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THE CONSERVATIVENESS OF GIRSANOV TRANSFORMED FOR SYMMETRIC JUMP-DIFFUSION PROCESS

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ABSTRACT. We study about the Girsanov transformed for symmetric Markov processes with jumps associated with regular Dirichlet form. We prove the conservativeness of it by dividing the regular Dirichlet form into the "small jump"part and the "big jump"part.

Key words and phrases: Conservative; Girsanov transformed process; Jump process.

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1. INTRODUCTION

In this paper, we study the Girsanov transformation of Markov process. A Markov process is said to be conservative if the associated particle stays at state space forever. This property is one of the important global properties of Markov process. The conservativeness of Markov processes has been considered by many authors (for example, see [4, 8, 9]). We give the conservativeness of Girsanov transform process associated with regular Dirichlet form. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; \mathfrak{m})$ and $\mathbf{X} = (\Omega, \mathfrak{F}_t, \theta_t, X_t, \mathbf{P}_x, \zeta)$ the $\mathfrak{m}$-symmetric Hunt process on E associated with $(\mathcal{E}, \mathcal{F})$. Let $\mu^c_{\langle u,v\rangle}$ be a bounded measure, (see [2, Lemma 3.2.3]), such that

$$\mathcal{E}^{(c)}(u, v) = \frac{1}{2}\mu^c_{\langle u,v\rangle}(X) \text{ for } u \in \mathcal{F}_e.$$

We define the family Θ of sequence of finely open sets defined by

$$\Theta = \{\{G_n\} : G_n \text{ is finely open for all } n, G_n \subset G_{n+1}, \cup_{n=1}^{\infty} G_n = E \text{ q.e.}\}.$$

A function $u \in \dot{\mathcal{F}}_{\text{loc}}$ if there exists $\{G_n\} \in \Theta$ and $\{u_n\} \in \mathcal{F}$ such that $u = u_n$ $\mathfrak{m}$-a.e on $\{G_n\}$ for each $n \in \mathbb{N}$. It is shown in [6, Theorem 4.1] that $\mathcal{F}_e \subset \dot{\mathcal{F}}_{\text{loc}}$.

We introduce the subclass $\dot{\mathcal{F}}^{\dagger}_{\text{loc}}$ of $\dot{\mathcal{F}}_{\text{loc}}$ as follows:

$$\dot{\mathcal{F}}^{\dagger}_{\text{loc}} := \left\{ u \in \dot{\mathcal{F}}_{\text{loc}} \,\middle|\, \int_{y\in E} (u(y) - u(x))^2 J(\mathrm{d}x, \mathrm{d}y) \text{ is the smooth measure} \right\}.$$

An increasing sequence $\{F_n\}$ of closed set of E is said to be a $strict$ $\mathcal{E}$-$nest$ if

$$\lim_{n\to\infty} \text{Cap}_{1,G_{1\varphi}}(E\setminus F_n) = 0,$$

where $\text{Cap}_{1,G_{1\varphi}}$ is the weighted capacity defined in [7, Chapter V, Definition 2.1] and family $\{F_n\}$ of closed sets is a strict $\mathcal{E}$-nest if and only if

$$\mathbf{P}_x(\lim_{n\to\infty} \sigma_{E\setminus F_n}) = 0 \text{ q. e.} x \in E$$

in view of [7, Chapter V, proposition 2.6]. A function u defined on E_∂ is said to be $strictly$ $\mathcal{E}$-$quasi\ continuous$ if there exists a strict $\mathcal{E}$-nest $\{F_n\}$ such that u is continuous on each $F_n \cup \{\partial\}$. Denote by $QC(E_\partial)$ the totality of strictly $\mathcal{E}$-quasi-continuous functions on E_∂. We assume that ρ is a non-negative function in $\dot{\mathcal{F}}^{\dagger}_{\text{loc}} \cap QC(E_\partial)$ such that $\mathfrak{m}(\{\rho > 0\}) > 0$ and $0 \le \rho(\partial) < \infty$. Set $N := \{x \in E : \rho(x) = 0\}$ and define a stopping time σ_N by $\sigma_N := \inf\{t > 0 : X_t \in N\}$. From the Fukushima decomposition,

$$\rho(X_t) - \rho(X_0) = M^{[\rho]}_t + N^{[\rho]}_t,\ t \in [0, \zeta[, \mathbf{P}\text{-a.s. for q.e. } x \in E,$$

where $M^{[\rho]}$ is an MAF locally of finite energy and $N^{[\rho]}$ is a CAF locally of zero energy. Define a local martingale M on the random interval $[\![0, \sigma_N \wedge \zeta]\!]$ by

$$M_t := \int_0^t \frac{1}{\rho(X_{s-})} \mathrm{d}M^{[\rho]}_s. \tag{1.1}$$

Let L^ρ_t be the Doléans-Dade exponential of M_t, that is, the unique solution of

$$L^\rho_t = 1 + \int_0^t L^\rho_{s-} \mathrm{d}M_s,\ \mathbf{P}_x\text{-a.s.}, x \in E\setminus N.$$

