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A MULTIVALUED VERSION OF THE RADON-NIKODÝM THEOREM, VIA THE SINGLE-VALUED GOULD INTEGRAL

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ABSTRACT. In this paper we consider a Gould type integral of real functions with respect to a compact and convex valued not necessarily additive measure. In particular we will introduce the concept of integrable multimeasure and, thanks to this notion, we will establish an exact Radon-Nikodým theorem relative to a fuzzy multisubmeasure which is new also in the finite dimensional case. Some results concerning the Gould integral are also obtained.

Key words and phrases: Set valued Radon-Nikodým theorem; Non-additive measure; Gould integrability.

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1. INTRODUCTION

Non-additive measures are an important field of research in measure theory. Due to its applications in economics, statistics, human decision making and medicine, the field of non-additive measures and of fuzzy measures has been intensively studied in the last years, while the theory of monotonicity is used in statistics, mathematical economy, game theory, probability and artificial intelligence (see for example [16, 29, 35, 39, 10]). In [36] Pap has recently studied multivalued integration, examining in particular the Gould integrability for multifunctions and multisubmeasures. The present research could be connected to his paper as a continuation.
Concerning with the theory of integration, the existence of a Radon-Nikodým derivative is an important tool. In fact it provides conditions for the existence of a certain integral representation of measures. The Radon-Nikodým theorem is used, for example, for converting actual probabilities into those of the risk neutral probabilities. Moreover it was approached by many authors in several different settings (e.g. [32, 30, 27, 28, 25]). In particular, in [12] an outline of the previous results is presented, together with quotations of the papers in this topic which have appeared since the late 60's. Similar problems were studied afterwards, e.g. in [33, 38] as an extension of [24, 34], later in [4, 5], and also recently deeply examined in [14, 18] both in the countably and the finitely additive case using different notions of integrals. Here we will undertake a similar investigation and we will consider fuzzy multisubmeasures defined on an algebra and taking compact and convex values in an arbitrary Banach space X.
In this paper essentially a Radon-Nikodým theorem is established, in order to represent a set-valued additive measure as the Gould type integral of a suitable real-valued function with respect to a fixed fuzzy multisubmeasure. We point out that this result is new also in the finite-dimensional case since additivity is requested only for one of the set valued measures involved.
The paper is organized as follows: in Section 2 some basic notions and results are given, while in section 3 some results and examples regarding Gould type integrability relative to a not necessarily additive measure m are obtained (see also [7, 36]). The target space for m is the Banach lattice of all real-valued continuous functions defined in a compact, Stonian space Ω (the space $C(\Omega)$). This is due to the fact that Banach lattices are often good models for applications, and also for studying set-valued measures or functions. In effect, thanks to the Rådström embedding Theorem, many important hyperspaces can be embedded in $C(\Omega)$ (for example the family of compact and convex subsets of a Banach space X, while for an exhaustive list of such hyperspaces see e.g. [31] and, for various applications in Banach lattices, [1, 40, 41, 8, 9, 13]). In section 4 a Radon-Nikodým type theorem will be obtained (Theorem 4.12) using the set-valued integral defined in [37, 36]. According to this result, a multimeasure Γ can be expressed as a Gould type set-valued integral of a function f with respect to a fuzzy multisubmeasure M, that is: $\Gamma(E) = \int_E f dM$, for every $E \in \mathscr{A}$, under a suitable exhaustion condition and the strong absolute continuity of Γ with respect to M. In this case, the construction of the Radon-Nikodým derivative makes use of the mentioned notion of exhaustion, introduced by Maynard [34] in the scalar case and extended by other authors to the vector-valued case: [24, 33, 38]. As an application of the Theorem 4.12 an integration by substitution theorem is obtained for fuzzy multimeasures.

2. BASIC FACTS AND DEFINITIONS

Unless stated otherwise, throughout this paper T is an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T, $\mathscr{A}$ an algebra of subsets of T and $\mu : \mathscr{A} \to [0, +\infty)$ an arbitrary set function, with $\mu(\emptyset) = 0$. A *partition* of T is a finite family of nonvoid sets $P = \{A_i\}_{i=1}^n \subset \mathscr{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$, and $\bigcup_{i=1}^n A_i = T$. Let $P = \{A_i\}_{i=1}^n$ and $P' = \{B_j\}_{j=1}^q$ be two partitions of T. The partition P' is said to be *finer than* P, denoted by $P \leq P'$ (or, $P' \geq P$), if for every

$j \in \{1,\dots,q\}$, there exists $i_j \in \{1,\dots,n\}$ so that $B_j \subseteq A_{i_j}$. The *common refinement* of two partitions $P = \{A_i\}_{i=1}^n$ and $P' = \{B_j\}_{j=1}^q$ is the partition $P \vee P' = \{A_i \cap B_j\}_{i\in\{1,\dots,n\},j\in\{1,\dots,q\}}$. Obviously, $P \vee P' \geq P$ and $P \vee P' \geq P'$. The class of all partitions of T will be denoted by $\mathscr{P}$, and if $A \in \mathscr{A}$ is fixed, $\mathscr{P}_A$ denotes the class of all partitions of A. Given μ, we will consider $\overline{\mu}, \mu^* : \mathcal{P}(T) \to [0,+\infty]$, *the variation, semivariation* of μ respectively and $\widetilde{\mu} : \mathcal{P}(T) \to [0,+\infty]$, given by $\widetilde{\mu}(E) = \inf\{\overline{\mu}(A); E \subseteq A, A \in \mathscr{A}\}$. For the properties of variation, semivariation and $\widetilde{\mu}$ see for example [17].
Let $(X, \|\cdot\|)$ be a Banach space, B_X its unit ball; the symbol m will be used for vector-valued set functions. For a vector measure $m : \mathscr{A} \to X$, its semivariation $m^* : \mathcal{P}(T) \to [0,+\infty]$ is defined by: $m^*(E) = \sup\{\|m(A)\|; A \in \mathscr{A}, A \subseteq E\}$. In an analogous way we can define the *variation* $\overline{m} := \overline{\|m\|}$. Thus, if $A \in \mathscr{A}$, then $\|m(A)\| \leq \overline{m}(A)$, which implies that $m^*(E) \leq \overline{m}(E)$, for every $E \in \mathcal{P}(T)$.
Let $ck(X)$ be the family of all nonempty compact and convex subsets of a real Banach space $(X, \|\cdot\|)$. By the symbol $+$ the Minkowski addition will be indicated. Let h be the Hausdorff metric on $ck(X)$. It is well-known that $(ck(X), h)$ is a complete metric space (see for example [15, Theorem II-14]). Finally, for any bounded set A, $|A|_h$ denotes the distance $h(A, \{0\})$, where 0 is the origin of X. With the symbol M we denote a set function with values in $ck(X)$. Now, several notions are recalled for further use.

Definition 2.1. ([24], [32, Definition 3.2]) Let $\mu : \mathscr{A} \to [0,\infty]$ be finitely additive.

