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LOCAL BOUNDEDNESS OF WEAK SOLUTIONS FOR SINGULAR PARABOLIC SYSTEMS OF p-LAPLACIAN TYPE

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ABSTRACT. We study the local boundedness of weak solutions for evolutional p-Laplacian systems in the singular case. The initial data is belonging to Lebesgue space $L^\infty(0,T;W^{(1,p)}(\Omega,\mathbb{R}^n))$. We use intrinsic scaling method to treat the boundedness of weak solutions. The main result is to make the local boundedness of weak solution for the systems well-worked in the intrinsic scaling.

Key words and phrases: Weak solutions; Singular parabolic systems; Intrinsic scaling.

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, with smooth boundary $\partial\Omega$, and let $\frac{2m}{m+2} < p < 2$. For a map $u : (0,T) \times \Omega \to \mathbb{R}^n$, $z = (t,x) = (t, x_1, x_2, ..., x_m)$, the unknown $u = (u^i)$, $i = 1, 2, ..., n$ is a vector-valued function on Q with values into $\mathbb{R}^n$, we consider p-Laplacian type, with principal term only, as below

$$\begin{cases} \partial_t u - \operatorname{div}(|Du|^{p-2} Du) = 0 & \text{in } (0,T) \times \Omega \\ u(0,x) = u_0(x) & \text{on } \partial_p(0,T) \times \Omega \end{cases} \tag{1.1}$$

To construct a weak solution of (1.1), we use a Galerkin method as in [1] or use a variational (like) method as in [9] by the Rothe-approximation to have the energy inequality of (1.1). The local boundedness of weak solutions p-Laplacian systems with only principal terms, where the unknown functions are real valued and scalar case, was studied by DiBenedetto et al. ([2, 3, 4, 5]), whose proof is based on De Giorgi's truncation and special scaling associated with inhomogeneity of the evolutionary p-Laplace operator. However, the regularity theory for p-Laplacian type of parabolic equations requires careful geometric techniques the so-called intrinsic scaling to resolve the inhomogeneity. Moreover the local boundedness of weak solutions where the unknown functions are vectorial case was studied in [11] using a perturbation estimate for degenerate case and singular case.

In early 2013's, a direct iteration scheme is introduced in [10] only in the degenerate case, using a geometrical progression based on an intrinsic scaling to the evolutionary p-Laplace operator, and the Hölder estimate of solutions for singular case was settled by [7] and the Gradient of its solutions also studied by [8], whose proof based on the intrinsic scaling. In this paper, we will study the local boundedness of weak solution for singular case such that it is well-worked by using intrinsic scaling. Here we point out that our intrinsic scaling is modified different for degenerate case and singular case in the original work by DiBenedetto [6] and Chen ([3, 4])

When one studies the existence of weak solutions of evolutional p-Laplacian systems, one needs to invoke a definition of weak solution itself. The weak solution is defined as usual.

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Definition 1.1. A vector-valued function u is a weak solution of (1.1), if and only if $u \in L^\infty(0,T; L^2(\Omega, \mathbb{R}^n)) \cap L^p(0,T;\, W^{1,p}(\Omega, \mathbb{R}^n))$ and satisfies

$$\int_{(0,T)\times\Omega} \partial_t u \cdot \varphi + |Du|^{p-2} Du \cdot D\varphi dz = 0, \tag{1.2}$$

for all $\varphi \in L^p\left(0, T; W_0^{1,p}\left(\Omega, \mathbb{R}^n\right)\right)$ with $\partial_t \varphi \in L^2(Q, \mathbb{R}^n)$ and $T > 0$.

While, our main theorem in this paper is the following:

Theorem 1. *Let $\frac{2m}{m+2} < p < 2$ and let u be a weak solution of (1.1). Then there exist positive constants $C_1 = C(m,p,\sigma)$ and $C_2 = C$ such that for every cylinder $Q(\rho^2, \lambda^{\frac{p-2}{2}\rho}(z_0) \subset Q$.*

$$\sup_{Q((\sigma\rho)^2, \lambda^{\frac{p-2}{2}}\sigma\rho)} |u| \leq C_1 (\lambda^{\frac{p}{2}}\rho)^{\frac{m(p-2)}{p(m+2)-2m}} \left(\fint_{Q(\rho^2, \lambda^{\frac{p-2}{2}}\rho)} |u|^2 dz \right)^{\frac{p}{p(m+2)-2m}} + C_2 \lambda^{\frac{p}{2}} \rho, \tag{1.3}$$

for any positive number $\tau < \frac{1}{2}$ *and* $\sigma = 2\tau/(1+\tau)$.

2. RESULTS

We consider u be a weak solution of (1.1) in $Q(\rho^2, \lambda^{\frac{p-2}{2}}\rho)(z_0) \subset Q$. First, we set the intrinsic scaling for $1 < p < 2$

$$t = t_0 + \rho^2 s, \qquad x = x_0 + \lambda^{\frac{p-2}{2}}\rho y, \qquad \widetilde{z} = (s, y), \tag{2.1}$$

$$v(s,y) = \frac{u\left(t_0 + \rho^2 s, x_0 + \lambda^{\frac{p-2}{2}}\rho y\right)}{\lambda^{\frac{p}{2}}\rho}, \qquad 0 \le \rho < 1, \tag{2.2}$$

so that our equation (1.1) in $Q\left(\rho^2, \lambda^{\frac{p-2}{2}}\rho\right)(z_0)$ reduced to the following equation in $Q(1,1)(0,0)$:

$$\partial_t v - \operatorname{div}\left(|Dv|^{p-2}Dv\right) = 0. \tag{2.3}$$

It implies that

$$\sup_{Q(\sigma^2,\sigma)} |v| \le C\left(\fint_{Q(1,1)} |v|^p dz\right)^{\frac{p}{p(m+2)-2m}} + C. \tag{2.4}$$

We have the following proposition holds for every cylinder $Q(1,1)(0) \subset Q$. Let $0 \le \eta \le 1$ be a piecewise smooth cutoff function in $B(r) \subset B(\rho) \subset\subset \Omega$ such that $\operatorname{supp}(\eta) \subset B(\rho)$, $\eta = 1$ in $B(r)$ and $|D\eta| \le \frac{C}{\rho - r}$.

