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A METHOD OF THE STUDY OF THE CAUCHY PROBLEM FOR A SINGULARLY PERTURBED LINEAR INHOMOGENEOUS DIFFERENTIAL EQUATION

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ABSTRACT. We construct a sequence that converges to a solution of the Cauchy problem for a singularly perturbed linear inhomogeneous differential equation of an arbitrary order. This sequence is also an asymptotic sequence in the following sense: the deviation (in the norm of the space of continuous functions) of its nth element from the solution of the problem is proportional to the $(n+1)$th power of the parameter of perturbation. This sequence can be used for justification of asymptotics obtained by the method of boundary functions.

Key words and phrases: Singular perturbations, Banach fixed-point theorem, Method of asymptotic iterations, Method of boundary functions, Routh–Hurwitz stability criterion.

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1. INTRODUCTION

We propose an algorithm of construction of a sequence

$$\psi_n(x;\varepsilon) = (y_n^1(x;\varepsilon), \dots, y_n^m(x;\varepsilon))$$

that converges for each $\varepsilon \in (0, \varepsilon_0]$ with respect to the norm of the space $C_m[0, X]$ of continuous m-dimensional vector-valued functions of the argument $x \in [0, X]$) to the function

$$\psi(x;\varepsilon) = \left(y(x;\varepsilon), \frac{d}{dx} y(x;\varepsilon), \dots, \frac{d^{m-1}}{dx^{m-1}} y(x;\varepsilon) \right),$$

where $y(x;\varepsilon)$ is a classical solution of the problem (2.1)–(2.2); for the value of ε_0 we obtain an explicit lower estimate. The construction and the proof of convergence of the sequence $\psi_n(x;\varepsilon)$ are based on the Banach fixed-point theorem for a contracting mapping of a complete metric space (see [4]). Since the contraction coefficient k of the mapping is a value of order ε ($k < \varepsilon/\varepsilon_0$), so that the deviation $y_n^i(x;\varepsilon)$ (with respect to the norm of $C[0, X]$) from $\dfrac{d^{i-1}}{dx^{i-1}} y(x;\varepsilon)$ is $O(\varepsilon^{n+1})$ (for $0 < \varepsilon \le \varepsilon_0$), we see that this result has also asymptotic character.

Note that each successive element of the sequence $\psi_n(x;\varepsilon)$ is the result of the action of a certain operator on the previous element. Elements of such sequences are usually called iterations and sequences themselves are said to be iterative. In our case, iterations approach to $\psi(x;\varepsilon)$ (in the norm of $C_m[0, X]$) sufficiently rapidly; the rate of approach is asymptotically reciprocal to ε. Therefore, the algorithm of construction of the sequence $\psi_n(x;\varepsilon)$ is a method of asymptotic iterations (for detail, see [1, 2]). The sequences $y_n^i(x;\varepsilon)$ are also called asymptotic iterative sequences of the $(i-1)$th derivative of the solution $y(x;\varepsilon)$ of the problem considered.

The possibility of application of the method of asymptotic iterations is related to the fulfillment of the condition (2.3) for coefficients of the right-hand side of the equation. However, the fulfillment of these conditions allows one to apply the method of boundary-layer functions (see, e.g., [7]). One can immediately verify that the deviation $y_n^1(x;\varepsilon)$ from the nth partial sum $Y_n(x;\varepsilon)$ (which is called the asymptotics or the asymptotic expansion of nth order) of the series $Y(x;\varepsilon)$ obtained by the method of boundary-layer functions has the form $O(\varepsilon^{n+1})$. Thus, the convergence of the sequence $y_n^1(x;\varepsilon)$ enables the using of the method of asymptotic iterations for the justification of asymptotic expansions obtained by the method of boundary-layer functions (i.e., to the proof of the fact that the difference of $Y_n(x;\varepsilon)$ and the solution $y(x;\varepsilon)$ has the form $O(\varepsilon^{n+1})$ uniformly with respect to $x \in [0, X]$).

Note that the convergence (uniform with respect to ε) as $\varepsilon \in (0, \varepsilon_0]$ of asymptotic sequences $y_n^i(x;\varepsilon)$ is a fundamental advantage of the method of asymptotic iterations over the method of boundary-layer functions, which allows one to construct an asymptotic series, which is, in general does not converge even for arbitrarily small ε. The reason is that the estimate of the deviation of $y_n^1(x;\varepsilon)$ from $Y_n(x;\varepsilon)$, which has the form $O(\varepsilon^{n+1})$, is not uniform with respect to n, so that this deviation may be not infinitesimal as $n \to \infty$ but even unboundedly increasing.

Another advantage of the sequence $\psi_n(x;\varepsilon)$ is the possibility of construction of all its terms under modest smoothness conditions for the functions a_i and b: for the construction of all $\psi_n(x;\varepsilon)$ it suffices that $a_i, b \in C^1[0, X]$, while for the construction of all terms of the series $Y(x;\varepsilon)$ the infinite differentiability of a_i and b is required.

2. STATEMENT OF THE PROBLEM AND AUXILIARY ESTIMATES

Consider the Cauchy problem for the linear, inhomogeneous, singularly perturbed differential equation of order m:

$$\varepsilon^m y^{(m)} = \varepsilon^{m-1} a_{m-1}(x) y^{(m-1)} + \cdots + a_0(x) y + b(x), \quad x \in (0, X]; \tag{2.1}$$

$$y(0;\varepsilon) = y^0, \quad \ldots, \quad y^{(m-1)}(0;\varepsilon) = \frac{y^{m-1}}{\varepsilon^{m-1}}, \tag{2.2}$$

where $\varepsilon > 0$ is the perturbation parameter, $X > 0$, $y^0, \ldots, y^{m-1} \in \mathbb{R}$, and $a_0, \ldots, a_{m-1}$, $b \in C^1[0, X]$. Moreover, we assume that the coefficients $a_i(x)$ satisfy the Routh–Hurwitz condition for all $x \in [0, X]$ (see, e.g., [3]):

$$\begin{gathered} -a_{00}(x) > 0, \quad \begin{vmatrix} a_{00}(x) & a_{01}(x) \\ a_{10}(x) & a_{11}(x) \end{vmatrix} > 0, \quad \ldots, \\ (-1)^m \begin{vmatrix} a_{00}(x) & \ldots & a_{0(m-1)}(x) \\ \vdots & \ddots & \vdots \\ a_{(m-1)0}(x) & \ldots & a_{(m-1)(m-1)}(x) \end{vmatrix} > 0, \end{gathered} \tag{2.3}$$

where

$$a_{ij}(x) := \begin{cases} a_{2i-j}(x) & \text{for } 0 \le 2i - j < m, \\ -1 & \text{for } 2i - j = m, \\ 0, & \text{for } 2i - j < 0 \text{ or } 2i - j > m. \end{cases}$$

Recall that for the fulfillment of the conditions (2.3) it is necessary (and for $m \in \{1, 2\}$ is also sufficiently) that all $a_i(x)$ be negative.

Let p be that mapping, which to each $x \in [0, X]$ puts in corresponding the polynomial

$$p(x) := \lambda^m - a_{m-1}(x)\lambda^{m-1} - \cdots - a_1(x)\lambda - a_0(x). \tag{2.4}$$

Since the degree of the polynomial $p(x)$ is m on the whole segment $[0, X]$, there exist functions $\lambda_1, \ldots, \lambda_m : [0, X] \to \mathbb{C}$ such that

$$p(x) = (\lambda - \lambda_1(x)) \ldots (\lambda - \lambda_m(x))$$

for each $x \in [0, X]$; the numbers $\lambda_1(x), \ldots, \lambda_m(x)$ are called roots of the polynomial $p(x)$. The ordered set $(\lambda_1, \ldots, \lambda_m)$ of the function λ_i is called the vector-function of roots of the mapping p. Note that there exist infinitely many vector-functions of roots since for each $x \in [0, X]$ we can list the roots of the polynomial $p(x)$ in various orders. We fix one of the possible orderings.