- **2.1.a):** A finite or countable family of pairwise disjoint sets $(E_i)_{i\in I} \subset \mathscr{A}$ will be called a μ- *exhaustion* of $E \in \mathscr{A}$ if $\mu(E_i) > 0$ for every $i \in I$ and for each $\varepsilon > 0$, there is $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that $\mu(E \setminus \bigcup_{i=1}^{n_0} E_i) < \varepsilon$.
- **2.1.b):** A set property $\mathfrak{P}$ is said to be μ*-exhaustive* on $E \in \mathscr{A}$ if there exists a μ-exhaustion $(E_i)_i$ of E, such that every E_i has $\mathfrak{P}$.
- **2.1.c):** A set property $\mathfrak{P}$ is called μ*-null difference* if whenever $A, B \in \mathscr{A}$ with $\mu(A) > 0$ and $\mu(B) > 0$, from $\mu(A \triangle B) = 0$, it follows that either A and B both have $\mathfrak{P}$ or neither does.
- **2.1.d):** A property $\mathfrak{P}$ about the points of T *holds* $\widetilde{\mu}$*-almost everywhere* (denoted $\widetilde{\mu}$*-a.e.*) if there exists $A \in \mathcal{P}(T)$ so that $\widetilde{\mu}(A) = 0$ and $\mathfrak{P}$ holds on $T \setminus A$.

For an arbitrary real function $f : T \to \mathbb{R}$, the symbol $\sigma_m(f,P)$ (or, if there is no doubt, $\sigma(f,P)$, $\sigma_m(P)$ or $\sigma(P)$) denotes the sum $\sum_{i=1}^n f(t_i)m(A_i)$, for every partition of T, $P = \{A_i\}_{i=1}^n$ and every $t_i \in A_i, i \in \{1,\dots,n\}$. With the same meaning we define $\sigma_M(f,P)$, for non-negative f and $ck(X)$-valued M.

3. GOULD INTEGRAL

We now introduce the definition of Gould integrability. The Gould integral was defined in [22] for real functions with respect to a finitely additive vector measure taking values in a Banach space. Different generalizations and topics were introduced and studied in [17, 19, 20, 37, 7, 36].
Moreover, since we want to study and consider mainly the multivalued case (i.e. set functions taking values in some space of bounded convex sets) we focus our attention on the Banach space $(C(\Omega), \|\cdot\|_\infty)$. This is due to the fact that, thanks to the Rådström Embedding Theorem, many important hyperspaces can be embedded in $C(\Omega)$ (for a list of such hyperspaces see e.g. [31]). We remember moreover that $C(\Omega)$ is also a Banach lattice in which the symbol $|\cdot|$ denotes the modulus. So, rather than considering a general Banach space $(X, \|\cdot\|)$, or a Banach lattice, from now on we restrict ourselves to $C(\Omega)$ and to mappings $m : \mathscr{A} \to C(\Omega)$, with Ω compact,

Hausdorff and we can give the notion of subadditivity for $C(\Omega)$-valued set functions in the usual way: $m(\emptyset) = 0$ and $m(A \cup B) \le m(A) + m(B)$ holds, when $A, B \in \mathscr{A}$, $A \cap B = \emptyset$

Definition 3.1. A real function $f : T \to \mathbb{R}$ is said to be

3.1.a) (*Gould*) *m integrable on T* if the net $(\sigma(P))_{P\in(\mathscr{P},\le)}$ is convergent in $C(\Omega)$, where $\mathscr{P}$ is ordered by the relation $\le$. If $(\sigma(P))_{P\in(\mathscr{P},\le)}$ is convergent, then its limit is called *the Gould integral of f on T with respect to m*, denoted by $(G)\int_T f dm$ (shortly $\int_T f dm$).
3.1.b) *m integrable on* $B \in \mathscr{A}$ if the restriction $f|_B$ of f to B is m integrable on $(B, \mathscr{A}_B, m_B)$.

Remark 3.1. Thus f is m integrable on T if and only if there exists $g \in C(\Omega)$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=1}^n$, with $P \ge P_\varepsilon$ and every choice of points $t_i \in A_i, i \in \{1,\dots,n\}$, one has $\|\sigma(P) - g\|_\infty < \varepsilon$. Moreover if f_1, f_2 are m integrable and α is any real constant, then αf_1 is m integrable, $f_1 + f_2$ is m integrable, and the integral is linear.

Proposition 3.1. *Let $f : T \to \mathbb{R}$ be any Gould integrable mapping with respect to m. Then, if A is any fixed element of $\mathscr{A}$, the mapping $f\mathbf{1}_A$ is integrable too.*

Proof. Given $\mathscr{P}_A$, it is not difficult to prove that the sums $\{\sigma(f,P) : P \in \mathscr{P}_A\}$ satisfy a Cauchy principle in $C(\Omega)$; since this space is complete with respect to its norm, the assertion follows. ∎

Example 3.1. Some examples of Gould integrable functions with respect to m are given here:

3.1.a) Let T be a finite set, $\mathscr{A} = \mathcal{P}(T)$, $m : \mathscr{A} \to C(\Omega)$ and $f : T \to \mathbb{R}$ be arbitrary. Then f is Gould m integrable and $\int_T f dm = \sum\limits_{t\in T} f(t)m(\{t\})$.
3.1.b) If $m : \mathscr{A} \to C(\Omega)$ is finitely additive and $f : T \to \mathbb{R}$ is simple, $f = \sum_{i=1}^n a_i 1_{A_i}$, then f is Gould m integrable and $\int_T f dm = \sum_{i=1}^n a_i m(A_i)$.

Moreover the previous example 3.1.b) can be improved as follows.

Proposition 3.2. *Let $\mathscr{A}$ be a σ-algebra, and $m : \mathscr{A} \to C(\Omega)$ be finitely additive, and assume that $(A_n)_{n\in\mathbb{N}}$ is a countable family of pairwise disjoint elements of $\mathscr{A}$, such that $\lim_n \overline{m}(\cup_{j>n} A_j) = 0$. Then, the function $f : T \to \mathbb{R}$ defined as $f = \sum_n c_n 1_{A_n}$ is Gould integrable as soon as the sequence $(c_n)_n$ is bounded in $\mathbb{R}$; in this case, $\int_T f dm = \sum_n c_n m(A_n)$.*

Proof. Under these assumptions, it is clear that the real-valued series $\sum_n |c_n| \overline{m}(A_n)$ is convergent, hence the series $\sum_n c_n m(A_n)$ is convergent in $C(\Omega)$. We will show that f is integrable and its integral coincides with $\sum_n c_n m(A_n)$. Define now $S := \bigcup_n A_n$, and fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\overline{m}(\cup_{j>N} A_j) < \varepsilon$. Therefore $\sum_{j>N} |c_j| \overline{m}(A_j) \le K\varepsilon$ where K is any bound for $|c_n|$, $n \in \mathbb{N}$. Now set $F := \bigcup_{j\le N} A_j$, and choose any partition P of T, finer than $\{F, S\setminus F, T\setminus S\}$. Setting $P = \{(B_i,t_i), i = 1..,k\}$, one then has $\sigma(f,P) := \sum_{i=1}^k f(t_i)m(B_i) = \sum_{i\in I_1} f(t_i)m(B_i) + \sum_{i\in I_2} f(t_i)m(B_i)$, where $I_1 = \{i : B_i \subset F\}$, $I_2 = \{i : B_i \subset S\setminus F\}$.
Of course $\sum_{i\in I_1} f(t_i)m(B_i) = \sum_{j=1}^N c_j m(A_j)$ and $\|\sum_{i\in I_2} f(t_i)m(B_i)\|_\infty \le K\overline{m}(\cup_{j>N} A_j)) \le K\varepsilon$.
So, $\|\sigma(f,P) - \sum_n c_n m(A_n)\|_\infty \le \|\sum_{i\in I_2} c_j m(A_j)\|_\infty + \sum_{j>N} |c_j| \overline{m}(A_j) \le 2K\varepsilon$. This concludes the proof. ∎

For more general functions, proceeding as in the proof of [6, theorem 1.4] and [11, Proposition 6], one can deduce the following proposition and the subsequent corollary. In this situation the absolute value replaces the norm of $C(\Omega)$ and u denotes an order unit.

