quad t\in[0,T] \tag{3.7}$$

and

$$\varrho(0,\xi) = \varrho_0(\xi), \quad v(0,\xi) = v_0(\xi) \quad \text{for} \quad \xi\in[0,1]. \tag{3.8}$$

3.1. **Estimate of energy.** We define a functional $\overline{\varphi}(t)$ such that

$$\overline{\varphi}(t)=\int_0^1\frac{v^2}{2}d\xi+\frac{h}{\gamma-1}\int_0^1\varrho^{(\gamma-1)}d\xi.$$

The application of Ito formula to the functional $\overline{\varphi}$ gives the following result.

Proposition 3.1. *Let be*

$$\sigma=\varrho-\frac{1}{\varrho}.$$

We have

$$\overline{\varphi}(t)-\overline{\varphi}(0)=-\eta\int_0^t\int_0^1\varrho(\partial_\xi v)^2d\xi ds-h\varepsilon(\gamma-1)\int_0^t\|\partial_\xi\sigma(s)\|^2_{L^2(0,1)}\,ds$$

$$+\int_0^t\langle v,dG_s\rangle+\frac{1}{2}\int_0^t\sum_{k=1}^{+\infty}\lambda_k^2\int_0^1 e_k^2d\xi ds. \tag{3.9}$$

Proof. Let

$$v=u+G.$$

Ito formula gives

$$\overline{\varphi}(t)-\overline{\varphi}(0) \tag{3.10}$$
$$=\int_0^t\frac{\partial\overline{\varphi}}{\partial u}du+\int_0^t\frac{\partial\overline{\varphi}}{\partial\varrho}d\varrho+\int_0^t\frac{\partial\overline{\varphi}}{\partial G_s}dG_s+\frac{1}{2}\int_0^t\frac{\partial^2\overline{\varphi}}{\partial^2G_s}dG_s^2,$$

where

$$\frac{\partial\overline{\varphi}}{\partial u},\ \frac{\partial\overline{\varphi}}{\partial\varrho},\ \frac{\partial\overline{\varphi}}{\partial G_s},\ \frac{\partial^2\overline{\varphi}}{\partial^2G_s}$$

are Fréchet derivatives. From (3.5), we have

$$\frac{\partial\overline{\varphi}}{\partial u}du=\int_0^1\left[\eta v\partial_\xi(\varrho\partial_\xi v)-hv(\partial_\xi\varrho^\gamma)\right]d\xi ds.$$

On the other hand, from (3.6), we get

$$\frac{\partial\overline{\varphi}}{\partial\varrho}d\varrho=-h\int_0^1\varrho^\gamma\partial_\xi v d\xi ds-h\varepsilon(\gamma+1)\int_0^1(\partial_\xi(\varrho-\frac{1}{\varrho})^2)d\xi ds.$$

Similarly, we have

$$\frac{\partial\overline{\varphi}}{\partial G_s}dG_s=\langle v,dG_s\rangle$$

and

$$\frac{\partial^2\overline{\varphi}}{\partial^2G_s}(dG_s,dG_s)=\frac{1}{2}\int_0^t\sum_{k=1}^{+\infty}\lambda_k^2\int_0^1e_k^2d\xi ds.$$

Subsisting these relations in Ito formula (3.10), we obtain the desired result. □

We have the following result.

Corollary 3.2. *We have the estimate*

$$\frac{1}{T}\mathbf{E}\int_0^T\|\partial_\xi v(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,1)}\,dt+\frac{1}{T}\mathbf{E}\int_0^T\|\partial_\xi\sigma(t)\|^2_{L^2(0,1)}\,dt\le\overline{C}, \tag{3.11}$$

where $\overline{C}$ is a positive constant.

Proof. As we have

$$\|\partial_\xi v\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,1)} = \int_0^1 |\partial_\xi v|^{\frac{4}{3}}\, d\xi = \int_0^1 (\frac{1}{\varrho})^{\frac{2}{3}} \varrho^{\frac{2}{3}} |\partial_\xi v|^{\frac{4}{3}}\, d\xi. \tag{3.12}$$

From Holder inequality, $(p = 3, q = \frac{3}{2})$, we have

$$\|\partial_\xi v\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,1)} \leq \left(\int_0^1 (\frac{1}{\varrho})^2 d\xi\right)^{\frac{1}{3}} \left(\int_0^1 \varrho\, |\partial_\xi v|^2\, d\xi\right)^{\frac{2}{3}}.$$