By the Routh–Hurwitz criterion (see [3]), the real parts of the roots of the polynomial $p(x)$ are negative if and only if its coefficients $a_i(x)$ satisfy the inequalities (2.3). Thus, for all $(i, x) \in \{1, \ldots, m\} \times [0, X]$, the inequality

$$\operatorname{Re} \lambda_i(x) < 0 \tag{2.5}$$

holds.

We prove that each of the function $\operatorname{Re} \lambda_i$ is bounded on the segment $[0, X]$ from the above by a certain negative constant.

Let P be the mapping that to each $M = (a_0, \ldots, a_{m-1}) \in \mathbb{C}^m$ puts in corresponding the polynomial

$$P(M) := \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0. \tag{2.6}$$

Denote by $\{\Lambda\}$ the set of all mappings $\Lambda : \mathbb{C}^m \to \mathbb{C}^m$, which to each $M \in \mathbb{C}^m$ put in correspondence an ordered set $(\lambda^1(M), \ldots, \lambda^m(M))$ of roots of the equation $P(M) = 0$ (we assume that each root is repeated as many times as its multiplicity). In fact, the choice of $\Lambda \in \{\Lambda\}$ means the choice of numbering of roots of the polynomial $P(M)$ for each $M \in \mathbb{C}^m$. It is easy to verify that for $m \geq 2$ the set $\{\Lambda\}$ contains no mappings continuous in the whole space $\mathbb{C}^m$. However, it is known that for each m and any point $M_0 \in \mathbb{C}^m$, there exists a mapping $\Lambda_{M_0} \in \{\Lambda\}$ continuous at this point (see, e.g., [6]).

Let φ be the mapping, which to each $\Lambda \in \{\Lambda\}$ puts in correspondence the vector-function $\lambda = (\lambda^1, \dots, \lambda^m)$ whose components λ^i to each $M \in \mathbb{C}^m$ put in correspondence the ith coordinates of $\Lambda(M)$: $\Lambda(M) = (\lambda^1(M), \dots, \lambda^m(M))$. Obviously, φ is a bijective correspondence between $\{\Lambda\}$ and $\{\lambda\} := \varphi(\{\Lambda\})$. Moreover, the continuity of the mapping Λ is equivalent to the continuity of the corresponding vector-function $\varphi(\Lambda)$, which, in its turn, is equivalent to the continuity of all its components.

Lemma 2.1. *Let* $\lambda = (\lambda^1, \dots, \lambda^m) \in \{\lambda\}$. *Then*

$$\overline{\Lambda} : \mathbb{C}^m \ni M \mapsto \max\{\operatorname{Re}\lambda^1(M), \dots, \operatorname{Re}\lambda^m(M)\}$$

is a continuous function.

Remark 2.1. For each point $M \in \mathbb{C}^m$, the unordered set of roots of the polynomial $P(M)$ and the value $\overline{\Lambda}(M)$ are independent of the choice of $\lambda \in \{\lambda\}$. Thus, to each $\lambda \in \{\lambda\}$ (i.e., to each way of numbering of roots of the polynomial $P(M)$) the same function $\overline{\Lambda}$ corresponds.

Proof of Lemma 2.1. Fix an arbitrary point $M_0 \in \mathbb{C}^m$ and choose a mapping $\lambda_{M_0} = (\lambda^1_{M_0}, \dots, \lambda^m_{M_0}) \in \{\lambda\}$ continuous at this point. Each of the functions $\lambda^i_{M_0}$ is also continuous at the point M_0. But the continuity of $\lambda^i_{M_0}$ implies the continuity of $\operatorname{Re}\lambda^i_{M_0}$, whereas the continuity of all $\operatorname{Re}\lambda^i_{M_0}$, in its turn, implies the continuity of the maximum of these functions. ∎

Corollary 2.2. *There exist positive* χ *(independent of* i *and* x*) such that*

$$\operatorname{Re}\lambda_i(x) < -\chi$$

for all $(i, x) \in \{1, \dots, m\} \times [0, X]$, *where* $\lambda_i(x)$ *is the* i*th root of the polynomial* $p(x)$ *(see* (2.4)*) for each* $x \in [0, X]$.

Remark 2.2. For each $x \in [0, X]$, the unordered set of roots of the polynomial $p(x)$ and the value $\overline{\lambda}(x) := \max\{\operatorname{Re}\lambda_1(x), \dots, \operatorname{Re}\lambda_m(x)\}$ are independent of the way of numbering of these roots.

Proof of Corollary 2.2. Let $\lambda = (\lambda^1, \dots, \lambda^m)$ be a mapping from $\{\lambda\}$. By the remark above, without loss of generality, we can assume that

$$\lambda_i(x) = \lambda^i(a_0(x), \dots, a_{m-1}(x)) \quad \forall (i, x) \in \{1, \dots, m\} \times [0, X].$$

Since the function $\overline{\lambda}(x)$, which is equal to $\overline{\Lambda}(a_0(x), \dots, a_{m-1}(x))$, is continuous (as a composite function) and negative (see (2.5)) on the whole segment $[0, X]$, by the Weierstrass extreme-value theorem, there exists $x_0 \in [0, X]$ such that

$$\begin{aligned} \chi : &= -\overline{\lambda}(x_0) = -\max_{[0,X]} \overline{\Lambda}\big(a_0(x), \dots, a_{m-1}(x)\big) \\ &= -\max_{[0,X]} \max\big\{ \operatorname{Re}\lambda_1(x), \dots, \operatorname{Re}\lambda_m(x)\big\} > 0. \end{aligned} \tag{2.7}$$

The proof is complete. ∎

Remark 2.3. One can prove that there exist continuous functions $\lambda_1, \dots, \lambda_m : [0, X] \mapsto \mathbb{C}$ that describe the set of all roots (with account of multiplicities) of the polynomial $p(x)$ for each $x \in [0, X]$; here the fact that the variable x is one-dimensional is substantial.

Consider the following auxiliary problem:

$$a_0(x)\bar{y}(x) + b(x) = 0, \quad x \in [0, X]; \tag{2.8}$$

$$\frac{d^m\Pi}{d\xi^m}(\xi) = a_{m-1}(0)\frac{d^{m-1}\Pi}{d\xi^{m-1}}(\xi) + \cdots + a_0(0)\Pi(\xi), \quad \xi \in \left(0, \frac{X}{\varepsilon}\right]; \tag{2.9}$$

$$\Pi(0) = y^0 - \bar{y}(0), \quad \frac{d\Pi}{d\xi}(0) = y^1, \quad \ldots, \quad \frac{d^{m-1}\Pi}{d\xi^{m-1}}(0) = y^{m-1}. \tag{2.10}$$

Equation (2.8) is an algebraic equation of the first degree with respect to $\bar{y}(x)$, whereas (2.9) is an autonomous homogeneous linear differential equation for $\Pi(\xi)$. The solution of the problem (2.8)–(2.10) has the form

$$\begin{aligned} \bar{y}(x) &= -\frac{b(x)}{a_0(x)}, \\ \Pi(\xi) &= \alpha_{11}e^{\lambda_1(0)\xi} + \cdots + \alpha_{1m_1}\xi^{m_1-1}e^{\lambda_{m_1}(0)\xi} + \cdots \\ &\quad + \alpha_{q1}e^{\lambda_{m_1+\cdots+m_{q-1}+1}(0)\xi} + \cdots + \alpha_{qm_q}\xi^{m_q-1}e^{\lambda_{m_1+\cdots+m_{q-1}+m_q}(0)\xi}, \end{aligned} \tag{2.11}$$

where $\lambda_1(0) = \cdots = \lambda_{m_1}(0)$, …, $\lambda_{m_1+\cdots+m_{q-1}+1}(0) = \cdots = \lambda_{m_1+\cdots+m_q}(0)$ are roots of the polynomial $p(0)$ (see (2.4)), α_{11}, …, α_{qm_q} are constants that are uniquely expressed through $y^0 - \bar{y}(0)$, y^1, …, y^{m-1} and $\lambda_1(0)$, …, $\lambda_m(0)$ (here $m_1 + \cdots + m_q = m$).