Proposition 3.3. *Let $f : T \to \mathbb{R}$ be any integrable function. Then there exists a sequence $(\Pi_n)_n$ of partitions such that, for every n it is $\sum_{E\in\Pi_n} Ob(f,E) \leq \frac{u}{n}$, where*

$$Ob(f,E) = \sup_{\Pi'_E,\Pi''_E} \left\{ \left| \sum_{F''\in\Pi''_E} f(t)m(F'') - \sum_{F'\in\Pi'_E} f(s)m(F') \right|, \forall\, t\in F'', s\in F' \right\},$$

and Π'_E, Π''_E run along all partitions of E.

Proof. First observe that, thanks to the Cauchy criterion, a sequence $(\Pi_n)_n$ of partitions exists, such that, for every integer n

$$\left| \sum_{F'\in\Pi'} f(s)m(F') - \sum_{F''\in\Pi''} f(t)m(F'') \right| \leq \frac{u}{n} \tag{3.1}$$

(with obvious meaning of symbols) holds, for all partitions Π', Π'' finer than Π_n. Now, take any integer n and, for each element E of Π_n, consider two arbitrary partitions Π'_E and Π''_E of E. Then, taking the *union* of the partitions Π'_E as E varies, and making the same operation with the partitions Π''_E, two partitions of T are obtained, finer than Π_n, for which (3.1) holds true. From (3.1), obviously it follows

$$\sum_{F'\in\Pi'} f(s)m(F') - \sum_{F''\in\Pi''} f(t)m(F'') \leq \frac{u}{n}. \tag{3.2}$$

Now, let E_1 be the first element of Π_n. In the summation at left-hand side, fix all the $F's$ and the $F''s$ that are not contained in E_1. Taking the supremum when the remaining $F's$ and $F''s$ vary in all possible ways, it follows

$$\sup_{\Pi'_{E_1}} \sigma(f,\Pi'_{E_1}) - \inf_{\Pi''_{E_1}} \sigma(f,\Pi''_{E_1}) + \sum_{\substack{F'\in\Pi',\\ F'\not\subset E_1}} f(s)m(F') - \sum_{\substack{F''\in\Pi'',\\ F''\not\subset E_1}} f(t)m(F'') \leq \frac{u}{n},$$

namely

$$Ob(f,E_1) + \sum_{\substack{F'\in\Pi',\\ F'\not\subset E_1}} f(s)m(F') - \sum_{\substack{F''\in\Pi'',\\ F''\not\subset E_1}} f(t)m(F'') \leq \frac{u}{n}.$$

In the same way , fixing all the F's and F''s that are not contained in the second subset of Π, (say E_2), and making the same operation, it follows

$$Ob(f,E_1) + Ob(f,E_2) + \sum_{\substack{F'\in\Pi',\\ F'\not\subset E_1\cup E_2}} f(s)m(F') - \sum_{\substack{F''\in\Pi'',\\ F''\not\subset E_1\cup E_2}} f(t)m(F'') \leq \frac{u}{n}$$

Now, it is clear how to deduce the assertion. ∎

Concerning the previous Proposition, we remark that, unless the space X is finite-dimensional, a similar conclusion fails to hold if the absolute value is replaced by the norm: this is noteworthy if one considers its consequences, in particular the corollary 3.6.

The following result states an easy consequence of Proposition 3.3 and it can be viewed as a Henstock Lemma result.

Corollary 3.4. *Let $g : T \to \mathbb{R}$ be any mapping, then g is Gould integrable if and only if there exists a sequence $(\Pi_n)_n$ of partitions, such that, for every n and every partition Π finer than Π_n*

3.4.1): $\sum_{E\in\Pi} |g(\tau_E)m(E) - \int_E g dm| \leq \frac{u}{n}$, *where τ_E is any point in the set E.*

Proof. The "if part" is a consequence of Proposition 3.3. ∎

Now we want to focus our attention on a particular type of set valued mappings m which will be useful in the last section, i.e. the *Gould integrable* ones. A similar notion was also given in [6, Definition 1.1], though for set functions taking values more generally in a vector lattice. Notice that we will use the symbol $\mathbf{1}$ to denote the real-valued function on T, defined by $\mathbf{1}(t) \equiv 1$, while the symbol u denotes the element of $C(\Omega)$ constantly equal to 1. We remember also that it is well-known that the norm $\|\cdot\|_\infty$ coincides with the *unit* norm $\|\cdot\|_u$.

Definition 3.2. Given a mapping $m : \mathscr{A} \to C(\Omega)$, such that $m(\emptyset) = 0$, m is said to be *Gould integrable* if the mapping $\mathbf{1} : T \to \mathbb{R}$ is Gould integrable with respect to m. We denote by $\upsilon_m(T) := \int_T \mathbf{1} dm$ its integral.

By Proposition 3.1, if m is Gould integrable, then m is integrable in every measurable set $A \subset T$. Moreover, denoting by $\upsilon_m(A)$ the integral of m in A, the mapping $A \mapsto \upsilon_m(A)$ is finitely additive, as will be proved in the Proposition 3.5. In other words, m is Gould integrable if and only if there exists $\upsilon_m : \mathscr{A} \to C(\Omega)$ such that, for every set $A \in \mathscr{A}$ and for every $\varepsilon > 0$ a partition $P \in \mathscr{P}$ can be found, such that $\|\sum_{I \in P'} m(I \cap A) - \upsilon_m(A)\|_\infty \leq \varepsilon$ holds, as soon as P' is finer than P. When this is the case, then υ_m is called the *integral function* of m.

Examples of non-additive set functions that are Gould integrable could be the following:

Example 3.2. Let $T = [0,1]$ endowed with the usual Borel σ-algebra Σ and Lebesgue measure λ.

3.2.a) Let $m(A) = \lambda^2(A) \cdot u$ for every $A \in \Sigma$. Clearly m is not additive (it is superadditive), but it has null integral: indeed, for any $\varepsilon > 0$ take any partition P of $[0,1]$ consisting of pairwise disjoint measurable sets A_i, each with measure less than ε. Then $\|\sum_i m(A_i)\|_\infty = \sum_i \lambda(A_i)^2 \leq \sum_i \lambda(A_i)\varepsilon = \varepsilon$. Of course, the same happens for every partition finer than P.

3.2.b) Let $\gamma(A) = (\lambda(A) - \lambda^2(A)) \cdot u$, then γ is non-additive (it is subadditive) and integrable too.