As

$$\left|\frac{1}{\varrho}\right|^2 \leq 2\,|\sigma|^2 + 1,$$

then

$$\|\partial_\xi v\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,1)} \leq \frac{1}{3} + \frac{2}{3}\int_0^1 |\sigma|^2 d\xi + \frac{2}{3}\int_0^1 \varrho\, |\partial_\xi v|^2\, d\xi. \tag{3.13}$$

Substituting the inequality (3.13) in the formula (3.9), applying the mathematical expectation and dividing by T, then we obtain the desired estimate. □

4. EXISTENCE OF THE SOLUTION FOR THE SYSTEM (3.5)-(3.7)

We consider the system of equations (3.5)-(3.8) for each $\omega \in \Omega$. It is a deterministic system for $u(\omega; t, x)$, $\varrho(\omega; t, x)$. Multiplying the equation (3.6) by $\varrho^{\frac{\gamma-3}{2}}$, by letting

$$\theta = \varrho^{\frac{\gamma-1}{2}}$$

and

$$\theta_0 = \varrho_0^{\frac{\gamma-1}{2}},$$

then, the system of equations (3.5)-(3.6) becomes

$$\partial_t u = \eta\partial_\xi(\theta^{\frac{2}{\gamma-1}}\partial_\xi u) + \eta\partial_\xi(\theta^{\frac{2}{\gamma-1}}\partial_\xi G) - h\partial_\xi\theta^{\frac{2\gamma}{\gamma-1}}, \tag{4.1}$$

$$\partial_t\theta + (\frac{\gamma-1}{2})\theta^{\frac{\gamma+1}{\gamma-1}}\partial_\xi(u + G) - (\frac{\gamma-1}{2})\varepsilon\theta\partial_\xi((\theta^{\frac{2(-\gamma+2)}{\gamma-1}} + 2\theta^{\frac{-2\gamma}{\gamma-1}})d\xi\theta^{\frac{-2(\gamma+2)}{\gamma-1}}) = 0. \tag{4.2}$$

with

$$u|_{\xi=0,1} = 0,\ t \in [0, T].$$

To study this system of equations, we use Galerkin approximate.

4.1. Galerkin method.

Let be V_m a space generated by

$$\{\cos k\pi x, \sin k\pi x\}_{k=0}^m, \qquad m \in \mathbf{N}.$$

In the space V_m, the system of equations (4.1)-(4.2) is written in the form

$$\partial_t u^{[m]} = \eta\partial_\xi(\theta^{[m]\,\frac{2}{\gamma-1}}\partial_\xi u^{[m]}) + \eta\partial_\xi(\theta^{[m]\,\frac{2}{\gamma-1}}\partial_\xi G) - h\partial_\xi\theta^{[m]\,\frac{2\gamma}{\gamma-1}}, \tag{4.3}$$

$$\begin{aligned} \partial_t\theta^{[m]} &+ (\frac{\gamma-1}{2})\theta^{[m]\,\frac{\gamma+1}{\gamma-1}}\partial_\xi(u^{[m]} + G) \\ &- (\frac{\gamma-1}{2})\varepsilon\theta^{[m]}\partial_\xi((\theta^{[m]\,\frac{2(-\gamma+2)}{\gamma-1}} + 2\theta^{[m]\,\frac{-2\gamma}{\gamma-1}})d\xi\theta^{[m]\,\frac{-2(\gamma+2)}{\gamma-1}}) = 0. \end{aligned} \tag{4.4}$$

4.1.1. *Existence and uniqueness of the solution.* Let $(\theta^{[m]}, u^{[m]})$ be the solution of the system to equations (4.3)-(4.4). The research of the solution $\theta^{[m]}$ given by

$$\theta^{[m]} - \theta_0 = \sum_{k=0}^{m} \alpha_k^m(t) \cos k\pi x + \beta_k^m(t) \sin k\pi x \tag{4.5}$$

is done by solving the system of a differential equations

$$\frac{d}{dt}\alpha_k^m(t) = -(\frac{\gamma+1}{2}) \int_0^1 (\varrho^{\frac{-\gamma+1}{2}} \partial_\xi (u^{[m]} + G) \cos k\pi x) d\xi \\ + F_0 k(\alpha^m, \beta^m),$$

$$\frac{d}{dt}\beta_k^m(t) = -(\frac{\gamma+1}{2}) \int_0^1 (\varrho^{\frac{-\gamma+1}{2}} \partial_\xi (u^{[m]} + G) \sin k\pi x) d\xi \\ + F_1 k(\alpha^m, \beta^m),$$