By the Doléans-Dade formula ([5, Theorem 9.39]) on $\{t < \sigma_N \wedge \zeta\}$,

$$L^\rho_t = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right) \prod_{0<s\le t} \frac{\rho(X_s)}{\rho(X_{s-})} \exp\left(1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right). \tag{1.2}$$

Since L_t^ρ is a positive local martingale on the random interval $[\![0, \sigma_N \wedge \zeta]\!]$, so is a positive supermartingale. Consequently, the formula

$$\mathrm{d}\widetilde{\mathbf{P}}_x = L_t^\rho \mathrm{d}\mathbf{P}_x \text{ on } \mathcal{F}_t \cap \{t < \sigma_N \wedge \zeta\} \text{ for } x \in E \backslash N, \tag{1.3}$$

uniquely determines a family of probability measures on $(\Omega, \mathcal{F}_\infty)$. Let $\widetilde{\mathbf{X}}^\rho := (\Omega, \mathcal{F}_t, \widetilde{X}_t, \widetilde{\mathbf{P}}_x, \widetilde{\zeta})$ be the transformed process of $\mathbf{X}$ by L_t^ρ. Here for $\omega \in \Omega$,

$$\widetilde{X}_t(\omega) := \begin{cases} X_t(\omega), & 0 \le t < \sigma_N, \\ \partial, & \sigma_N \le t \le \infty, \end{cases} \quad \widetilde{\zeta}(\omega) := \sigma_N(\omega) \wedge \zeta(\omega).$$

The semigroup $\{\tilde{P}_t\}$ of $\widetilde{\mathbf{X}}^\rho$ equals

$$\widetilde{P}_t f(x) = \widetilde{\mathbf{E}}_x[f(\widetilde{X}_t) : t < \widetilde{\zeta}] = \mathbf{E}_x[L_t^\rho f(X_t); t < \sigma_N \wedge \zeta]. \tag{1.4}$$

The transformed process $\widetilde{\mathbf{X}}^\rho$ by L_t^ρ is a $\rho^2\mathfrak{m}$-symmetric right process by [8, Lemma 3.1]. We denote by $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ the Dirichlet form on $L^2(E, \rho^2\mathfrak{m})$ associated with $\widetilde{\mathbf{X}}^\rho$. It is known that $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ is a *quasi-regular* (see [7]). For a closed subset F of E, $\mathcal{D}_b(\mathcal{E})_F$ is the space defined by

$$\mathcal{D}_b(\mathcal{E})_F = \{u \in \mathcal{D}_b(\mathcal{E}) : u = 0 \ \mathfrak{m}\text{-a.e. on } E \backslash F\},$$

where $\mathcal{D}_b(\mathcal{E})$ is the set of bounded functions in $\mathcal{F}$. The following theorem has been proved in [8, Theorem 3.4].

Theorem 1. *Suppose that $\rho > 0$ q.e belongs to $\dot{\mathcal{F}}_{loc}^+ \cap QC(E_\partial)$. Then there exists an $\mathcal{E}$-nest $\{F_n\}$ of compact sets such that $\cup_{n \ge 1} \mathcal{D}_b(\mathcal{E})_{F_n} \subset \widetilde{\mathcal{F}}^\rho$ and for $u \in \mathcal{D}_b(\mathcal{E})_{F_n}$,*

$$\begin{aligned} \widetilde{\mathcal{E}}^\rho(u, u) = & \frac{1}{2} \int_E \rho(x)^2 \mu_{\langle u \rangle}^c(\mathrm{d}x) + \int_{E \times E \backslash \mathsf{d}} (u(x) - u(y))^2 \rho(x)\rho(y) J(\mathrm{d}x, \mathrm{d}y) \\ & + \rho(\partial) \int_E u(x)^2 \rho(x) \kappa(\mathrm{d}x) \end{aligned} \tag{1.5}$$

We assume that $\kappa = 0$ in (1.5). This means that the corresponding symmetric Hunt process has no killing inside. For $u, v \in \mathcal{F}$, then we have the following.

$$\widetilde{\mathcal{E}}^\rho(u, v) = \frac{1}{2} \int_E \rho(x)^2 \mu_{\langle u,v \rangle}^c(\mathrm{d}x) + \int_{E \times E \backslash \mathsf{d}} (u(x) - u(y))(v(x) - v(y)) \rho(x)\rho(y) J(\mathrm{d}x, \mathrm{d}y). \tag{1.6}$$

By [2, Theorem 1.6.6] that $(\mathcal{E}, \mathcal{F})$ is conservative if and only if there exists a sequence $\{u_n\} \subset \mathcal{F}$ satisfying

$$0 \le u_n \le 1, \ \lim_{n \to \infty} u_n = 1 \ \mathfrak{m}\text{-a.e.}$$

such that

$$\lim_{n \to \infty} \mathcal{E}(u_n, v) = 0 \text{ for any } v \in \mathcal{F} \cap L^1(E; \mathfrak{m}).$$

By the same argument as in [9, Lemma 2.1], we obtain the following lemma.