3.2.c) Let X be any finite-dimensional Banach space. Let $(W_t)_t$ denote the standard scalar Brownian motion, $t \in [0, T^\star]$, and set $(B_t)_t = (W_t B_X)_t$, $t \in [0, T^\star]$. This clearly defines a set-valued process. If U denotes the Rådström embedding of the family of compact and convex subsets of X into $C(\Omega)$, as we will recall in Theorem 4.2, then $U(B_X) = u$, where u is the element of $C(\Omega)$ constantly equal to 1. Therefore, $t \mapsto W_t u$ defines a $C(\Omega)$-valued process. Now, let $\mathscr{A}$ be the algebra in $[0, T^\star]$ generated by all (half-open) subintervals, and define $m : \mathscr{A} \to C(\Omega)$ in the following way:

$$m(A) = \begin{cases} (W_b - W_a)^2 \cdot u & A =]a,b] \\ \sum_i (W_{b_i} - W_{a_i})^2 \cdot u & A \text{ is the finite union of (maximal) disjoint intervals} \\ &]a_i, b_i]. \end{cases}$$

Then, for any partition P of $[0, T^\star]$, $P := \{I_1, ..., I_k\}$ into pairwise disjoint elements of $\mathscr{A}$, define $S(P) = \sum_{j=1}^k m(I_j)$ and observe that, thanks to well-known properties of the Brownian motion, this quantity tends to $T^\star \cdot u$ in L^2 when the maximum length of the partitions tends to 0. Therefore, at least for this type of convergence, the measure m has integral $T^\star \cdot u$, and in every interval $[a,b] \subset [0, T^\star]$ the integral is $(b-a) \cdot u$.

Proposition 3.5. *If m is Gould integrable then its integral function υ_m, defined in $\mathscr{A}$ as $\upsilon_m(A) = \int_T \mathbf{1}_A \, dm$, is additive.*

Proof. It follows immediately from the Remark 3.1. ∎

So the Gould integrability of m allows to link m with υ_m which is an additive set function and clearly, m is additive if and only if it is integrable and $m = \upsilon_m$. Moreover, for bounded functions, the following characterization can be given:

Corollary 3.6. *Assume that m is integrable. Then a bounded function $f : T \to \mathbb{R}$ is Gould integrable with respect to m if and only if it is with respect to υ_m, and the two integrals agree.*

Proof. Assume that f is integrable with respect to m, and denote by K any majorant for $|f|$. Now, fix arbitrarily $\varepsilon > 0$: correspondingly, there exists a partition P_1 such that

$$\left\|\sum_{I\in P} f(t_I)m(I) - \int_T f dm\right\|_\infty \leq \varepsilon \qquad i.e. \qquad \left|\sum_{I\in P} f(t_I)m(I) - \int_T f dm\right| \leq \varepsilon u$$

holds, for every partition P, finer than P_1. Let n be such that $1/n \leq \varepsilon$, by the Corollary 3.4, for $g = 1$, there exists also a partition P_2 such that $\sum_{E\in\Pi} |m(E) - \upsilon_m(E)| \leq \varepsilon u$ holds, for every partition Π finer than P_2. So, if P is any partition finer than $P_1 \vee P_2$, one gets

$$\left|\sum_I f(t_I)\upsilon_m(I) - \int_T f dm\right| \leq \left|\sum_I [f(t_I)\upsilon_m(I) - f(t_I)m(I)]\right| + \left|\sum_I f(t_I)m(I) - \int_T f dm\right| \leq$$
$$\leq \sum_I |f(t_I)\,(\upsilon_m(I) - m(I))| + \varepsilon u \leq K\sum_I |m(I) - \upsilon_m(I)| + \varepsilon u \leq (1+K)\varepsilon u.$$

So $\|\sum_I f(t_I)\upsilon_m(I) - \int_T f dm\|_\infty \leq (1+K)\varepsilon$. This clearly suffices to conclude that f is integrable with respect to υ_m and the two integrals agree. A similar argument can be used to prove also the reverse implication. Hence the proof is finished. ∎

We remark that, for bounded functions, the Corollary 3.6 allows to deal with the non-additive case by means of the additive one, similarly as the Stone estension Theorem which connects $L^1(m)$, when m is finitely additive, with $L^1(\nu)$, where ν is the countably additive *transform* of m.

We can observe that the results obtained in Propositions 3.1, 3.2 and 3.5 are still valid in an arbitrary Banach space and not only in $C(\Omega)$ and we remember also that notions of order-type integrals have also been investigated, for functions taking their values in ordered vector spaces, and in Banach lattices: see for example [20, 11, 2].

4. A RADON-NIKODÝM TYPE THEOREM

This section deals with a Radon-Nikodým type theorem for multimeasures using the notion of exhaustion, following a method of Maynard [34, 24, 33, 38]. We recall that, given A and B nonempty sets in X, the Hausdorff distance h is defined by $h(A,B) = \max\{e(A,B), e(B,A)\}$, where the *excess* $e(A,B)$ is defined as $e(A,B) := \sup_{a\in A} d(a,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|$. In particular

Remark 4.1. If $A \subset B$, then $e(A,B) = 0$ and $h(A,B) = e(B,A)$. Moreover, observe that for any nonempty bounded set $A \subset X$, and any pair t,s of elements of X, $|d(t,A) - d(s,A)| \leq \|s-t\|$. Indeed, fix arbitrarily $\varepsilon > 0$, then there exists $a \in A$ such that $d(t,A) \geq \|t-a\| - \varepsilon \geq \|s-a\| - \|s-t\| - \varepsilon \geq d(s,A) - \varepsilon - \|s-t\|$. By the arbitrariness of ε, it follows that $d(t,A) - d(s,A) \geq -\|s-t\|$, i.e. $d(s,A) - d(t,A) \leq \|s-t\|$. Exchanging the roles between t and s, one obtains $d(t,A) - d(s,A) \leq \|s-t\|$ and therefore $|d(t,A) - d(s,A)| \leq \|s-t\|$. Another useful fact is the following: for every pair of bounded subsets $A,B \subset X$, $e(B,A) = e(cl(B),A)$. Of course, since $B \subset cl(B)$, it is clear that $e(B,A) \leq e(cl(B),A)$.
Viceversa, fix $\varepsilon > 0$: then $j \in cl(B)$ exists, such that $d(j,A) \geq e(cl(B),A) - \varepsilon/2$. Now, let $b \in B$ be such that $\|b-j\| \leq \varepsilon/2$: then $|d(b,A) - d(j,A)| \leq \varepsilon/2$, and so $e(B,A) \geq d(b,A) \geq$

$d(j,A) - \varepsilon/2 \geq e(cl(B),A) - \varepsilon$. By the arbitrariness of ε, this gives the reverse inequality and the proof is complete.

By [26, Proposition 1.19 Chapter 7] we have that

Proposition 4.1. *Let $(A_n)_n$ be any increasing sequence of compact and convex subsets of X, and assume that a compact and convex set K exists, such that $A_n \subset K$ for all n. Then $\lim_n h(A_n, J) = 0$, where $J := cl(\bigcup_n A_n)$.*

For set valued functions we recall from the following concept:

Definition 4.1. A set function $M : \mathscr{A} \to ck(X)$ is said to be a *multisubmeasure* if: $M(\emptyset) = \{0\}$ and $M(A \cup B) \subset M(A) + M(B)$, for every $A, B \in \mathscr{A}$, with $A \cap B = \emptyset$. M is said to be a *fuzzy multisubmeasure* if moreover: $M(A) \subset M(B)$, for every $A, B \in \mathscr{A}$, with $A \subset B$ (that is, M is *monotone* on $\mathscr{A}$). If $M(A \cup B) = M(A) + M(B)$, for every $A, B \in \mathscr{A}$, with $A \cap B = \emptyset$ then M is said to be a *multimeasure* that is, M is finitely additive.