We see from (2.11) and (2.7) that for sufficiently large $\tilde{C}$ the functions $\Pi^{(i)}$ satisfy the estimate

$$\left|\Pi^{(i)}(\xi)\right| \le \tilde{C}(1 + \xi^{m-1})e^{-\chi\xi}, \quad (i, \xi) \in \{0, \ldots, m-1\} \times [0, +\infty). \tag{2.12}$$

In the problem (2.1)–(2.2), we perform the following change of variables:

$$\begin{aligned} x &= \varepsilon\xi, \\ y(x;\varepsilon) &= \tilde{y}(\xi, x) + \varepsilon z^1(\xi;\varepsilon), \\ \frac{d^{i-1}y}{dx^{i-1}}(x;\varepsilon) &= \varepsilon^{1-i}\frac{d^{i-1}\Pi}{d\xi^{i-1}}(\xi) + \varepsilon^{2-i}z^i(\xi;\varepsilon), \quad i = \overline{2, m}, \end{aligned} \tag{2.13}$$

where $\tilde{y}(\xi, x) := \bar{y}(x) + \Pi(\xi)$.

For the new functions $z^i(\xi;\varepsilon)$ we obtain the following initial-value problem:

$$\frac{dz^1}{d\xi} = z^2 - \bar{y}'(\varepsilon\xi), \quad \xi \in \left(0, \frac{X}{\varepsilon}\right]; \tag{2.14}$$

$$\frac{dz^i}{d\xi} = z^{i+1}, \quad (i, \xi) \in \{2, \ldots, m-1\} \times \left(0, \frac{X}{\varepsilon}\right]; \tag{2.15}$$

$$\frac{dz^m}{d\xi} = a_{m-1}(\varepsilon\xi)z^m + \cdots + a_0(\varepsilon\xi)z^1 + f(\xi;\varepsilon), \quad \xi \in \left(0, \frac{X}{\varepsilon}\right]; \tag{2.16}$$

$$z^1(0;\varepsilon) = \ldots = z^m(0;\varepsilon) = 0 \tag{2.17}$$

((2.14) only for $m \ge 2$, (2.15) only for $m \ge 3$), where

$$f(\xi;\varepsilon) := \begin{cases} \varepsilon^{-1}\Big\{\big[a_{m-1}(\varepsilon\xi) - a_{m-1}(0)\big] \\ \qquad \times\Pi^{(m-1)}(\xi) + \cdots + \big[a_0(\varepsilon\xi) - a_0(0)\big]\Pi(\xi)\Big\} & \text{for } m \ge 2; \\ \varepsilon^{-1}\big[a_0(\varepsilon\xi) - a_0(0)\big]\Pi(\xi) - \bar{y}'(\varepsilon\xi) & \text{for } m = 1. \end{cases} \tag{2.18}$$

We transform Eq. (2.16) adding the variable x as a new parameter:

$$\frac{dz^m}{d\xi} = a_{m-1}(x)z^m + \dots + a_0(x)z^1 + \big[a_{m-1}(\varepsilon\xi) - a_{m-1}(x)\big]z^m + \dots + \big[a_0(\varepsilon\xi) - a_0(x)\big]z^1 + f(\xi;\varepsilon), \quad (\xi, x) \in \left(0, \frac{X}{\varepsilon}\right] \times [0, X]. \tag{2.19}$$

The problem (2.14), (2.15), (2.19), (2.17) is equivalent to the following system of integral equations:

$$\begin{aligned} z^i(\xi;\varepsilon) = &-\int_0^\xi \Phi^1_{\xi^{i-1}}(\xi-\zeta;x)\bar{y}'(\varepsilon\zeta)d\zeta \\ &+ \int_0^\xi \Phi^m_{\xi^{i-1}}(\xi-\zeta;x)\Big\{\big[a_{m-1}(\varepsilon\zeta) - a_{m-1}(x)\big]z^m(\zeta;\varepsilon) \\ &+ \dots + \big[a_0(\varepsilon\zeta) - a_0(x)\big]z^1(\zeta;\varepsilon) + f(\zeta;\varepsilon)\Big\}d\zeta, \\ &(i, \xi, x) \in \overline{1, m} \times \left[0, \frac{X}{\varepsilon}\right] \times [0, X], \end{aligned} \tag{2.20}$$

where $\Phi^j_{\xi^{i-1}}(\xi-\zeta;x) = K^i_j(\xi,\zeta;x)$ are the entries of the Cauchy matrix

$$K(\xi,\zeta;x) := \begin{bmatrix} \Phi^1(\xi-\zeta;x) & \Phi^2(\xi-\zeta;x) & \dots & \Phi^m(\xi-\zeta;x) \\ \Phi^1_\xi(\xi-\zeta;x) & \Phi^2_\xi(\xi-\zeta;x) & \dots & \Phi^m_\xi(\xi-\zeta;x) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^1_{\xi^{m-1}}(\xi-\zeta;x) & \Phi^2_{\xi^{m-1}}(\xi-\zeta;x) & \dots & \Phi^m_{\xi^{m-1}}(\xi-\zeta;x) \end{bmatrix}$$

of the corresponding homogeneous system

$$\frac{dz^1}{d\xi} = z^2, \quad \dots, \quad \frac{dz^{m-1}}{d\xi} = z^m, \quad \frac{dz^m}{d\xi} = a_{m-1}(x)z^m + \dots + a_0(x)z^1.$$

Note that the functions $\Phi^1(\xi;x)$ and $\Phi^m(\xi;x)$ used in (2.20), due to the definition of the Cauchy matrix, are the solutions of the following initial-value problems:

$$\frac{d^m\Phi^1}{d\xi^m} = a_{m-1}(x)\frac{d^{m-1}\Phi^1}{d\xi^{m-1}} + \dots + a_0(x)\Phi^1, \quad (\xi, x) \in \mathbb{R} \times [0, X]; \tag{2.21}$$

$$\Phi^1(0;x) = 1, \quad \frac{d\Phi^1}{d\xi}(0;x) = \ldots = \frac{d^{m-1}\Phi^1}{d\xi^{m-1}}(0;x) = 0, \quad x \in [0, X]; \tag{2.22}$$

$$\frac{d^m\Phi^m}{d\xi^m} = a_{m-1}(x)\frac{d^{m-1}\Phi^m}{d\xi^{m-1}} + \dots + a_0(x)\Phi^m, \quad (\xi, x) \in \mathbb{R} \times [0, X]; \tag{2.23}$$

$$\Phi^m(0;x) = \ldots = \frac{d^{m-2}\Phi^m}{d\xi^{m-2}}(0;x) = 0, \quad \frac{d^{m-1}\Phi^m}{d\xi^{m-1}}(0;x) = 1, \quad x \in [0, X]. \tag{2.24}$$

From (2.21)–(2.24) and the theorems on the continuity and differentiability with respect to parameters of solutions of initial-value problems we conclude that $\Phi^1(\xi;x)$, $\Phi^m(\xi;x) \in C^{\infty,1}(\mathbb{R} \times [0, X])$.