Lemma 1. *Let $\{\varphi_n\}$ denote an increasing sequence of non negative function in $L^2(E; \mathfrak{m}) \cap L^\infty(E; \mathfrak{m})$ such that*

$$\lim_{n \to \infty} \varphi_n = 1 \ \mathfrak{m}\text{-a.e.}$$

Assume that there exists $t_0 > 0$ *such that*

$$\lim_{n\to\infty}\int_E (f(x) - \tilde{P}_t f(x))\varphi_n(x)\rho\mathfrak{m}(\mathrm{d}x) = 0$$

for any $t \in (0, t_0]$ *and any* $f \in \mathcal{F} \cap C_0(E)$. *Then* $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ *is conservative.*

By considering the Beurling-Deny formula of Girsanov transformed process $\widetilde{\mathbf{X}}^\rho$ in [8, Theorem 3.4], we prove the conservativeness by using a similar way to [9]. More precisely, we first divide the regular Dirichlet form $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ into the "small jump"part $\widetilde{\mathcal{E}}^{\rho,(1)}$ and the "big jump"part $\widetilde{\mathcal{E}}^{\rho,(2)}$ (see Section 2 for definition). Here we assume that $\widetilde{\mathcal{E}}^{\rho,(2)}$ is regarded as a bounded perturbation, which implies that $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ is conservative if and only if so is $(\widetilde{\mathcal{E}}^{\rho,(1)}, \widetilde{\mathcal{F}}^\rho)$. We then adapt the Davies method as in [1] to $(\widetilde{\mathcal{E}}^{\rho,(1)}\widetilde{\mathcal{F}}^\rho)$.

2. MAIN RESULTS

In this section, we prove the conservativeness of Girsanov transformed for symmetric jump-diffusion processes, $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$. Firstly, we divide it into "small jump"part $\widetilde{\mathcal{E}}^{\rho,(1)}$ and the "big jump"part $\widetilde{\mathcal{E}}^{\rho,(2)}$. We now impose assumption on the measure $J(\mathrm{d}x, \mathrm{d}y)$.

Assumption 1.

(i) The measure $J(\mathrm{d}x, \mathrm{d}y)$ on $E \times E \backslash \mathsf{d}$ is expressed by

$$J(\mathrm{d}x, \mathrm{d}y) = J(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x)$$

for some kernel $J(x, \mathrm{d}y)$ which associates a positive Radon measure on $\mathcal{B}(E)$ for each $x \in E$.

(ii) There exists a strictly positive function $F(x, y)$ on $E \times E\backslash \mathsf{d}$ such that

$$F(x, y) = F(y, x) \ \text{ for any } (x, y) \in E \times E\backslash \mathsf{d}$$

and

$$M := \sup_{x\in E}\int_{d(x,y)\geq F(x,y)} J(x, \mathrm{d}y) < \infty.$$

Let $F(x, y)$ be a function on $E \times E\backslash \mathsf{d}$ satisfying Assumption 1 (ii). For $u, v \in \mathcal{F}$ and assume ρ is bounded, we divide the integral in (1.6) into:

$$\begin{aligned}
\widetilde{\mathcal{E}}^\rho(u, v) &:= \frac{1}{2}\int_E \rho(x)^2\mu^c_{\langle u,v\rangle}(\mathrm{d}x) + \frac{1}{2}\iint_{E\times E\backslash \mathsf{d}} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)J(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x)\\
&= \frac{1}{2}\int_E \rho(x)^2\mu^c_{\langle u,v\rangle}(\mathrm{d}x)\\
&\quad + \frac{1}{2}\iint_{E\times E\backslash \mathsf{d}} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)(J^{(1)}(x, \mathrm{d}y) + J^{(2)}(x, \mathrm{d}y))\mathfrak{m}(\mathrm{d}x)\\
&= \frac{1}{2}\int_E \rho(x)^2\mu^c_{\langle u,v\rangle}(\mathrm{d}x)\\
&\quad + \frac{1}{2}\iint_{0<d(x,y)<F(x,y)} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)J(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x)\\
&\quad + \frac{1}{2}\iint_{d(x,y)\geq F(x,y)} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)J(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x)\\
&= \widetilde{\mathcal{E}}^{\rho,(1)}(u, v) + \widetilde{\mathcal{E}}^{\rho,(2)}(u, v).
\end{aligned}$$

where $J^{(1)}(x, \mathrm{d}y = \mathbf{1}_{\{0<d(x,y)<F(x,y)\}} J(x, \mathrm{d}y)$ and $J^{(2)}(x, \mathrm{d}y = \mathbf{1}_{\{d(x,y) \geq F(x,y)\}} J(x, \mathrm{d}y)$.