Examples of fuzzy multisubmeasures M are given in [20], moreover we can consider also $M(A) = [0, \lambda(A) - \lambda^2(A)] \cdot u$, where λ and u are as given in Example 3.2.

Definition 4.2. Let $M : \mathscr{A} \to ck(X)$ be a multivalued set function, with $M(\emptyset) = \{0\}$. Consider the following set functions associated to M:

4.2.a) $|M(\cdot)|_h$ defined by $|M(A)|_h = h(M(A), \{0\}) = \sup\{\|x\| : x \in M(A)\}$ for every $A \in \mathscr{A}$.

4.2.b) $v_M(\cdot)$ defined by $v_M(A) = \sup\{\sum_{i=1}^{n} |M(E_i)|_h\}$, for every $A \in \mathscr{A}$, where the supremum is extended over all finite partitions $\{E_i\}_{i=1}^n$ of A. $v_M(\cdot)$ is said to be *the variation* of M. The multivalued set function M is said to be *of finite variation* if $v_M(T) < \infty$.

In the sequel, let $M : \mathscr{A} \to ck(X)$ be a fuzzy multisubmeasure and f a non negative real-valued function. Let $\sigma(P) = \sigma_{f,M}(P) = \sum_{i=1}^n f(t_i)M(A_i)$, for every partition $P = \{A_i\}_{i=1,\dots,n}$ of T and every $t_i \in A_i, i = 1, \dots, n$. Then

Definition 4.3. A function f is said to be *M-integrable (on T)* if the net $(\sigma(P))_{P \in (\mathscr{P}, \leq)}$ is convergent in $(ck(X), h)$, where $\mathscr{P}$ is the set of all partitions of T and $\leq$ is the order relation on $\mathscr{P}$ given in Definition 3.1.a). Its limit is called *the integral of f on T* with respect to the fuzzy multisubmeasure M and is denoted by $\int_T f dM$.
If $B \in \mathscr{A}$, then f is said to be *M-integrable on B* if the restriction $f|_B$ of f to B is M integrable on $(B, \mathscr{A}_B, M_B)$.

In other words, f is M integrable in T if there exists an element $J \in ck(X)$, such that for every $\varepsilon > 0$ there exists a partition $P \in \mathscr{P}$ with the property that $h(\sigma(P'), J) \leq \varepsilon$ holds true, for every partition P' finer than P.

As well highlighted in [31] the space $ck(X)$ is a sub-near vector lattice of $cwk(X)$ (nonempty, weakly compact and convex subsets of X) with respect to the operations of additions and multiplication by positive scalars and to order induced by $cwk(X)$; moreover if X is not finite dimensional, this hyperspace can be considered as a subset of $S_1 = cbf(X)$ (nonempty, convex, closed, bounded subset of X) and it can be embedded, using the structure of S_1, provided that $u = B_X$, $\vec{0} = \{0\}$, in such a way that the norm of the embedding space is a Riesz norm. So, using Kakutani's M-space representation theorem, the near vector lattice $ck(X)$ with order units, endowed with the Hausdorff metric can be represented in terms of $C(\Omega)$ spaces, as shown in:

Theorem 4.2. ([31, Theorem 5.7]). *Let X be any Banach space. Then there exist a compact, Stonian, Hausdorff space Ω and an isometry $U : ck(X) \to C(\Omega)$ such that*

4.2.1) $U(\alpha A+\beta C)=\alpha U(A)+\beta U(C)$ *for all* $\alpha,\beta\in]0,\infty[$ *and* $A,C\in ck(X)$.
4.2.2) $h(A,C)=\|U(A)-U(C)\|_\infty$ *for all* $A,C\in ck(X)$.
4.2.3) $U(ck(X))$ *is norm-closed in* $C(\Omega)$.
4.2.4) $U(\overline{co}(A\cup C))=\max\{U(A),U(C)\}$, *for all* $A,C\in ck(X)$.

Observe now that the embedding theorem can be used in order to replace the multivalued integral above with a single-valued one, at least for *positive* integrands f. This leads to the following

Definition 4.4. Define $U_M:\mathscr{A}\to C(\Omega)$ as $U_M(E)=U(M(E))$ for all $E\in\mathscr{A}$. The mapping U_M will be called the *embedded* mapping of M. Moreover, thanks to **4.2.4)**, the embedded mapping U_M is a fuzzy submeasure if M is a fuzzy multisubmeasure.

Thanks to the Theorem 4.2, it is clear that $\|U_M(E)\|_\infty=|M(E)|_h$ for every $E\in\mathscr{A}$, and so M is of bounded variation if and only if U_M is, as a $C(\Omega)$-valued set function.

Since we can consider also Gould integrability with respect to U_M (according to Definition 3.1) for mappings $f:T\to\mathbb{R}_0^+$, then the following result holds:

Theorem 4.3. *A function f is M integrable, if and only if it is Gould integrable with respect to U_M. Moreover if we denote by J and j the M integral and the U_M integral of f respectively, then $U(J)=j$. Finally, in these cases, $f\mathbf{1}_A$ is integrable for every $A\in\mathscr{A}$.*

Proof. First, assume that f is Gould integrable with respect to U_M, and denote by j its integral. This means that the filtering net $(U(\sigma_M(f,P))_{P\in\mathscr{P}}$ is convergent to j. Hence it is Cauchy in $C(\Omega)$. Then, also the net $(\sigma_M(f,P))_{P\in\mathscr{P}}$ is Cauchy in $ck(X)$: by completeness of this space, $(\sigma_M(f,P))_{P\in\mathscr{P}}$ has limit J in $ck(X)$. By continuity of U, it is then clear that $U(J)=j$.
A similar argument can be used to prove the converse implication. So to conclude the proof it only remains to deduce integrability of f in every subset $A\in\mathscr{A}$, and this is a consequence of integrability of f with respect to U_M: indeed, fixing any subset $A\in\mathscr{A}$ and any positive ε in $\mathbb{R}$, a partition P exists, finer than $\{A,T\setminus A\}$, such that $\|\sigma_{U_M}(f,P')-\sigma_{U_M}(f,P'')\|_\infty\le\varepsilon$ holds, for all partitions P' and P'' finer than P. So, choosing two partitions of A, say Π'_A and Π''_A, both finer than P_A (i.e. P restricted to A), and *extending* them to A^c with a unique partition finer than P_{A^c}, then two partitions, P' and P'', can be found, both finer than P, and coincident in the set A^c: these partitions satisfy $\varepsilon>\|\sigma_{U_M}(f,P')-\sigma_{U_M}(f,P'')\|_\infty=\|\sigma_{U_M}(f,\Pi'_A)-\sigma_{U_M}(f,\Pi''_A)\|_\infty$. By the completeness of $C(\Omega)$, this is enough to deduce integrability of $f\mathbf{1}_A$. ∎

Following Definition 3.2, a multisubmeasure $M:\mathscr{A}\to ck(X)$, it is said to be *integrable* if the function $f(x)\equiv\mathbf{1}_A$ is M integrable for every $A\in\mathscr{A}$. Then the notation

$$M_0(A):=\int_T\mathbf{1}_A dM:=\int_A\mathbf{1}\,dM,$$

is used, for $A\in\mathscr{A}$.
This means that, for every element $A\in\mathscr{A}$ there exists an element $M_0(A)\in ck(X)$ such that, for every $\varepsilon>0$ a partition $P\in\mathscr{P}$ can be found with the property that $h(\sum_{I\in P'}M(I\cap A),M_0(A))\le\varepsilon$ holds, as soon as P' is a partition finer than P.