Since the solution $(z^1, \dots, z^m)$ of the system (2.20) is clearly independent of x, we can replace x in (2.20) by an arbitrary function ξ and ε with values in $[0, X]$. Then, setting $x = \varepsilon\xi$, we arrive at the following equations for $z^i(\xi;\varepsilon)$:

$$(2.25)\quad z^i(\xi;\varepsilon) = -\int_0^\xi \Phi^1_{\xi^{i-1}}(\xi-\zeta;\varepsilon\xi)\bar{y}'(\varepsilon\zeta)d\zeta + \int_0^\xi \Phi^m_{\xi^{i-1}}(\xi-\zeta;\varepsilon\xi)$$
$$\times\Big\{\big[a_{m-1}(\varepsilon\zeta)-a_{m-1}(\varepsilon\xi)\big]z^m(\zeta;\varepsilon)+\cdots+\big[a_0(\varepsilon\zeta)-a_0(\varepsilon\xi)\big]z^1(\zeta;\varepsilon)+f(\zeta;\varepsilon)\Big\}d\zeta$$
$$=: \widehat{A}_i(\varepsilon)[z^1,\dots,z^m](\xi;\varepsilon),\quad (i,\xi)\in\overline{1,m}\times\left[0,\frac{X}{\varepsilon}\right],$$

(the first integral only for $m\ge 2$) or briefly

$$(2.26)\quad \big(z^1(\xi;\varepsilon),\dots,z^m(\xi;\varepsilon)\big)$$
$$=\big(\widehat{A}_1(\varepsilon)[z^1,\dots,z^m](\xi;\varepsilon),\ \dots,\ \widehat{A}_m(\varepsilon)[z^1,\dots,z^m](\xi;\varepsilon)\big)=$$
$$=: \widehat{A}(\varepsilon)[z^1,\dots,z^m](\xi;\varepsilon),\quad \xi\in\left[0,\frac{X}{\varepsilon}\right],$$

where for each fixed $\varepsilon\in(0,+\infty)$ by the domain of the operator $\widehat{A}(\varepsilon)$ we mean the space $C_m[0,X/\varepsilon]$ of m-dimensional vector-functions continuous on the segment $[0,X/\varepsilon]$:

$$\widehat{A}(\varepsilon): C_m\left[0,\frac{X}{\varepsilon}\right]\to C_m\left[0,\frac{X}{\varepsilon}\right].$$

In the sequel we need one auxiliary property of the solution w of the Cauchy problem for a linear differential equation with constant coefficients considered as parameters for w:

$$(2.27)\qquad \frac{d^m w}{d\xi^m}=a_{m-1}\frac{d^{m-1}w}{d\xi^{m-1}}+\cdots+a_0w,\quad \xi\in(0,+\infty);$$

$$(2.28)\qquad w(0;M_m,N_m)=w^0,\quad \dots,\quad \frac{d^{m-1}w}{d\xi^{m-1}}(0;M_m,N_m)=w^{m-1},$$

where $M_m=(a_0,\dots,a_{m-1})\in\mathbb{C}^m$ and $N_m=(w^0,\dots,w^{m-1})\in\mathbb{C}^m$.

Introduce the following notation:

$$(2.29)\qquad \overline{\Lambda}_m(M_m):=\max\{\operatorname{Re}\lambda^1(M_m),\dots,\operatorname{Re}\lambda^m(M_m)\},$$

where $\lambda^1(M_m)$, …, $\lambda^m(M_m)$ are the roots of the characteristic polynomial of Eq. (2.27) (see also (2.6)),

$$\Pi_m(C):=\big\{(x_1,\dots,x_m)\in\mathbb{C}^m:|x_1|\le C,\ \dots,\ |x_m|\le C\big\}.$$

Lemma 2.3. *Let $C_a\ge 0$ and $C_w\ge 0$. Then there exists $\tilde{C}_m\ge 0$ such that*

$$(2.30)\qquad \left|\frac{d^i w}{d\xi^i}(\xi;M_m,N_m)\right|\le \tilde{C}_m(1+\xi^{m-1})e^{\overline{\Lambda}_m(M_m)\xi}$$

for all $(i,\xi,M_m,N_m)\in\{0,\dots,m-1\}\times[0,+\infty)\times\Pi_m(C_a)\times\Pi_m(C_w)$, where $w(\xi;M_m,N_m)$ is a solution of the problem (2.27)–(2.28)*.*

Proof. Apply induction by m. Denote by S_m the assertion of the lemma. Since the validity of S_1 is obvious, it remains to verify that for any integer $m\ge 1$ the assertion S_m implies S_{m+1}.

Consider the Cauchy problem for the equation of $(m+1)$th order:

$$(2.31)\qquad \frac{d^{m+1}w}{d\xi^{m+1}}=a_m\frac{d^m w}{d\xi^m}+\cdots+a_0w,\quad \xi\in(0,+\infty);$$

$$(2.32)\qquad w(0;M_{m+1},N_{m+1})=w^0,\quad \dots,\quad \frac{d^m w}{d\xi^m}(0;M_{m+1},N_{m+1})=w^m$$

and fix arbitrary nonnegative C_a and C_w. The assertion S_{m+1} is as follows: there exists sufficiently large $\tilde{C}_{m+1}$ such that

$$\left|\frac{d^i w}{d\xi^i}(\xi; M_{m+1}, N_{m+1})\right| \leq \tilde{C}_{m+1}(1+\xi^m)e^{\overline{\Lambda}_{m+1}(M_{m+1})\xi}$$

for all

$$(i, \xi, M_{m+1}, N_{m+1}) \in \overline{0,m} \times [0,+\infty) \times \Pi_{m+1}(C_a) \times \Pi_{m+1}(C_w),$$

where $w(\xi; M_{m+1}, N_{m+1})$ is a solution of the problem (2.31)–(2.32).

To verify the validity of S_{m+1} (under the validity of S_m), we perform the change of the dependent variable in the problem (2.31)–(2.32):

$$w(\xi; M_{m+1}, N_{m+1}) = e^{\lambda^*(M_{m+1})\xi} u(\xi; M_{m+1}, N_{m+1}), \tag{2.33}$$

where λ^* is the function, which to each $M_{m+1} = (a_0, \dots, a_m) \in \mathbb{C}^{m+1}$ puts in correspondence an arbitrary root $\lambda_i(M_{m+1})$ of the characteristic polynomial of Eq. (2.31) whose real part $\operatorname{Re}\lambda_i(M_{m+1})$ coincides with $\overline{\Lambda}_{m+1}(M_{m+1})$:

$$\operatorname{Re}\lambda^*(M_{m+1}) = \overline{\Lambda}_{m+1}(M_{m+1}). \tag{2.34}$$