We set

$$\widetilde{\mathcal{E}}^{\rho,(1)}(u, v) := \frac{1}{2} \int_E \rho(x)^2 \mu^c_{\langle u,v \rangle}(\mathrm{d}x) + \frac{1}{2} \iint_{0<d(x,y)<F(x,y)} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)J(\mathrm{d}x, \mathrm{d}y)$$

and

$$\widetilde{\mathcal{E}}^{\rho,(2)}(u, v) := \frac{1}{2} \int_E \left(\int_{d(x,y) \geq F(x,y)} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\rho(x)\rho(y)J(x, \mathrm{d}y) \right) \mathfrak{m}(\mathrm{d}x).$$

Then by the symmetry of J, the boundedness of ρ and Assumption 1 (ii) we have

$$0 \leq \widetilde{\mathcal{E}}^{\rho,(2)}(u, u) \leq \|\rho\|^2_\infty \mathcal{E}^{(2)}(u, u) \leq 2M\|\rho\|^2_\infty \|u\|^2_{L^2(E;\mathfrak{m})} \text{ for any } u \in \mathcal{F},$$

where $\mathcal{E}^{(2)}(u, u) = \frac{1}{2} \int_E \left(\int_{d(x,y) \geq F(x,y)} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))J(x, \mathrm{d}y) \right) \mathfrak{m}(\mathrm{d}x)$. Since this implies that

$$\widetilde{\mathcal{E}}^{\rho,(1)}_1(u, u) \leq \widetilde{\mathcal{E}}^{\rho}_1(u, u) \leq \widetilde{\mathcal{E}}^{\rho,(1)}_{2M\|\rho\|^2_\infty+1}(u, u) \text{ for any } u \in \mathcal{F},$$

$(\widetilde{\mathcal{E}}^{\rho,(1)}_1, \widetilde{\mathcal{F}}^\rho)$ is also a *regular Dirichlet form* on $L^2(E; \mathfrak{m})$. Moreover, by the same way as in [9, Lemma 2.3], we have the following.

Lemma 2. *The form $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ is conservative if and only if so is $(\widetilde{\mathcal{E}}^{\rho,(1)}, \widetilde{\mathcal{F}}^\rho)$.*

Then we consider the conservativeness of $(\widetilde{\mathcal{E}}^{\rho,(1)}, \widetilde{\mathcal{F}}^\rho)$ in order to show the conservativeness of $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$. In the following, we consider the conservativeness of $(\widetilde{\mathcal{E}}^{\rho,(1)}, \widetilde{\mathcal{F}}^\rho)$. To this end, we drop the suffix "(1)"for simplicity until the end of this section.

Define $\mu^c_{\langle \xi \rangle}$ for any non negative $\xi \in \dot{\mathcal{F}}_{\mathrm{loc}}$. Let $C(E)$ be the totality of continuous functions on E and define

$$\dot{\mathcal{F}}_{\text{loc, ac}} := \{\xi \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap C(E) \mid \mu^c_{\langle \xi \rangle} << \mathfrak{m}\}.$$

For non negative $\xi \in \dot{\mathcal{F}}_{\text{loc, ac}}$, we denote by $\Gamma^c(\rho)$ the density function of $\mu^c_{\langle \xi \rangle}$ with respect to the measure $\mathfrak{m}$. For each $r > 0$ and $\xi \in \dot{\mathcal{F}}_{\text{loc, ac}}$, we set

$$K_\xi(r) := \{x \in E \mid \xi(r) \leq r\}.$$

To state another assumption, we introduce a function class $\mathcal{A}$ defined by

$$\mathcal{A} := \{\xi \in \dot{\mathcal{F}}_{\text{loc, ac}} \mid \lim_{x \to \Delta} \xi(x) = \infty \text{ and } K_\xi(r) \text{ is compact for each } r > 0\}.$$

Assumption 2. There exist a function $F(x, y)$ satisfying Assumption 1 (ii) and a function $\xi \in \mathcal{A}$ such that the following hold.