The following theorem states a necessary and sufficient condition for the integrability of a $ck(X)$-valued fuzzy multisubmeasure of bounded variation. The technique takes into account previous results and it is inspired by the notions of [2, Definition 3.4] and [3, Definition 3.13]. This equivalence could be also useful in order to study differential inclusions.

Theorem 4.4. *Let M be a fuzzy multisubmeasure of bounded variation. If M is $ck(X)$-valued, then M is integrable if and only if there exists a compact and convex set K such that $\sigma(1,P) := \sigma_M(1,P) \subset K$ for all partitions $P \in \mathscr{P}$. When this is the case, then*

$$\int_T M = cl\left(\bigcup \{\sigma(1,P) : P \in \mathscr{P}\}\right).$$

Proof. Sufficiency: first of all, thanks to bounded variation, all the sums $\sigma(1,P) := \sum_{I\in P} M(I)$ are compact and convex sets contained in K for all partitions P. Moreover, thanks to subadditivity, the sums above are a filtering family in $ck(X)$. In order to prove the existence of the integral, it is enough to show that the map $P \mapsto \sigma(1,P)$ is Cauchy, since $ck(X)$ is a complete space with respect to the Hausdorff distance. Assume by contradiction that the Cauchy property does not hold: then there exists a positive number ε such that, as soon as P is any partition of T, a couple (P',P'') of finer partitions exists, satisfying $h(\sigma(1,P'),\sigma(1,P'')) \geq \varepsilon$. Since the refinement order is filtering, there exists a sequence $(P_n)_n$ of partitions, increasing in the refinement order, and such that $h(\sigma(1,P_n),\sigma(1,P_{n+1})) \geq \varepsilon$ for all n. Now, since $\sigma(1,P_n)$ is an increasing sequence of elements of $ck(X)$, Proposition 4.1 applies, and the limit $\lim_n \sigma(1,P_n)$ exists, with respect to the Hausdorff distance: but this contradicts the fact that $h(\sigma(1,P_n),\sigma(1,P_{n+1})) \geq \varepsilon$ for all n. So this part of the theorem is proved.
Necessity: choose any partition P of T, $P = \{E_1,...,E_k\}$, and fix arbitrarily $\varepsilon > 0$. By integrability, there exists a partition P_ε such that, for every finer partition P' it holds $h\left(\sigma(1,P'), \int_T M\right) \leq \varepsilon$, from which $\sigma(1,P') \subset \int_T M + \varepsilon B_X$. Therefore, choosing $P' := P_\varepsilon \vee P$, and thanks to subadditivity of M, it follows $\sigma(1,P) \subset \sigma(1,P') \subset \int_T M + \varepsilon B_X$. By the arbitrariness of P, one gets

$$\bigcup_{P\in\mathscr{P}} \sigma(1,P) \subset \int_T M + \varepsilon B_X,$$

where P ranges over all the possible partitions of T. By the arbitrariness of $\varepsilon > 0$ and compactness of $\int_T M$, it is obvious that $\bigcup_{P\in\mathscr{P}} \sigma(1,P) \subset \int_T M$, and so the necessity is proven. Observe that, since $\int_T M$ is closed, the inclusion

$$cl\left(\bigcup_{P\in\mathscr{P}} \sigma(1,P)\right) \subset \int_T M$$

follows from the last formula.
In order to finish the proof, it only remains to prove the reverse inclusion, assuming that M is Gould integrable. To this aim, fix $\varepsilon > 0$ and any partition P of T such that $\int_T M \subset \sigma(1,P') + \varepsilon B_X$ holds, for all partitions P' finer than P. Then

$$\int_T M \subset cl\left(\bigcup_{P\in\mathscr{P}} \sigma(1,P)\right) + \varepsilon B_X.$$

Then, by the arbitrariness of ε, the desired conclusion follows. ∎

Now, assuming that M is integrable, the additivity of its *integral mapping* M_0 will be proven together with the equivalence between the integrability of positive functions f with respect to M and with respect to M_0, at least when f is bounded. So, at least for bounded positive mappings f, integrability with respect to M is equivalent to integrability with respect to an additive multimeasure (with the same integral).

Remark 4.2. Observe that, since the embedding theorem applies, M can be always identified with U_M, so that M can be viewed as a single-valued mapping, taking values in $C(\Omega)$ and so all the results concerning Gould integrability in Section 3 can be applied to M.

H_0**):** From now on the fuzzy multisubmeasure $M : \mathscr{A} \to ck(X)$ will always be assumed of bounded variation and satisfying the conditions in Theorem 4.4, namely there exists a compact and convex set K such that $\sigma(1,P) \subset K$ for all partitions P, so that M is integrable in $ck(X)$.

First of all, observe that, in those conditions, $\int_T \mathbf{1}_A \, dM$ exists, for every set $A \in \mathscr{A}$, essentially with the same proof of Theorem 4.4. Next,

Proposition 4.5. *The function M_0, defined in $\mathscr{A}$ as $M_0(A) = \int_T \mathbf{1}_A \, dM$, is additive.*

Proof. The result is a consequence of Proposition 3.5, and Theorem 4.3. ∎

Now, notice that, under the above conditions, the $C(\Omega)$-valued measure $U_{M_0} - U_M$ (which is non-negative, of course) has null integral. This almost immediately implies the following result.

Theorem 4.6. *Let $f : T \to [0,\infty[$ be any bounded mapping. Then f is M integrable if and only if it is M_0-integrable, and the integrals coincide.*

Proof. Assume that f is integrable with respect to M. Then it is U_M-integrable, and $U\left(\int_T f dM\right) = \int_T f dU_M$. Now, it is sufficient to apply Corollary 3.6 to deduce that f is integrable also with respect to U_{M_0} with the same integral, so $U\left(\int_T f dM\right) = \int_T f dU_M = \int_T f dU_{M_0}$, which also shows that $\int_T f dU_{M_0}$ is in the range of U. Thanks to Theorem 4.3 this is enough to conclude that f is integrable with respect to M_0 and the two integrals agree. A reverse argument also shows the opposite implication, so the proof is complete. ∎

Also integrability on measurable subsets can be deduced, in the usual manner.

Proposition 4.7. *Let $f : T \to [0,\infty[$ be any Gould integrable mapping with respect to M. Then, if A is any fixed element of $\mathscr{A}$, the mapping $f\mathbf{1}_A$ is integrable too.*

Proof. It is analogous to Proposition 3.1. ∎

Moreover

Proposition 4.8. *M, M_0 have the same variation measure.*

Proof. Fix arbitrarily $A \in \mathscr{A}$, and denote by $\mathscr{P}_A$ the family of all finite partitions of A. Then, thanks to Theorem 4.4, $M_0(A) = cl(\bigcup\{\sum_i M(B_i) : (B_i)_i \in \mathscr{P}_A\})$. In particular, $M(A) \subset M_0(A)$, and, for any partition $(B_i)_i$ in $\mathscr{P}_A$, one has $\sum_i |M(B_i)|_h \le \sum_i |M_0(B_i)|_h \le v_{M_0}(A)$. This clearly implies that $v_M(A) \le v_{M_0}(A)$.