where

$$F_0 k(\alpha^m, \beta^m) = (\frac{\gamma-1}{2})\varepsilon \int_0^1 \theta^{[m]} (\partial_\xi (\theta^{[m]\,\frac{-(\gamma+2)}{\gamma-1}} + 2\theta^{[m]\,\frac{-2\gamma}{\gamma-1}} + \theta^{[m]\,\frac{-2(\gamma+2)}{\gamma-1}})) \\ \partial_\xi (\theta^{[m]} \frac{2}{\gamma-1}) \cos k\pi x dx$$

$$F_1 k(\alpha^m, \beta^m) = (\frac{\gamma-1}{2})\varepsilon \int_0^1 \theta^{[m]} (\partial_\xi (\theta^{[m]\,\frac{-(\gamma+2)}{\gamma-1}} + 2\theta^{[m]\,\frac{-2\gamma}{\gamma-1}} + \theta^{[m]\,\frac{-2(\gamma+2)}{\gamma-1}})) \\ \partial_\xi (\theta^{[m]} \frac{2}{\gamma-1}) \sin k\pi x dx.$$

4.2. **Estimate for the convergence.** If we multiply the equation (4.3) by $u^{[m]}$ and the equation (4.4) by $\theta^{[m]}$, after integrating from 0 to 1, the system (4.3)-(4.4) becomes

$$\frac{1}{2}\frac{d}{dt} \left\|\theta^{[m]}\right\|^2_{L^2(0,1)} + (\frac{\gamma-1}{2}) \int_0^1 \theta^{[m]\,\frac{2\gamma}{\gamma-1}} \partial_\xi u^{[m]} d\xi \\ + (\frac{\gamma-1}{2}) \int_0^1 \theta^{[m]\,\frac{2\gamma}{\gamma-1}} \partial_\xi W d\xi + \frac{(\gamma-1)^2}{2} \varepsilon \left\|\partial_\xi \sigma\right\|^2_{L^2(0,1)} = 0 \tag{4.6}$$

and

$$\frac{1}{2}\frac{d}{dt} \left\|u^{[m]}\right\|^2_{L^2(0,1)} = -\eta \int_0^1 \theta^{[m]\,\frac{2}{\gamma-1}} (\partial_\xi u^{[m]})^2 d\xi \\ -\eta \int_0^1 \theta^{[m]\,\frac{2}{\gamma-1}} (\partial_\xi u^{[m]} \partial_\xi W) d\xi + h \int_0^1 \theta^{[m]\,\frac{2\gamma}{\gamma-1}} \partial_\xi u^{[m]} d\xi. \tag{4.7}$$

Then, we have the following result.

Proposition 4.1. *There exists a constant K independent of m such that*

$$\sup_{0\le t\le t_1} \int_0^1 \rho^{[m]\,\gamma-1} d\xi + \sup_{0\le t\le t_1} \left\|u^{[m]}\right\|^2_{L^2(0,1)} \\ + \left\|\partial_\xi \sigma(\rho^{[m]})\right\|^2_{L^2(0,t_1;L^2(0,1))} + \left\|\partial_\xi u^{[m]}\right\|^2_{L^{\frac{4}{3}}(0,t_1;L^{\frac{4}{3}}(0,1))} + \int_0^{t_1}\int_0^1 \rho^{[m]} (\partial_\xi u^{[m]})^2 d\xi \le K. \tag{4.8}$$

Proof. Multiplying (4.6) by h and (4.7) by $\frac{\gamma-1}{2}$ and summing the two equations, we have

$$\frac{d}{dt}[\frac{h}{2}\left\|\theta^{[m]}\right\|^2_{L^2(0,1)}+\frac{\gamma-1}{4}\left\|u^{[m]}\right\|^2_{L^2(0,1)}]+\eta\frac{(\gamma-1)}{4}\int_0^1\theta^{[m]\,\frac{2}{\gamma-1}}(\partial_\xi u^{[m]})^2d\xi \tag{4.9}$$
$$\leq -h\frac{(\gamma-1)}{2}\int_0^1\theta^{[m]\,\frac{2\gamma}{\gamma-1}}\partial_\xi W d\xi - h\frac{(\gamma-1)^2}{2}\varepsilon\left\|\partial_\xi\sigma\right\|^2_{L^2(0,1)}$$
$$-\eta\frac{(\gamma-1)}{4}\int_0^1\theta^{[m]\,\frac{2}{\gamma-1}}(\partial_\xi u^{[m]})^2d\xi-\eta\frac{(\gamma-1)}{2}\int_0^1\theta^{[m]\,\frac{2}{\gamma-1}}(\partial_\xi u^{[m]}\partial_\xi W)d\xi.$$

Substituting the inequality (3.13) in (4.9), we obtain the desired result. □

4.3. **Existence of the solution.** We have the following result.