(i) For any $(x, y) \in E \times E \backslash \mathsf{d}$ with $d(x, y) < F(x, y)$,

$$|\xi(x) - \xi(y)| < 1.$$

(ii) The function ξ satisfies

$$\frac{1}{\rho(x)} \int_{0<d(x,y)<F(x,y)} (\xi(x) - \xi(y))^2 \rho(y)J(x, \mathrm{d}y) < \infty$$

for any $x \in E$ and $\rho(x) \neq 0$.

The main result of this section is the following.

Theorem 2. *Assume Assumptions 1 and 2 on* $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$. *If there exists a sequence* $\{a_n\}$ *such that*

$$\varliminf_{n\to\infty} M_\xi(n+3)\mathfrak{m}(K_\xi(n+3)) \exp(-na_n + a_n^2 e^{a_n p_n} M_\xi(n+1)T) = 0 \tag{2.1}$$

for some $T > 0$, *then* $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ *is conservative.*

We fix $F(x,y)$ and $\xi \in \mathcal{A}$ satisfying Assumptions 1 and 2. Let $\{w_n\} \subset C_0(\mathbb{R})$ be an increasing sequence such that

$$w_n(t) := \begin{cases} 1, & |t| \le n+1, \\ n+2-|t|, & n+1 \le |t| \le n+2, \\ 0, & |t| \ge n+2. \end{cases}$$

Let us define a sequence of cut-off functions $\{\varphi_n\}$ by

$$\varphi_n(x) := w_n(\xi(x)) \text{ for } n = 1, 2, 3, \ldots.$$

By [9, Lemma 3.1], we can see that $\varphi_n \in \mathcal{F} \cap C_0(E)$ for any $n \ge 1$. We know by [2, Theorem 3.2.2] that the measure $\mu^{(c)}_{\langle \varphi_n \rangle}$ is absolutely continuous with respect to $\mathfrak{m}$. Moreover, if we denote by $\Gamma^c(\varphi_n)$ the density function of $\mu^{(c)}_{\langle \varphi_n \rangle}$ with respect to $\mathfrak{m}$, then we obtain

$$\Gamma^c(\varphi_n)(x) = w_n'(\xi(x))^2 \cdot \Gamma^c(\xi)(x) \le \Gamma^c(\xi)(x) \cdot \mathbf{1}_{\{n+1 \le \xi(x) \le n+2\}}. \tag{2.2}$$

We denote by $\Gamma^j(\varphi_n)$ the density function of the jumping energy measure

$$\mu^{(j)}_{\langle \varphi_n \rangle}(\mathrm{d}x) := \int_E (\varphi_n(x) - \varphi_n(y))^2 \rho(x)\rho(y) J(\mathrm{d}x, \mathrm{d}y)$$

with respect to $\rho(x)^2\mathfrak{m}(\mathrm{d}x)$. By Assumption 2 we have

$$|\varphi_n(x) - \varphi_n(y)| = |w_n(\xi(x)) - w_n(\xi(y))| \le |\xi(x) - \xi(y)| \cdot \mathbf{1}_{\{n \le \xi(x) \le n+3\}}$$

for any $(x,y) \in E \times E \backslash \mathsf{d}$ with $d(x,y) < F(x,y)$, then we obtain

$$\begin{aligned} \Gamma^j(\varphi_n)(x) &:= \int_{0<d(x,y)<F(x,y)} (\varphi_n(x) - \varphi_n(y))^2 \rho(x)\rho(y) J(x, \mathrm{d}y) \\ &\le \int_{0<d(x,y)<F(x,y)} (\xi(x) - \xi(y))^2 \rho(x)\rho(y) J(x, \mathrm{d}y) \cdot \mathbf{1}_{\{n \le \xi(x) \le n+3\}} \\ &= \Gamma^j(\xi)(x) \cdot \mathbf{1}_{\{n \le \xi(x) \le n+3\}}. \end{aligned} \tag{2.3}$$

Hence we obtain from Eq. (2.2) and (2.3) the following result.

Lemma 3. *The following inequality holds:*

$$\Gamma^c(\varphi_n)(x) + \Gamma^j(\varphi_n)(x) \le (\Gamma^c(\xi)(x) + \Gamma^j(\xi)(x)) \cdot \mathbf{1}_{\{n \le \xi(x) \le n+3\}}.$$

For the proof of Theorem 2, we need the following two lemmas.