Conversely, fix any $\varepsilon > 0$ and $A \in \mathscr{A}$. For every partition $P \equiv (B_i)_{i=1}^N \in \mathscr{P}_A$, and every index i, since M is integrable, there exists a partition $P_i \equiv (B'_{i,j})_j \in \mathscr{P}_{B_i}$ such that $|M_0(B_i)|_h \le |\sum_j M(B'_{i,j})|_h + \varepsilon/N \le \sum_j |M(B'_{i,j})|_h + \varepsilon/N$ and so $\sum_i |M_0(B_i)|_h \le \sum_i \sum_j |M(B'_{i,j})|_h + \varepsilon \le v_M(A) + \varepsilon$. By the arbitrariness of $P \in \mathscr{P}_A$ and of $\varepsilon > 0$, it follows $v_{M_0}(A) \le v_M(A)$. ∎

In the sequel, assume that $\mathscr{A}$ is a σ-algebra. Also let $\Gamma : \mathscr{A} \to ck(X)$ be any fixed fuzzy multimeasure. Following [24, 33] for $\alpha > 0$ and $E \in \mathscr{A}$, let $A_\Gamma(E,\alpha)$ be the *α-approximate range* defined by: $A_\Gamma(E,\alpha) = \{r \in [0,+\infty) : h(\Gamma(H), rM(H)) \le \alpha v_M(H), \ \forall H \in \mathscr{A} \cap E\}$.

Remark 4.3. Observe that, by Theorem 4.2, using the embedding U, it is possible to formulate the α-approximate range in the following way:

$$A_\Gamma(E,\alpha) \quad = \quad \{r \in [0,+\infty) : \|U_\Gamma(H) - rU_M(H)\|_\infty \le \alpha \overline{m}(H), \forall H \in \mathscr{A} \cap E\},$$

where $m := U_M$. In fact $h(\Gamma(H), rM(H)) = \|U_\Gamma(H) - rU_M(H)\|_\infty$ and

$$\overline{m}(H) = \sup \sum_i \|U_M(E_i)\|_\infty = \sup \sum_i |M(E_i)|_h = v_M(H).$$

Theorem 4.9. ([33, Lemma 3.3]) *Let $\Gamma \ll v_M$ (i.e. $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon) = \delta > 0$ such that for every $E \in \mathscr{A}$ with $v_M(E) < \delta$, it follows $v_\Gamma(E) < \varepsilon$). Then, for every $\alpha > 0$, the property $A_\Gamma(E, \alpha) \neq \emptyset$ is v_M-null difference.*

Theorem 4.10. *Let $\mathfrak{P}$ be a v_M-null difference property such that $\mathfrak{P}$ is v_M-exhaustive on T. Then there exists a v_M-exhaustion of T, $(B_i)_i$, such that every B_i has P and $T = \bigcup\limits_i B_i$.*

Proof. Since $\mathfrak{P}$ is v_M-exhaustive on T, there exists a v_M-exhaustion of T, denoted by $(E_i)_{i \in I}$, such that every E_i has $\mathfrak{P}$. Thus

$$\forall \varepsilon > 0, \exists\, n_0(\varepsilon) = n_0 \in \mathbb{N} \text{ such that } v_M(T \setminus \bigcup_{i=1}^{n_0} E_i) < \varepsilon. \tag{4.1}$$

Let $E_0 = T \setminus \bigcup\limits_{i \in I} E_i$. From the previous inequality it results $v_M(E_0) = 0$. Let $(B_i)_{i \in I}$ be the family of sets defined by: $B_1 = E_0 \cup E_1 \in \mathscr{A}, B_i = E_i \in \mathscr{A}$ for $i \geq 2$. Then $v_M(B_1) = v_M(E_1) > 0$ since $v_M(E_1) \leq v_M(B_1) \leq v_M(E_1) + v_M(E_0) = v_M(E_1)$ and $v_M(B_i) = v_M(E_i) > 0$, for every $i \geq 2$. Obviously, $T = \cup_{i \in I} B_i$. It is $\cup_{i=1}^{n_0} B_i = E_0 \cup (\cup_{i=1}^{n_0} E_i)$. Since $v_M(T \setminus \cup_{i=1}^{n_0} B_i) \leq v_M(T \setminus \cup_{i=1}^{n_0} E_i) < \varepsilon$, then $(B_i)_{i \in I}$ is a v_M-exhaustion of T. Now, for every $i \geq 2$, $B_i = E_i$ has $\mathfrak{P}$. So, it only remains to prove that B_1 has $\mathfrak{P}$. By the relations:

$$B_1 \triangle E_1 = (E_0 \cup E_1) \triangle E_1 = E_0 \backslash E_1 \subset E_0 \Rightarrow 0 \leq v_M(B_1 \triangle E_1) \leq v_M(E_0) = 0,$$

it follows that $v_M(B_1 \triangle E_1) = 0$. Since $\mathfrak{P}$ is v_M-null difference and E_1 has $\mathfrak{P}$, one concludes that B_1 has $\mathfrak{P}$. ∎

Definition 4.5. A multimeasure $\Gamma : \mathscr{A} \to ck(X)$ is said to be *dominated by M* if there is a constant $b > 0$ such that $|\Gamma(E)|_h \leq b v_M(E)$, for every $E \in \mathscr{A}$.

Observe that if Γ is dominated by M then $\Gamma \ll v_M$.

Lemma 4.11. (see [32, Lemma 2.9] for semivariation). *For every $E \in \mathscr{A}$ with $v_M(E) > 0$, there exists $B \in \mathscr{A} \cap E$, such that $v_M(B) < 2|M(B)|_h$.*

Proof. By contradiction, there exists an element $E \in \mathscr{A}$, with $v_M(E) > 0$ such that $v_M(B) \geq 2|M(B)|_h$ for all measurable sets $B \subset E$, with $v_M(B) > 0$. Fix $\varepsilon > 0$ arbitrarily and pick a disjoint family $B_1, \ldots, B_k$ of measurable subsets of E, such that $v_M(B_i) > 0$ for all i and $\sum_i |M(B_i)|_h + \varepsilon \geq v_M(E)$. Since v_M is additive, from the contradiction assumed we obtain $v_M(E) = \sum_i v_M(B_i) \geq 2\sum_i |M(B_i)|_h \geq 2v_M(E) - 2\varepsilon$. Since ε is arbitrarily small it follows that $v_M(E) \leq 0$, giving a contradiction. ∎

We state now our main theorem, which is inspired by [32, Lemma 3.1].

Theorem 4.12. [Radon Nikodým] *Let M and Γ be $ck(X)$-valued fuzzy multisubmeasures, satisfying the condition H_0. Suppose moreover that Γ is additive and*

4.12.1) Γ *is dominated by M*;
4.12.2) *for every $\varepsilon > 0$, the set property $A_\Gamma(E, \varepsilon) \neq \emptyset$ is v_M-exhaustive on every $E \in \mathscr{A}$.*

Then there exists an M integrable bounded function $f : T \to [0, \infty[$ such that for every $E \in \mathscr{A}$ it is $\Gamma(E) = \int_E f dM$.