For the new function $u(\xi; M_{m+1}, N_{m+1})$ we obtain the following initial-value problem:

$$\frac{d^{m+1}u}{d\xi^{m+1}} = b_m(M_{m+1})\frac{d^m u}{d\xi^m} + \dots + b_1(M_{m+1})\frac{du}{d\xi}, \quad \xi \in (0,+\infty); \tag{2.35}$$

$$\begin{gathered} u(0; M_{m+1}, N_{m+1}) = u^0(M_{m+1}, N_{m+1}), \quad \dots, \\ \frac{d^m u}{d\xi^m}(0; M_{m+1}, N_{m+1}) = u^m(M_{m+1}, N_{m+1}), \end{gathered} \tag{2.36}$$

where

$$\begin{gathered} b_i(M_{m+1}) = \tilde{b}_i(\lambda^*(M_{m+1}), M_{m+1}), \\ u^i(M_{m+1}, N_{m+1}) = \tilde{u}^i(\lambda^*(M_{m+1}), N_{m+1}), \end{gathered}$$

$\tilde{b}_i$ and $\tilde{u}^i$ are known functions of λ^*, $M_{m+1} = (a_0, \dots, a_m)$, and $N_{m+1} = (w^0, \dots, w^m)$ (they are polynomial functions with respect to λ^* and linear functions with respect to $a_0, \dots, a_m$ and $w^0, \dots, w^m$). The characteristic polynomial of Eq. (2.35) for any $M_{m+1} \in \mathbb{C}^{m+1}$ has the zero root (see (2.37)); hence the coefficient $b_0(M_{m+1})$ of the function u is identical zero.

Due to (2.33), for each $M_{m+1} \in \mathbb{C}^{m+1}$, the roots of the characteristic polynomial of Eq. (2.35) are as follows:

$$\mu_i(M_{m+1}) := \lambda_i(M_{m+1}) - \lambda^*(M_{m+1}), \quad i \in \{1, \dots, m+1\}. \tag{2.37}$$

This and the definition of $\lambda^*(M_{m+1})$ imply

$$\operatorname{Re}\mu_i(M_{m+1}) \leq 0 \tag{2.38}$$

for all $(i, M_{m+1}) \in \{1, \dots, m+1\} \times \mathbb{C}^{m+1}$.

Since we assume that the points $M_{m+1} = (a_0, \dots, a_m)$ belong to the finite parallelepiped $P_{m+1}(C_a)$, all roots $\lambda_i(M_{m+1})$ of the characteristic polynomial of Eq. (2.31) satisfy the condition

$$\left|\lambda_i(M_{m+1})\right| \leq 1 + C_a \tag{2.39}$$

(see, e.g., [5]). Then there exist nonnegative constants C_b and C_u such that

$$\left|b_i(M_{m+1})\right| \leq C_b, \quad \left|u^i(M_{m+1}, N_{m+1})\right| \leq C_u \tag{2.40}$$

for all $(i, M_{m+1}, N_{m+1}) \in \overline{0,m} \times \Pi_{m+1}(C_a) \times \Pi_{m+1}(C_w)$.

We reduce the order of Eq. (2.35) by the following change of the dependent variable:

$$\frac{du}{d\xi}(\xi; M_{m+1}, N_{m+1}) = v(\xi; M_{m+1}, N_{m+1}). \tag{2.41}$$

The function $v(\xi; M_{m+1}, N_{m+1})$ satisfies the following initial-value problem:

$$\begin{gathered} \frac{d^m v}{d\xi^m} = b_m(M_{m+1})\frac{d^{m-1}v}{d\xi^{m-1}} + \cdots + b_1(M_{m+1})v, \quad \xi \in (0, +\infty); \\ v(0; M_{m+1}, N_{m+1}) = u^1(M_{m+1}, N_{m+1}), \quad \ldots, \\ \frac{d^{m-1}v}{d\xi^{m-1}}(0; M_{m+1}, N_{m+1}) = u^m(M_{m+1}, N_{m+1}). \end{gathered} \tag{2.42}$$

Let $\nu_1(M_{m+1})$, ..., $\nu_m(M_{m+1})$ be roots of the characteristic polynomial of Eq. (2.42). Since each of the roots $\nu_i(M_{m+1})$ is at the same time a root of the characteristic polynomial of Eq. (2.35), they, similarly to $\mu_i(M_{m+1})$ (see (2.38)), satisfy the following inequality for all $M_{m+1} \in \mathbb{C}^{m+1}$:

$$\operatorname{Re}\nu_i(M_{m+1}) \le 0. \tag{2.43}$$

Note also that

$$\begin{gathered} M_m = (b_1(M_{m+1}), \ldots, b_m(M_{m+1})) \in \Pi_m(C_b), \\ N_m = (u^1(M_{m+1}, N_{m+1}), \ldots, u^m(M_{m+1}, N_{m+1})) \in \Pi_m(C_u) \end{gathered}$$

for all $M_{m+1} \in \Pi_{m+1}(C_a)$ and $N_{m+1} \in \Pi_{m+1}(C_w)$ (see (2.40)). The last estimates allow one to apply the inductive hypothesis to the function v: there exists $\tilde{C}_m \ge 0$ such that

$$\left|\frac{d^i v}{d\xi^i}(\xi; M_{m+1}, N_{m+1})\right| \le \tilde{C}_m(1 + \xi^{m-1}) \tag{2.44}$$

for all $(i, \xi, M_{m+1}, N_{m+1}) \in \{0, \ldots, m-1\} \times [0, +\infty) \times \Pi_{m+1}(C_a) \times \Pi_{m+1}(C_w)$ (see (2.30), (2.29), and (2.43)).

From (2.41) and (2.44) we obtain for the first m derivatives of the function u the relation

$$\left|\frac{d^i u}{d\xi^i}(\xi; M_{m+1}, N_{m+1})\right| = \left|\frac{d^{i-1} v}{d\xi^{i-1}}(\xi; M_{m+1}, N_{m+1})\right| \le \tilde{C}_m(1 + \xi^{m-1}); \tag{2.45}$$

here up to the end of the proof we assume that $(\xi, M_{m+1}, N_{m+1}) \in [0, +\infty) \times \Pi_{m+1}(C_a) \times \Pi_{m+1}(C_w)$.

To estimate the function u, we integrate (2.41) and then apply (2.36), (2.40), and (2.44) and the monotonicity property and the estimate of the absolute value of the definite integral:

$$\begin{aligned} &\left|u(\xi; M_{m+1}, N_{m+1})\right| \\ &= \left|u(0; M_{m+1}, N_{m+1}) + \int_0^\xi v(\zeta; M_{m+1}, N_{m+1})d\zeta\right| \le \left|u^0(M_{m+1}, N_{m+1})\right| \\ &\quad + \int_0^\xi \left|v(\zeta; M_{m+1}, N_{m+1})\right|d\zeta \le C_u + \int_0^\xi \tilde{C}_m(1 + \xi^{m-1})d\zeta \le \tilde{C}_u(1 + \xi^m) \end{aligned} \tag{2.46}$$

for sufficiently large $\tilde{C}_u$.