Lemma 4. *The following inequality holds:*

$$\begin{aligned} &\int_E \psi_n(x)^{-2} \Gamma^c(\varphi_n)(x)\mathfrak{m}(\mathrm{d}x) + \int_E \psi_n(x)^{-2} \Gamma^j(\varphi_n)(x)\mathfrak{m}(\mathrm{d}x) \\ &\qquad \le e^{-na} M_\xi(n+3)\rho^2 \mathfrak{m}(K_\xi(n+3)). \end{aligned} \tag{2.4}$$

Proof. Fix a constant $a > 0$. Let $\{v_n\} \subset C_b(\mathbb{R})$ be a sequence such that

$$v_n(t) = \begin{cases} 0, & |t| \le \frac{n}{2}, \\ a(|t| - \frac{n}{2}), & \frac{n}{2} \le |t| \le n, \\ na/2, & |t| \le n. \end{cases}$$

Let us define

$$\psi_n(x) := \exp(v_n(\xi(x))) \text{ for } n = 1, 2, 3, ...$$

By the definition of ψ_n, we have

$$\begin{aligned}
&\int_E \psi_n(x)^{-2}\Gamma^c(\varphi_n)(x)\rho(x)^2\mathfrak{m}(\mathrm{d}x) + \int_E \psi_n(x)^{-2}\Gamma^j(\varphi_n)(x)\rho(x)^2\mathfrak{m}(\mathrm{d}x) \\
&= \int_{n+1\le\xi(x)\le n+2} e^{-na}\Gamma^c(\varphi_n)(x)\rho^2\mathfrak{m}(\mathrm{d}x) + \int_{n\le\xi(x)\le n+3} e^{-na}\Gamma^j(\varphi_n)(x)\rho^2\mathfrak{m}(\mathrm{d}x) \\
&\le e^{-na}\int_{n\le\xi(x)\le n+3}(\Gamma^c(\varphi_n)(x) + \Gamma^j(\varphi_n)(x))\rho^2\mathfrak{m}(\mathrm{d}x) \\
&\le e^{-na}\left(\operatorname{ess\,sup}_{n\le\xi(x)\le n+3}(\Gamma^c(\varphi_n)(x)) + \operatorname{ess\,sup}_{n\le\xi(x)\le n+3}(\Gamma^j(\varphi_n)(x))\right)\rho^2\mathfrak{m}\left(K_\xi(n+3)\right).
\end{aligned}$$

Put

$$M_\xi(r) := \operatorname{ess\,sup}_{x\in K_\xi(r)}(\Gamma^c(\xi)(x)) + \operatorname{ess\,sup}_{x\in K_\xi(r)}(\Gamma^j(\xi)(x)) \text{ for each } r > 0,$$

hence we obtain the assertion. ∎

Lemma 5. *The following inequality holds:*

$$\begin{aligned}
&\frac{1}{2}\int_0^t\left(\int_E \psi_n(x)^2\rho(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)\right)\mathrm{d}s \\
&+\frac{1}{2}\int_0^t\left(\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)\mathrm{d}s \\
&\le 2\exp(a^2e^{ap_n}M_\xi(n+1)t)\|\rho\psi_n\|^2_{L^2(E;\mathfrak{m})}. \qquad (2.5)
\end{aligned}$$

Proof. The proof is based on an idea in [9]. Fix $f \in \mathcal{F} \cap C_0(E)$ and set $u_t := \widetilde{P}^{\rho,(1)}_t$ for $t > 0$, where $\{\widetilde{P}^{\rho,(1)}_t\}_{t>0}$ is the Markovian semigroup on $L^2(E;\mathfrak{m})$ associated with the *Dirichlet form* $(\widetilde{\mathcal{E}}^{\rho,(1)}, \widetilde{\mathcal{F}}^\rho)$. By similar way in [9, Lemma 3.1], we have $\psi_n \in \dot{\mathcal{F}}_{b,loc}$. Furthermore, the measure $\mu^{(c)}_{\langle\psi_n\rangle}$ is absolutely continuous with respect to $\mathfrak{m}$ by [2, Theorem 3.2.2]. We denote by $\Gamma^c(\psi_n)$ the density function of $\mu^{(c)}_{\langle\psi_n\rangle}$ with respect to $\mathfrak{m}$ then by the Schwarz inequality ([6, lemma 5.2]), we have for any $\lambda > 0$,

$$\begin{aligned}
&-2\int_E \rho(x)^2u_s(x)\psi_n(x)\mu^c_{\langle u_s,\psi_n\rangle}(\mathrm{d}s) \\
&\le 2\sqrt{\int_E \rho(x)^2u_s(x)^2\mu^c_{\langle\psi_n\rangle}(\mathrm{d}x)}\cdot\sqrt{\int_E \rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)} \\
&\le \lambda\int_E \rho(x)^2u_s(x)^2\mu^c_{\langle\psi_n\rangle}(\mathrm{d}x) + \frac{1}{\lambda}\int_E \rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x) \\
&= \lambda\int_E \rho(x)^2u_s(x)^2\Gamma^c(\psi_n)\mathfrak{m}(\mathrm{d}x) + \frac{1}{\lambda}\int_E \rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x) \qquad (2.6)
\end{aligned}$$

and

$$
\begin{aligned}
&-\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))(\psi_n(x)^2-\psi_n(y)^2)u_s(y)\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&\leq \lambda\iint_{0<d(x,y)<F(x,y)}(\psi_n(x)-\psi_n(y))^2u_s(y)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&+\frac{1}{4\lambda}\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))^2(\psi_n(x)+\psi_n(y))^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&\leq \lambda\iint_{0<d(x,y)<F(x,y)}(\psi_n(x)-\psi_n(y))^2u_s(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&+\frac{1}{2\lambda}\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))^2(\psi_n(x)^2+\psi_n(y)^2)\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)
\end{aligned}
$$