Proof. Thanks to Theorems 4.3 and 4.6, it will be sufficient to prove that, for the single-valued mappings U_Γ and U_{M_0}, there exists a bounded measurable Radon-Nikodým derivative f. Indeed, since Γ is dominated by M, the same property holds with respect to M_0, since $v_M = v_{M_0}$ by Proposition 4.8. Of course, then the single-valued additive mapping U_Γ is dominated by U_{M_0}. Now, it will be proved that Γ and M_0 satisfy the condition of exhaustivity of the property

$A_\Gamma(E,\varepsilon) \neq \emptyset$. To this aim, the notation $A_{\Gamma,M}(E,\varepsilon)$ will be used, in order to stress the role of M in $A_\Gamma(E,\varepsilon)$. Then, assuming that the set $A_{\Gamma,M}(E,\varepsilon)$ is nonempty, it also turns out that $A_{\Gamma,M_0}(E,\varepsilon)$ is nonempty. Indeed, let $\rho > 0$ and $r \in A_{\Gamma,M}(E,\varepsilon)$ be fixed. Then, for every measurable $H \subset E$, there exists a finite partition $\{H_1,...,H_l\}$ of H, such that $h(rM_0(H), r\sum_{i=1}^l M(H_i)) \leq \rho$. Now, one has $h(\Gamma(H), rM_0(H)) \leq h(\Gamma(H), r\sum_{i=1}^l M(H_i)) + \rho = h(\sum_{i=1}^l \Gamma(H_i), r\sum_{i=1}^l M(H_i)) + \rho \leq \sum_{i=1}^l h(\Gamma(H_i), rM(H_i)) + \rho \leq \varepsilon \sum_{i=1}^l v_M(H_i) + \rho = \varepsilon v_M(H) + \rho = \varepsilon v_{M_0}(H) + \rho$. Since ρ is arbitrary, this shows that $r \in A_{\Gamma,M_0}(E,\varepsilon)$, and, in turn, this implies the exhaustivity of the property $A_\Gamma(E,\varepsilon) \neq \emptyset$ also with respect to M_0. Of course, since U is an isometric embedding, the measures U_Γ and U_{M_0} also enjoy the same properties of absolute continuity and exhaustivity.

Once a bounded measurable Radon-Nikodým derivative f has been found, for U_Γ with respect to U_{M_0}, then one has integrability of f with respect to M_0 by the Theorem 4.3 and $U\left(\int_E f dM_0\right) = \int_E f dU_{M_0} = U_\Gamma(E)$, while $\int_E f dM_0 = \int_E f dM$, holds true for every measurable set E thanks to Theorem 4.6.
So, for all $E \in \mathscr{A}$ one has $U\left(\int_E f dM\right) = U_\Gamma(E)$ and therefore $\int_E f dM = \Gamma(E)$. So, the problem is to find a bounded measurable mapping f, derivative of U_Γ with respect to U_{M_0}. In order to prove this we can observe that if M has bounded variation, then U_{M_0} is obviously strongly bounded; moreover the pair U_{M_0}, U_Γ satisfies the assumptions of [32, Lemma 3.1] and so an integrable function f in the sense of [32] can be found. Finally, since such f is non negative, bounded and v_{M_0}-totally measurable then, by [21, Theorem 4.9] it is M_0 integrable. ∎

We remark that the result is new even in the single-valued case, since additivity is requested just from one of the measures involved, so Theorem 4.12 extends theorems given in [32, 33].

Moreover, as an application, we can consider a Gould integrable multifunction (in the sense of [36, Definition 16]) $\varphi : T \to ck(X)$ with respect to a probability μ; let M be its Gould integral: $M : \mathscr{A} \to ck(X)$. Thanks to [36, Theorem 11] the set valued function M is additive, and satisfies the condition H_0. Observe moreover that, thanks to the Rådström's embedding and [36, Definition 16], an analogous version of Corollary 3.4 holds when the measure involved is scalar and the integrand is $C(\Omega)$-valued. Then we have

Corollary 4.13. *Let Γ be any $ck(X)$-valued fuzzy multimeasure, satisfying H_0,* 4.12.1) *and* 4.12.2). *Then there exists a scalar M integrable bounded mapping $f : T \to \mathbb{R}^+$ such that, for every $A \in \mathscr{A}$,*

$$\Gamma(A) = \int_A f(t)\varphi(t)d\mu. \tag{4.2}$$

Proof. Thanks to Theorem 4.12, there exists a bounded M integrable mapping $f : T \to \mathbb{R}^+$, such that, for all $A \in \mathscr{A}$, $\int_A f(t)dM = \Gamma(A)$. By [36, Theorem 5] it is enough to prove the assertion for $A = T$. Without loss of generality we also can consider φ, M, Γ as objects with values in $C(\Omega)$, as pointed out in Remark 4.2 and in Theorem 4.2. In this new setting, we can use Corollary 3.4 (in both versions: scalar functions and $C(\Omega)$-valued measures and viceversa) and so, there exists a sequence of partitions $(P_n)_n$ in T such that, for every $n \in \mathbb{N}$ and for every partition P finer than P_n, one has: $\sum_{J\in P} |M(J) - \varphi(t_J)\mu(J)| \leq \dfrac{u}{n}$, $\sum_{J\in P} |\Gamma(J) - f(t_J)M(J)| \leq \dfrac{u}{n}$ for every choice of $t_J \in J$. Then, for every P finer than P_n,

$$\begin{aligned}|\sum_{J\in P} f(t_J)\varphi(t_J)\mu(J)-\Gamma(T)| &\leq \sum_{J\in P}|f(t_J)\varphi(t_J)\mu(J)-\Gamma(J)| \leq \\ \leq \sum_{J\in P}|f(t_J)\varphi(t_J)\mu(J)- f(t_J)M(J)| &+ \sum_{J\in P}|f(t_J)M(J)-\Gamma(J)| \leq \\ \leq \sum_{J\in P}|f(t_J)|\cdot|\varphi(t_J)\mu(J)-M(J)| &+ \frac{u}{n} \leq \frac{u}{n}(1+\sup|f|).\end{aligned}$$

So $(\sigma(f\varphi,P))_P$ is convergent in $C(\Omega)$ to $\Gamma(T)$, and so $f\varphi$ is μ-integrable and

$$\Gamma(T)=\int_T f(t)\varphi(t)d\mu.$$

∎

The last result of Corollary 4.13 can be viewed as an integration by substitution for fuzzy multisubmeasures and (4.2) can be written as

$$\int_A f dM = \int_A f\varphi d\mu, \quad \forall A\in\mathscr{A}.$$

5. CONCLUSIONS

We have studied the Gould integrability of a scalar function f with respect to a set-valued, non necessarily additive measure m. In particular we have focused our attention on compact and convex-valued measures. In this case, thanks to the well-known Rådström's embedding theorem, m can be considered as a measure taking values in the Banach lattice $C(\Omega)$. In addition, the notion of *integrability* has been introduced for m, with the purpose to avoid the requirement of additivity. In fact, thanks to the Rådström's embedding, we are able to establish a Henstock-type theorem for this kind of integral. This, in turn, implies that any *integrable* measure m can be seen as an additive measure, plus a *negligible* one. Finally a Radon-Nikodým Theorem is obtained in this situation which is new also in the finite dimensional case, since one of the involved set-valued measures is non-additive. Moreover, a u-substitution result for fuzzy multimeasures is established.

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