Now we turn to w. From (2.33), (2.46), (2.45), (2.39), and (2.34) and the Leibniz formula for the ith derivative of the product of two functions, for each $i \in \overline{0, m}$ and sufficiently large $\tilde{C}_{m+1}$

we have

$$\left|\frac{d^i w}{d\xi^i}(\xi; M_{m+1}, N_{m+1})\right|$$
$$\leq \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} \left|u^{(j)}(\xi; M_{m+1}, N_{m+1})\right| \left|\lambda^*(M_{m+1})\right|^{i-j} \left|e^{\lambda^*(M_{m+1})\xi}\right|$$
$$\leq \left[\tilde{C}_u(1+\xi^m)(1+C_a)^i + \sum_{j=1}^{i} \frac{i!}{j!(i-j)!}\tilde{C}_m(1+\xi^{m-1})(1+C_a)^{i-j}\right] e^{\operatorname{Re}\lambda^*(M_{m+1})\xi}$$
$$\leq \tilde{C}_{m+1}(1+\xi^m) e^{\overline{\Lambda}_{m+1}(M_{m+1})\xi}.$$

The proof is complete. ∎

Corollary 2.4. *There exist* $\chi > 0$ *and* $C_\Phi > 0$ *such that*

$$|\Phi^1_{\xi^i}(\xi; x)|,\ |\Phi^m_{\xi^i}(\xi; x)| \leq C_\Phi(1+\xi^{m-1})e^{-\chi\xi} \tag{2.47}$$

for all $(i, \xi, x) \in \{0, \ldots, m-1\} \times [0, +\infty) \times [0, X]$, *where* $\Phi^1(\xi; x)$ *and* $\Phi^m(\xi; x)$ *are the solutions of the problems* (2.21)–(2.22) *and* (2.23)–(2.24), *respectively.*

Proof. To prove the estimate (2.47) it suffices to set

$$\chi := -\max_{[0,X]} \max\left\{\operatorname{Re}\lambda_1(x), \ldots, \operatorname{Re}\lambda_m(x)\right\}$$

(see (2.7)) and apply the Weierstrass extreme-value theorem on the boundedness of a continuous function for a_i and Lemma 2.3. ∎

3. CONSTRUCTION AND PROOF OF CONVERGENCE OF ITERATIVE SEQUENCE

Let

$$O(\vartheta, C_0; \varepsilon) := \left\{(z^1, \ldots, z^m) \in C_m\left[0, \frac{X}{\varepsilon}\right] : \forall \xi \in \left[0, \frac{X}{\varepsilon}\right]\right.$$
$$\left.(z^1(\xi), \ldots, z^m(\xi)) \in [-C_0, +C_0]^m\right\}$$

be a closed C_0-neighborhood of the vector-function $(z^1, \ldots, z^m) \equiv (0, \ldots, 0) =: \vartheta$ in the space $C_m[0, X/\varepsilon]$.

Proposition 3.1. *There exist* $\varepsilon_0 > 0$ *and* $C_0 \geq 0$ *such that*

$$\widehat{A}(C_0; \varepsilon) : O(\vartheta, C_0; \varepsilon) \to O(\vartheta, C_0; \varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_0]$, *where* $\widehat{A}(C_0; \varepsilon) = \big(\widehat{A}_1(C_0; \varepsilon), \ldots, \widehat{A}_m(C_0; \varepsilon)\big)$ *is the restriction of the operator* $\widehat{A}(\varepsilon)$ *to* $O(\vartheta, C_0; \varepsilon)$.

Proof. We fix arbitrary $\varepsilon > 0$ and $C_0 \geq 0$, apply the operators $\widehat{A}_i(C_0; \varepsilon)$ to an arbitrary vector-function $(z^1(\xi), \ldots, z^m(\xi)) \in O(\vartheta, C_0; \varepsilon)$ and, taking into account (2.25) and (2.47), estimate the result obtained:

$$\text{(3.1)}\quad \left|\widehat{A}_i(C_0;\varepsilon)[z^1,\dots,z^m](\xi)\right| \le C_\Phi e^{-\chi\xi}\Big\{C_0\int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]\big[|a_{m-1}(\varepsilon\zeta)-a_{m-1}(\varepsilon\xi)|+\dots + |a_0(\varepsilon\zeta)-a_0(\varepsilon\xi)|\big]d\zeta + \int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]\big[|f(\zeta;\varepsilon)|+|\bar{y}'(\varepsilon\zeta)|\big]d\zeta\Big\},\quad i=\overline{1,m}$$

(the term $|\bar{y}'(\varepsilon\zeta)|$ only for $m\ge 2$).

For the first integral in (3.1) we have

$$\text{(3.2)}\quad \int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]\big[|a_{m-1}(\varepsilon\zeta)-a_{m-1}(\varepsilon\xi)|+\dots+|a_0(\varepsilon\zeta)-a_0(\varepsilon\xi)|\big]d\zeta$$
$$= \varepsilon\int_0^\xi e^{\chi\zeta}\big[(\xi-\zeta)+(\xi-\zeta)^m\big]\Big\{\big|a'_{m-1}(\varepsilon[(1-\theta_{m-1})\zeta+\theta_{m-1}\xi])\big|+\dots + \big|a'_0(\varepsilon[(1-\theta_0)\zeta+\theta_0\xi])\big|\Big\}d\zeta$$
$$\le \varepsilon\big\{\|a'_{m-1}(x)\|+\dots+\|a'_0(x)\|\big\}\int_0^\xi e^{\chi\zeta}\big[(\xi-\zeta)+(\xi-\zeta)^m\big]d\zeta$$
$$= \varepsilon\alpha\Big\{\tfrac{1}{\chi^2}\big[e^{\chi\xi}-1-\chi\xi\big]+\tfrac{m!}{\chi^{m+1}}\big[e^{\chi\xi}-1-\chi\xi-\dots-\tfrac{1}{m!}(\chi\xi)^m\big]\Big\}\le\varepsilon\beta e^{\chi\xi},$$

where $\theta_i=\theta_i(\varepsilon\zeta,\varepsilon\xi)\in(0,1)$, $\|\cdot\|$ is the norm of the space $C[0,X]$, and

$$\alpha:=\|a'_{m-1}(x)\|+\dots+\|a'_0(x)\|,\quad \beta:=\alpha\frac{\chi^{m-1}+m!}{\chi^{m+1}}.$$

For the second integral in (3.1) we have (see (2.18) and (2.12))

$$\text{(3.3)}\quad \int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]\big[|f(\zeta;\varepsilon)|+|\bar{y}'(\varepsilon\zeta)|\big]d\zeta \le \int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]$$
$$\times\Big\{\tilde{C}\big[|a'_{m-1}(\varepsilon\theta_{m-1}\zeta)|+\dots+|a'_0(\varepsilon\theta_0\zeta)|\big](\zeta+\zeta^m)e^{-\chi\zeta}+|\bar{y}'(\varepsilon\zeta)|\Big\}d\zeta$$
$$\le\Big\{\tilde{C}\alpha\max_{\zeta>0}\big[(\zeta+\zeta^m)e^{-\chi\zeta}\big]+\|\bar{y}'(x)\|\Big\}\int_0^\xi e^{\chi\zeta}\big[1+(\xi-\zeta)^{m-1}\big]d\zeta$$
$$=\Big\{\tilde{C}\alpha\max_{\zeta>0}\big[(\zeta+\zeta^m)e^{-\chi\zeta}\big]+\|\bar{y}'(x)\|\Big\}\Big\{\tfrac{1}{\chi}\big[e^{\chi\xi}-1\big] + \tfrac{(m-1)!}{\chi^m}\big[e^{\chi\xi}-1-\chi\xi-\dots-\tfrac{1}{(m-1)!}(\chi\xi)^{m-1}\big]\Big\}\le\gamma e^{\chi\xi},$$

where $\theta_i=\theta_i(\varepsilon\zeta)\in(0,1)$,

$$\gamma:=\Big\{\tilde{C}\alpha\max_{\zeta>0}\big[(\zeta+\zeta^m)e^{-\chi\zeta}\big]+\|\bar{y}'(x)\|\Big\}\tfrac{\chi^{m-1}+(m-1)!}{\chi^m}.$$

From (3.1), (3.2), and (3.3) we see that if C_0 and ε satisfy the inequalities

$$\text{(3.4)}\quad 0\le C_0\varepsilon C_\Phi\beta+C_\Phi\gamma\le C_0,$$

hence $\widehat{A}(C_0;\varepsilon)[z^1,\dots,z^m](\xi)\in O(\vartheta,C_0;\varepsilon)$.