Defining

$$\Gamma^j(\psi_n)(x):=\frac{1}{\rho(x)}\int_{0<d(x,y)<F(x,y)}(\psi_n(x)-\psi_n(y))^2\rho(y)J(x,\mathrm{d}y),\ x\in E,$$

and by Assumption 1 the following holds.

$$
\begin{aligned}
&-\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))(\psi_n(x)^2-\psi_n(y)^2)u_s(y)\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&=\lambda\int_E u_s(x)^2\Gamma^j(\psi_n)(x)\rho(x)^2\mathfrak{m}(\mathrm{d}x)\\
&+\frac{1}{\lambda}\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y). \qquad (2.7)
\end{aligned}
$$

Since $\psi_n\in\dot{\mathcal{F}}_{loc}\cap C_b(E)$, we have $\rho u_n\psi_n^2\in\mathcal{F}$ by Assumption 2 and [9, Eq. 3.5] in similar way to [4, Lemma 3.5]. Then by [3, Theorem 4.9(iv)] and [2] we see that

$$
\begin{aligned}
&\frac{d}{ds}\|\rho u_s\psi_n\|^2_{L^2(E;\mathfrak{m})}=-2\widetilde{\mathcal{E}}^{\rho,(1)}(u_s,u_s\psi_n^2)\\
&=-\int_E\rho(x)^2\mu^c_{\langle u_s,u_s\psi_n^2\rangle}(\mathrm{d}x)-\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))\\
&\times(u_s(x)\psi_n(x)^2-u_s(y)\psi_n(y)^2)\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&=-\int_E\rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)-2\int_E\rho(x)^2u_s(x)\psi_n(x)\mu^c_{\langle u_s,\psi_n\rangle}(\mathrm{d}x)\\
&-\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\
&-\iint_{0<d(x,y)<F(x,y)}(u_s(x)-u_s(y))\\
&\times(\psi_n(x)^2-\psi_n(y)^2)u_s(y)\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y). \qquad (2.8)
\end{aligned}
$$

Take $\lambda=1$ in Eq. (2.6) and (2.7) we have

$$\frac{d}{ds}\|\rho u_s\psi_n\|^2_{L^2(E;\mathfrak{m})}\leq\int_E\rho(x)^2u_s(x)^2\Gamma^c(\psi_n)\mathfrak{m}(\mathrm{d}x)+\int_E\rho(x)^2u_s(x)^2\Gamma^j(\psi_n)(x)\mathfrak{m}(\mathrm{d}x)$$

Let $\{p_n\}$ be a sequence defined by

$$p_n := 2\sup\{|\xi(x)-\xi(y)| : \frac{n}{2}-1 \le \xi(x) \le n+1, 0 < d(x,y) < F(x,y)\}.$$

By [9, Lemma 3.3] we then have the following.

$$\begin{aligned}\int_E \rho(x)^2 u_s(x)^2 \Gamma^c(\psi_n)\mathfrak{m}(\mathrm{d}x) + \int_E \rho(x)^2 u_s(x)^2 \Gamma^j(\psi_n)(x)\mathfrak{m}(\mathrm{d}x)\\ \le a^2 e^{ap_n} M_\xi(n+1)\|\rho u_s \psi_n\|^2_{L^2(E;\mathfrak{m})}.\end{aligned}$$

By the Gronwall lemma, we obtain

$$\|\rho u_t \psi_n\|^2_{L^2(E;\mathfrak{m})} \le \exp(a^2 e^{ap_n} M_\xi(n+1)t\|\rho\psi_n\|^2_{L^2(E;\mathfrak{m})}). \tag{2.9}$$