Propotation 1. *Let v be a weak solution of (2.3). There exists a positive constant $C = C(p)$ such that*

$$\begin{aligned} &\sup_{-r^2<t<0} \int_{B(\rho)} |v|^2\eta^p\xi dx + \int_{-r^2}^{0}\int_{B(\rho)} |Dv|^p\eta^p\xi dxdt \\ &\le \int_{Q(\rho)} |v|^2\eta^p\partial_t\xi dz + C\int_{Q(\rho)} |v|^p|D\eta|^p dz. \end{aligned} \tag{2.5}$$

By using the Sobolev inequality for function and the reverse Hölder inequality, we have

$$\begin{aligned} \int_{Q(r)} |v|^{1+\frac{2(p(m+1)-2m}{mp}} dz &\le \int_{-r^2}^{0}\left(\int_{B(r)} |v|^{\frac{p(m+1)-m}{mp}(2)\frac{mp}{p(m+1)-m}} dx\right)^{\frac{p(m+1)-m}{mp}}\left(\int_{B(r)} (|v|^p)^{\frac{m}{m-p}} dx\right)^{\frac{m-p}{mp}} dt \\ &\le \sup_{-r^2<t<0}\left(\int_{B(r)} |v|^2 dx\right)^{\frac{p(m+1)-m}{mp}} \int_{-r^2}^{0}\left(\int_{B(r)} (|v|^p)^{\frac{m}{m-p}} dx\right)^{\frac{m-p}{mp}} dt \\ &\le \left\{C\frac{1}{(r-\rho)^p}\left(\int_{Q(\rho)} |v|^2 dz + 1\right)\right\}^{1+\frac{1}{m}}. \end{aligned} \tag{2.6}$$

Now let

$$\alpha_k = \frac{p(m+2)-2m}{p}\Big((1+\frac{1}{m})^k - 1\Big) + 2; \quad \theta = 1 + \frac{1}{m};$$
$$R_k = \sigma + \frac{1-\sigma}{2^k}; \quad R_0 = 1,$$

then

$$\left(\fint_{Q(r)} |v|^{\alpha_{k+1}} dz\right)^{\frac{1}{\theta^{k+1}}} \leq C \frac{|Q(\rho)|^{\frac{1}{\theta^k}}}{|Q(r)|^{\frac{1}{\theta^{k+1}}}} \frac{1}{(r-\rho)^{\frac{p}{\theta^k}}} \left(\fint_{Q(\rho)} |v|^{\alpha_k} dz + 1\right)^{\frac{1}{\theta^k}}.$$

In above we choose $r = \frac{\rho}{2}$ and make iteration on $k = 0, 1, 2, ...$ to have

(2.7)
$$\left(\fint_{Q(R_{k+1})} |v|^{\alpha_{k+1}} dz + 1\right)^{\frac{1}{\theta^{k+1}}} \leq \prod_{i=0}^{\infty} C \frac{|Q(R_i)|^{\frac{1}{\theta^i}}}{|Q(R_{i+1})|^{\frac{1}{\theta^{i+1}}}} \frac{1}{(R_i - R_{i+1})^{\frac{p}{\theta^i}}} \left(\fint_{Q(R_0)} |v|^{\alpha_0} dz + 1\right)^{\frac{1}{\theta^0}}.$$

Since

$$\alpha_i = \frac{p(m+2)-2m}{p}\Big((1+\frac{1}{m})^i - 1\Big) + 2; \quad \theta = 1 + \frac{1}{m};$$
$$R_i = \sigma + \frac{1-\sigma}{2^i}; \quad R_0 = 1.$$

We see that the constant in (2.7) is computed as

$$\overline{C}(m,p,\sigma) \leq (C_1)^{p(m+1)} (2)^{p\tilde{c}} (1)^{m+2}, \tag{2.8}$$

where $C_1 = 2C(1-\sigma)^{-1}$ and we use

$$\lim_{i\to\infty} \frac{(i+1)}{\theta^{i+1}} \Big/ \frac{i}{\theta^i} = \frac{1}{\theta} < 1.$$

Thus $\forall i$ it holds that

$$\left(\fint_{Q(R_i)} |v|^{\alpha_i} dz\right)^{\frac{1}{\theta^i}} \leq \overline{C} \fint_{Q(R_0)} |v|^2 dz + C. \tag{2.9}$$

In fact, we use

$$\lim_{i\to\infty} \frac{\alpha_i}{\theta^i} = \frac{p(m+2)-2m}{p}.$$

From this estimate it follows that for any $\frac{2m}{m+2} < p < 2$

$$\sup_{Q(\sigma^2,\sigma)} |v| \leq \overline{C} \left(\fint_{Q(1)} |v|^2 dz\right)^{\frac{p}{p(m+2)-2m}} + C. \tag{2.10}$$

Rescaling (2.10) by (2.2) to have (1.3).

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