Proposition 4.2. *There exists a couple (u,ρ) satisfying the equalities*

$$-\int_0^T\int_0^1(\partial_\xi\phi)ud\xi ds-\int_0^1\phi(x,0)u(x,0)dx \tag{4.10}$$
$$=-\eta\int_0^T\int_0^1(\partial_\xi\phi)\rho\partial_\xi v d\xi ds+h\int_0^T\int_0^1(\partial_\xi\phi)\rho^\gamma d\xi ds$$

and

$$-\int_0^T\int_0^1(\partial_t\phi)\log\rho-\int_0^1\phi(x,0)\log(x,0) \tag{4.11}$$
$$=-\int_0^T\int_0^1(\rho\partial_\xi v)\phi d\xi ds-\varepsilon\int_0^T\int_0^1((\rho^{-(\gamma-1)}+2\rho^{-\gamma}+\rho^{-(\gamma+2)})\partial_\xi\rho)d\xi ds,$$

for any function ϕ sufficiently regular satisfying

$$\phi(0)=\phi(1)=0. \tag{4.12}$$

Proof. Recall that

$$\partial_\xi\sigma=\partial_\xi(\rho-\frac{1}{\rho})=(1+\frac{1}{\rho^2})\partial_\xi\rho.$$

As we have

$$((\rho^{-(\gamma-1)+2\rho^{-\gamma}+\rho^{-(\gamma+2)}})\partial_\xi\rho)=\rho^{2-\gamma}(1+\frac{1}{\rho^2})+\rho^{-\gamma}(1+\frac{1}{\rho^2})=(\rho^{2-\gamma}+\rho^{-\gamma})\partial_\xi\sigma,$$

then, (4.11) becomes

$$-\int_0^T\int_0^1(\partial_t\phi)\log\rho-\int_0^1\phi(x,0)\log(x,0)$$
$$=-\int_0^T\int_0^1(\rho\partial_\xi v)\phi d\xi ds-\varepsilon\int_0^T\int_0^1(\rho^{2-\gamma}+\rho^{-\gamma})\partial_\xi\sigma\partial_\xi\phi d\xi ds.$$

□

To prove the existence of the solution for the system of equations (3.5)-(3.8), we have the following theorem.

Theorem 4.3. *Let $T>0$. Assume that $\sqrt{\varrho_0}v_0\in L^2(0,1)$ and $\varrho_0\in L^\gamma(0,1)$, then there exists a couple (v,ρ) with value in $L^2(0,T;H_0^1(0,1))$ and $L^2(0,T;L^p(0,1)),p>1$ respectively satisfying in $[0,T]$ the system of equations* (3.5)-(3.8).

To prove this theorem, we need the following compactness lemma (for more details, see [6, page 57 – 59] (Ch. 1, Theorem 5.1)).

Lemma 4.4. [6] *Consider the Banach spaces B_0, B_1 and B such that*
(1) $B_0 \subset B \subset B_1$ *with continuous injections and,* B_0, B_1 *are reflexives,*
(2) *the injection of* B_0 *in* B_1 *is compact.*
Let

$$W = \{u \in \mathbf{L}^{p_0}(0,T;B_0) | \frac{du}{dt} \in \mathbf{L}^{p_1}(0,T;B_1)\}, \tag{4.13}$$

where T *is positive number while* $1 < p_i < +\infty$ *for* $i = 0, 1$*, equipped with the norm*

$$\|u\|_{\mathbf{L}^{p_0}(0,T;B_0)} + \|\frac{du}{dt}\|_{\mathbf{L}^{p_1}(0,T;B_1)},$$

then the injection of the Banach space W *in* $\mathbf{L}^{p_0}(0,T;B)$ *is compact.*

Proof of the theorem. It is clear that the inequalities (4.10) and (4.11) are satisfied for $\phi \in V_m$. For the passage to the limit, we use a compactness Lemma with

$$B_0 = H^1(0,1), \qquad B = L^p(0,1) \qquad \text{and} \qquad B_1 = H^{-2}(0,1).$$

It is easy to control that all conditions of compactness lemma are satisfied, then, we have

$$\begin{aligned} \log \rho^{[m]} &\rightharpoonup \log \rho \qquad \text{in } L^1(0,1), \\ {\rho^{[m]}}^{2-\gamma} &\rightharpoonup \rho^{2-\gamma} \qquad \text{in } L^2(0,1), \\ {\rho^{[m]}}^{-\gamma} &\rightharpoonup \rho^{-\gamma} \qquad \text{in } L^2(0,1). \end{aligned}$$

□

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