By (2.6) and (2.7) with $\lambda = 2$, we see from (2.8) that

$$\begin{aligned}\frac{d}{ds}\|\rho u_s\psi_n\|^2_{L^2(E;\mathfrak{m})} &\le -\frac{1}{2}\int_E \psi_n(x)^2\rho(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)\\ &\quad + 2\int_E u_s(x)^2\rho(x)^2\Gamma^c(\psi_n)\mathfrak{m}(\mathrm{d}x)\\ &\quad - \frac{1}{2}\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\ &\quad + 2\int_E u_s(x)^2\rho(x)^2\Gamma^j(\psi_n)\mathfrak{m}(\mathrm{d}x)\\ &\le -\frac{1}{2}\int_E \psi_n(x)^2\rho(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)\\ &\quad - \frac{1}{2}\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\\ &\quad + 2a^2e^{ap_n}M_\xi(n+1)\exp(a^2e^{ap_n}M_\xi(n+1)s)\|\rho\psi_n\|^2_{L^2(E;\mathfrak{m})}.\end{aligned}$$

By integrating with respect to s, we have the assertion. ∎

Proof of Theorem 2. By [3, Theorem 4.9(iv)], we have the following.

$$\begin{aligned}&\int_E (u_t(x)-f(x))\varphi_n(x)\rho(x)\mathfrak{m}(\mathrm{d}x)\\ &= -\int_0^t \widetilde{\mathcal{E}}^{\rho,(1)}(u_s,\varphi_n)\mathrm{d}s\\ &= -\frac{1}{2}\int_0^t\left(\int_E \rho(x)^2\mu^c_{\langle u_s,\varphi_n\rangle}(\mathrm{d}x)\right)\mathrm{d}s\\ &\quad -\frac{1}{2}\int_0^t\left(\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))(\varphi_n(x)-\varphi_n(y))\rho(x)\rho(y)J(\mathrm{d}x,\mathrm{d}y)\right)\mathrm{d}s.\end{aligned}$$

By Schwarz inequality [6, Lemma 5.2], we then have

$$\begin{aligned}\left(\int_E \rho(x)^2\mu^c_{\langle u_s,\varphi_n\rangle}(\mathrm{d}x)\right)^2 &= \left(\int_E \rho(x)^2\psi_n(x)^{-1}\psi_n(x)\mu^c_{\langle u_s,\varphi_n\rangle}(\mathrm{d}x)\right)^2\\ &\le \int_E \rho(x)^2\psi_n(x)^{-2}\mu^c_{\langle\varphi_n\rangle}(\mathrm{d}x)\int_E\rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x),\end{aligned}$$

and

$$\left(\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))(\varphi_n(x)-\varphi_n(y))\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)^2$$
$$\leq \iint_{0<d(x,y)<F(x,y)} (\varphi_n(x)-\varphi_n(y))^2\psi_n(x)^{-2}\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)$$
$$\times \iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y).$$

We thus have

$$\left(\int_E (u_t(x)-f(x))\varphi_n(x)\rho(x)\mathfrak{m}(\mathrm{d}x)\right)^2$$
$$\leq t\left(\int_E \rho(x)^2\psi_n(x)^{-2}\mu^c_{\langle\varphi_n\rangle}(\mathrm{d}x)\right)\left(\frac{1}{2}\int_0^t\left(\int_E \rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)\right)\mathrm{d}s\right)$$
$$+t\left(\int_E (\varphi_n(x)-\varphi_n(y))^2\psi_n(x)^{-2}\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)$$
$$\times\left(\frac{1}{2}\int_0^t\left(\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)\mathrm{d}s\right)$$
$$\leq t\left(\int_E \rho(x)^2\psi_n(x)^{-2}\mu^c_{\langle\varphi_n\rangle}(\mathrm{d}x)+\int_E (\varphi_n(x)-\varphi_n(y))^2\psi_n(x)^{-2}\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)$$
$$\times\left(\frac{1}{2}\int_0^t\left(\int_E \rho(x)^2\psi_n(x)^2\mu^c_{\langle u_s\rangle}(\mathrm{d}x)\right)\mathrm{d}s\right.$$
$$\left.+\frac{1}{2}\int_0^t\left(\iint_{0<d(x,y)<F(x,y)} (u_s(x)-u_s(y))^2\psi_n(x)^2\rho(x)\rho(y)J(\mathrm{d}x\mathrm{d}y)\right)\mathrm{d}s\right).$$

If we take $a=a_n$, then by Lemmas 4 and 5 that the last expression is less than

$$2te^{-na_n}M_\xi(n+3)\mathfrak{m}(K_\xi(n+3))\exp(a_n^2e^{a_np_n}M_\xi(n+1)t)\|\rho\psi_n\|^2_{L^2(E;\mathfrak{m})}. \tag{2.10}$$

Moreover, we can take a subsequence $\{n_k\}$ such that the expression in (2.10) goes to 0 as $k\to\infty$. Namely, we have

$$\lim_{k\to\infty}\int_X (u_t(x)-f(x))\,\varphi_{n_k}(x)\rho\mathfrak{m}(\mathrm{d}x)=0 \quad \text{for any } t\in(0,T].$$

This equality and Lemma 1 complete the proof. ∎

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