We set

$$\text{(3.5)}\quad \varepsilon_0:=\gamma_0(C_\Phi\beta)^{-1},$$

where γ_0 is an arbitrary number from the interval $(0,1)$ (if $\beta = 0$, i.e., $a_i(x) = \mathrm{const}$ on $[0, X]$, then $\varepsilon_0 := +\infty$) and $C_0 := C_\Phi\gamma/(1-\gamma_0)$. Then the inequalities (3.4) hold for any $\varepsilon \in (0, \varepsilon_0]$. ∎

Assume that for any fixed positive ε and any $\varphi_1(\xi) = (z_1^1(\xi), \dots, z_1^m(\xi))$ and $\varphi_2(\xi) = (z_2^1(\xi), \dots, z_2^m(\xi))$ from $C_m[0, X/\varepsilon]$, the distance ρ_ε between φ_1 and φ_2 is defined:

$$\rho_\varepsilon(\varphi_1, \varphi_2) := \|\varphi_2 - \varphi_1\|_{C_m[0,X/\varepsilon]} := \max_{\xi\in X(\varepsilon)} \max_{1\le i\le m} |z_2^i(\xi) - z_1^i(\xi)|, \tag{3.6}$$

where $X(\varepsilon) := [0, X/\varepsilon]$. Note that $C_m[0, X/\varepsilon]$ and $O(\vartheta, C_0; \varepsilon)$ with ρ_ε defined above are complete metric spaces.

Proposition 3.2. *The operator $\widehat{A}(\varepsilon)$ is a contractive operator for any $\varepsilon \in (0, \varepsilon_0]$.*

Proof. Let ρ_ε be the metric (3.6) of the space $C_m[0, X/\varepsilon]$. Take two arbitrary functions $\varphi_1(\xi) = (z_1^1(\xi), \dots, z_1^m(\xi))$ and $\varphi_2(\xi) = (z_2^1(\xi), \dots, z_2^m(\xi))$ from this space and, taking into account (2.25) and (2.47), estimate the distance between $\widehat{A}(\varepsilon)[\varphi_1]$ and $\widehat{A}(\varepsilon)[\varphi_2]$:

$$\begin{aligned} \rho_\varepsilon\Big(\widehat{A}(\varepsilon)[\varphi_1], \widehat{A}(\varepsilon)[\varphi_2]\Big) &= \max_{\xi\in X(\varepsilon)} \max_{1\le i\le m} \Big|\widehat{A}_i(\varepsilon)[\varphi_2](\xi) - \widehat{A}_i(\varepsilon)[\varphi_1](\xi)\Big| \\ &= \max_{\xi\in X(\varepsilon)} \max_{1\le i\le m} \Big| \int_0^\xi \Phi^m_{\xi^{i-1}}(\xi-\zeta;\varepsilon\xi)\Big\{ \big[a_{m-1}(\varepsilon\zeta) - a_{m-1}(\varepsilon\xi)\big]\big[z_2^m(\zeta) - z_1^m(\zeta)\big] + \dots \\ &\qquad + \big[a_0(\varepsilon\zeta) - a_0(\varepsilon\xi)\big]\big[z_2^1(\zeta) - z_1^1(\zeta)\big]\Big\} d\zeta\Big| \\ &\le \rho_\varepsilon(\varphi_1,\varphi_2) C_\Phi \max_{\xi\in X(\varepsilon)} \int_0^\xi e^{\chi(\zeta-\xi)}\big[1 + (\xi-\zeta)^{m-1}\big] \\ &\qquad \times \Big[\big|a_{m-1}(\varepsilon\zeta) - a_{m-1}(\varepsilon\xi)\big| + \dots + \big|a_0(\varepsilon\zeta) - a_0(\varepsilon\xi)\big|\Big] d\zeta. \end{aligned} \tag{3.7}$$

From (3.7), (3.2), and (3.5) we conclude that for any $\varepsilon \in (0, \varepsilon_0]$ the contraction coefficient $k(\varepsilon)$ of the operator $\widehat{A}(\varepsilon)$ satisfies the estimate

$$k(\varepsilon) \le \varepsilon C_\Phi \beta = \gamma_0 \frac{\varepsilon}{\varepsilon_0} \le \gamma_0 < 1. \tag{3.8}$$

The proof is complete. ∎

Since the contraction coefficient $k(C_0;\varepsilon)$ of the operator $\widehat{A}(C_0;\varepsilon)$ certainly does not exceed $k(\varepsilon)$, the estimate (3.8) is also valid for it:

$$k(C_0;\varepsilon) \le \gamma_0 \frac{\varepsilon}{\varepsilon_0} \le \gamma_0 < 1. \tag{3.9}$$

Thus, we can apply the Banach fixed-point theorem to the operator $\widehat{A}(C_0;\varepsilon)$ and conclude that for any $\varepsilon \in (0, \varepsilon_0]$ the solution $(z^1(\xi;\varepsilon), \dots, z^m(\xi;\varepsilon)) =: \varphi(\xi;\varepsilon)$ of the problem (2.14)–(2.17) (which is equivalent to Eq. (2.26)) belongs to $O(\vartheta, C_0; \varepsilon)$. We emphasize that the existence and the global uniqueness (i.e., uniqueness on the set $[0, X/\varepsilon] \times \mathbb{R}^m$) of the solution $\varphi(\xi;\varepsilon)$ (for all $\varepsilon \in \mathbb{R}$) are immediately implied by the linearity of the problem (2.14)–(2.17) (the linearity of Eq. (2.26)).

The contractive property of the operator $\widehat{A}(C_0;\varepsilon)$ also allows one to construct the iterative sequence $\varphi_n(\xi;\varepsilon) = (z_n^1(\xi;\varepsilon), \dots, z_n^m(\xi;\varepsilon))$ converging with respect to the norm of the space $C_m[0, X/\varepsilon]$ to the exact solution $\varphi(\xi;\varepsilon)$ of the problem (2.14)–(2.17) uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$:

$$\big\|\varphi - \varphi_n\big\|_{C_m[0,X/\varepsilon]} := \max_{\xi\in X(\varepsilon)} \max_{1\le i\le m} \big|z^i(\xi;\varepsilon) - z_n^i(\xi;\varepsilon)\big| \to 0, \quad n\to\infty.$$

We set $\varphi_0(\xi;\varepsilon) \equiv (0,\dots,0) =: \vartheta$. Since $\varphi(\xi;\varepsilon) \in O(\vartheta, C_0;\varepsilon)$, we have

$$\big\|\varphi(\xi;\varepsilon) - \varphi_0(\xi;\varepsilon)\big\|_{C_m[0,X/\varepsilon]} = \big\|\varphi(\xi;\varepsilon)\big\|_{C_m[0,X/\varepsilon]} \le C_0 \tag{3.10}$$

for all $\varepsilon \in (0,\varepsilon_0]$.

Further, for any natural n we set

$$\varphi_n(\xi;\varepsilon) := \widehat{A}(C_0;\varepsilon)[\varphi_{n-1}](\xi;\varepsilon). \tag{3.11}$$

Then, taking into account (3.9) and (3.10), we have for each $n \in \{0\} \cup \mathbb{N} =: \mathbb{N}_0$ and each $\varepsilon \in (0,\varepsilon_0]$

$$\begin{aligned}\big\|\varphi(\xi;\varepsilon) - \varphi_n(\xi;\varepsilon)\big\|_{C_m[0,X/\varepsilon]} \\ \le k(C_0;\varepsilon)^n \big\|\varphi(\xi;\varepsilon) - \varphi_0(\xi;\varepsilon)\big\|_{C_m[0,X/\varepsilon]} \le C_0\left(\gamma_0\frac{\varepsilon}{\varepsilon_0}\right)^n.\end{aligned} \tag{3.12}$$

We turn to the problem (2.1)–(2.2). Due to (2.13), we obtain the iterative sequences $y_n^1(x;\varepsilon)$, …, $y_n^m(x;\varepsilon)$, respectively, for the solution $y(x;\varepsilon)$ of the original problem and its derivatives $\frac{d}{dx}y(x;\varepsilon)$, …, $\frac{d^{m-1}}{dx^{m-1}}y(x;\varepsilon)$:

$$y_n^1(x;\varepsilon) := \tilde{y}\left(\frac{x}{\varepsilon}, x\right) + \varepsilon z_n^1\left(\frac{x}{\varepsilon};\varepsilon\right), \quad n \in \mathbb{N}_0; \tag{3.13}$$

$$y_n^i(x;\varepsilon) := \varepsilon^{1-i}\Pi^{(i-1)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^{2-i} z_n^i\left(\frac{x}{\varepsilon};\varepsilon\right), \quad (i,n) \in \overline{2,m} \times \mathbb{N}_0. \tag{3.14}$$

For $n \ge 1$, the values $y_n^i(x;\varepsilon)$ can be immediately expressed through $y_{n-1}^i(x;\varepsilon)$:

$$\begin{aligned} y_n^1(x;\varepsilon) &= \tilde{y}\left(\frac{x}{\varepsilon}, x\right) + \varepsilon \widehat{A}_1(C_0;\varepsilon)[z_{n-1}^1,\dots,z_{n-1}^m]\left(\frac{x}{\varepsilon};\varepsilon\right) \\ &=: \widehat{B}_1(\varepsilon)[y_{n-1}^1,\dots,y_{n-1}^m](x;\varepsilon), \\ y_n^i(x;\varepsilon) &= \varepsilon^{1-i}\Pi^{(i-1)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^{2-i}\widehat{A}_i(C_0;\varepsilon)[z_{n-1}^1,\dots,z_{n-1}^m]\left(\frac{x}{\varepsilon};\varepsilon\right) \\ &=: \widehat{B}_i(\varepsilon)[y_{n-1}^1,\dots,y_{n-1}^m](x;\varepsilon), \quad i \in \overline{2,m}, \end{aligned}$$

where

$$\begin{gathered} z_{n-1}^1(\xi;\varepsilon) = \varepsilon^{-1}\Big[y_{n-1}^1(\varepsilon\xi;\varepsilon) - \tilde{y}(\xi,\varepsilon\xi)\Big], \\ z_{n-1}^i(\xi;\varepsilon) = \varepsilon^{i-2} y_{n-1}^i(\varepsilon\xi;\varepsilon) - \varepsilon^{-1}\Pi^{(i-1)}(\xi), \quad i \in \overline{2,m} \end{gathered}$$

(see (3.13), (3.14), and (3.11)) or briefly

$$\psi_n(x;\varepsilon) := \widehat{B}(\varepsilon)[\psi_{n-1}](x;\varepsilon),$$

where $\psi_n(x;\varepsilon) := (y_n^1(x;\varepsilon),\dots,y_n^m(x;\varepsilon))$ and $\widehat{B}(\varepsilon) := (\widehat{B}_1(\varepsilon),\dots\widehat{B}_m(\varepsilon))$. Note that the operator $\widehat{B}(\varepsilon)$ is contractive for $\varepsilon \in (0,\varepsilon_0]$ (i.e., for the same ε as $\widehat{A}(C_0;\varepsilon)$) and the operator $\widehat{B}(\varepsilon)$ satisfies the condition

$$\widehat{B}(\varepsilon) : O(\tilde{\psi}, C_0;\varepsilon) \to O(\tilde{\psi}, C_0;\varepsilon)$$

for $\varepsilon \in (0, \varepsilon_0]$, where

$$O(\tilde{\psi}, C_0; \varepsilon) := \Big\{ (y^1, \ldots, y^m) \in C_m[0, X] : \forall x \in [0, X]$$
$$y^1(x) \in \left[\tilde{y}\left(\frac{x}{\varepsilon}, x\right) - \varepsilon C_0, \tilde{y}\left(\frac{x}{\varepsilon}, x\right) + \varepsilon C_0 \right],$$
$$y^2(x) \in \left[\varepsilon^{-1}\Pi'\left(\frac{x}{\varepsilon}\right) - C_0, \varepsilon^{-1}\Pi'\left(\frac{x}{\varepsilon}\right) + C_0 \right], \ldots,$$
$$y^m(x) \in \left[\varepsilon^{1-m}\Pi^{(m-1)}\left(\frac{x}{\varepsilon}\right) - \varepsilon^{2-m}C_0, \varepsilon^{1-m}\Pi^{(m-1)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^{2-m}C_0 \right] \Big\}$$

is a closed $(\varepsilon C_0, C_0, \ldots, \varepsilon^{2-m}C_0)$-neighborhood of the vector-function

$$\tilde{\psi}\left(\frac{x}{\varepsilon}; \varepsilon\right) := \left(\tilde{y}\left(\frac{x}{\varepsilon}, x\right), \varepsilon^{-1}\Pi'\left(\frac{x}{\varepsilon}\right), \ldots, \varepsilon^{1-m}\Pi^{(m-1)}\left(\frac{x}{\varepsilon}\right) \right)$$

in the space $C_m[0, X]$.

We estimate the accuracy of the approximation of $\frac{d^{i-1}}{dx^{i-1}}y(x;\varepsilon)$ by $y_n^i(x;\varepsilon)$. For each $n \in \mathbb{N}_0$ and $\varepsilon \in (0, \varepsilon_0]$ we have (see (3.13), (3.14), (2.13), and (3.12)):

$$\left\| y(x;\varepsilon) - y_n^1(x;\varepsilon) \right\| = \left\| y(x;\varepsilon) - \tilde{y}\left(\frac{x}{\varepsilon}, x\right) - \varepsilon z_n^1\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\|$$
$$= \varepsilon \left\| z^1\left(\frac{x}{\varepsilon}; \varepsilon\right) - z_n^1\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\| \le \varepsilon \left\| \varphi\left(\frac{x}{\varepsilon}; \varepsilon\right) - \varphi_n\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\|_{C_m[0,X]} \le C_0 \varepsilon \left(\gamma_0 \frac{\varepsilon}{\varepsilon_0} \right)^n,$$
$$\left\| \frac{d^{i-1}}{dx^{i-1}} y(x;\varepsilon) - y_n^i(x;\varepsilon) \right\| = \left\| \frac{d^{i-1}}{dx^{i-1}} y(x;\varepsilon) - \varepsilon^{1-i}\Pi^{(i-1)}\left(\frac{x}{\varepsilon}\right) - \varepsilon^{2-i} z_n^i\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\|$$
$$= \varepsilon^{2-i} \left\| z^i\left(\frac{x}{\varepsilon}; \varepsilon\right) - z_n^i\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\| \le \varepsilon^{2-i} \left\| \varphi\left(\frac{x}{\varepsilon}; \varepsilon\right) - \varphi_n\left(\frac{x}{\varepsilon}; \varepsilon\right) \right\|_{C_m[0,X]}$$
$$\le C_0 \varepsilon^{2-i} \left(\gamma_0 \frac{\varepsilon}{\varepsilon_0} \right)^n, \quad i \in \overline{2, m}.$$

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