The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 15, Issue 2, Article 5, pp. 1-9, 2018

EXISTENCE OF OPTIMAL PARAMETERS FOR DAMPED SINE-GORDON EQUATION WITH VARIABLE DIFFUSION COEFFICIENT AND NEUMANN BOUNDARY CONDITIONS

N. THAPA

Received 16 March, 2018; accepted 05 July, 2018; published 1 August, 2018.

DEPARTMENT OF MATHEMATICAL SCIENCES, CAMERON UNIVERSITY, 2800 WEST GORE BLVD, 73505 LAWTON, OKLAHOMA, USA.
nthapa@cameron.edu
URL: http://www.cameron.edu/~nthapa/

ABSTRACT. The parameter identification problem for sine-Gordon equation is of a major interests among mathematicians and scientists. In this work we the consider sine-Gordon equation with variable diffusion coefficient and Neumann boundary data. We show the existence and uniqueness of weak solution for sine-Gordon equation. Then we show that the weak solution continuously depends on parameters. Finally we show the existence of optimal set of parameters.

Key words and phrases: Identification problem; Weak solution; Approximate solution; Optimal parameters.

2000 *Mathematics Subject Classification.* 49J50, 49J20.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. Let us consider the following sine-Gordon equation with variable coefficient $\beta(x)$ with Neumann boundary data.

$$
\begin{aligned}
&u_{tt}(x,t) + \alpha u_t(x,t) - \nabla(\beta(x)\nabla u(x,t)) + \delta \sin u(x,t) = f(x,t);\ (t,x) \in Q \\
&\frac{\partial u}{\partial n}(t,x)|_{x\in\Gamma} = 0,\ t \in\ (0,T) \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega
\end{aligned}
\tag{1.1}
$$

where $T > 0,\ Q = (0,T) \times \Omega,\ f \in L^2(Q),\ u_0 \in V = H^1(\Omega)$ and $u_1 \in H = L^2(\Omega)$. The diffusion coefficient $\beta(x) \in \mathcal{B} = \{\beta \in L^\infty(\Omega) : 0 < m \le \beta(x) \le M\ a.e.\ in\ \Omega\}$. Throughout this work we assume that $\mathcal{B}$ is equipped with $L^1(\Omega)$ topology.

For equation (1.1) with constant parameters and Dirischlet boundary conditions, Ha and Gutman estimated the parameters. For details, see [6]. Similarly for constant parameters with Neumann boundary data, Thapa estimated parameters. For details, see [9]. In this paper we consider $\beta(x) \in L^\infty(\Omega)$ along with Neumann boundary data and establish the optimality conditions such that equation (1.1) exhibits the desired behavior listed below.

Let

$$
\mathcal{P}_{ad} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times \mathcal{B} \times [\delta_{min}, \delta_{max}]\}, \tag{1.2}
$$

Define the cost functional $J(q)$ by

$$
J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{1.3}
$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \ge 0$ for $i = 1,2$ with $k_1+k_2 > 0$. The data z_d^1 and z_d^2 can be thought of as the targeted behavior of (1.1). We claim that there exist $q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}_{ad}$ such that

$$
J(q^*) = \inf_{q\in\mathcal{P}_{ad}} J(q) \tag{1.4}
$$

Let $q \to u(q)$ from $\mathcal{P}_{ad} \to C([0,T];H)$ be the solution map. The existence and uniqueness of solution map is established in Section 2. In Section 3 we establish the continuity of solution map with respect to parameters so that the equation (1.4) has a solution if the minimization is restricted to a compact subset of $\mathcal{P}_{ad}$.

2. EXISTENCE AND UNIQUENESS OF WEAK SOLUTION

In this section, we use the standard argument outlined in [6, 7, 9] for the existence and uniqueness of weak solution of (1.1). Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$
(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad |\phi| = (\phi,\phi)^{\frac{1}{2}} \tag{2.1}
$$

for all ϕ , $\psi \in L^2(\Omega)$. Let $V = H^1(\Omega)$ be a Hilbert space with following inner product and norm

$$
((\phi, \psi)) = (\phi, \psi) + (\nabla\phi, \nabla\psi), \qquad \|\phi\| = ((\phi,\phi))^{\frac{1}{2}} \tag{2.2}
$$

for all ϕ , $\psi \in H^1(\Omega)$. The dual H' is identified with H leading to $V \subset H \subset V'$ with compact, continuous, and dense injections. For details, see [1] Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1\|w\| \quad \text{for any} \quad w \in V. \tag{2.3}$$

Given $\beta \in \mathcal{B}$, we define the following bilinear, continuous, and coercive form.

$$a_\beta(u, v) = \int_\Omega uvdx + \int_\Omega \beta(x)\nabla u(x)\nabla v(x)dx \tag{2.4}$$

Let $< u, v >_{V,V'}$ denotes the duality pairing between V and V' and the assoicated linear operator form V to V' defined by $< a_\beta u, v > = a_\beta(u, v)$ is an isomorphim from V onto V'. Let $\{\lambda_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ are nonzero eigenvalues and eigenfunctions for the operator $-\Delta + I$ defined in V such that $\{w_k\}_{k=1}^\infty$ forms an orthonormal basis in H. Then the functions $\{\frac{w_k}{\sqrt{\lambda_k}}\}_{k=1}^\infty$ form an orthonormal basis in V. For details, see [2]. From now on, the dependency on x is supressed and we use $'$ and $''$ for the time derivatives.
Let

$$W(0, T) = \{u \,:\, u \in L^2(0, T; V), u' \in L^2(0, T; H), u'' \in L^2(0, T; V')\}. \tag{2.5}$$

u' and u'' are the derivatives in the distributional sense. That is, $u' \in L^2(0, T; H)$ is derivative of $u \in L^2(0, T; V)$ in the distributional sense if for any $\phi \in C_0^\infty(0, T)$ and $v \in V$

$$\int_0^T (u'(t), v)\phi(t)dt = -\int_0^T (u(t), v)\phi'(t)dt \tag{2.6}$$

similarly, $u'' \in L^2(0, T; V')$ is second derivative of $u \in L^2(0, T; V)$ in the distributional sense if for any $\phi \in C_0^\infty(0, T)$ and $v \in V$

$$\int_0^T (u''(t), v)\phi(t)dt = \int_0^T (u(t), v)\phi''(t)dt. \tag{2.7}$$

Let $\{c_j\}_{j=1}^\infty$ be the eigenfunctions of the operator A_β. The weak solution of (1.1) is a function $u \in W(0, T)$ satisfying

$$\begin{aligned} &\langle u'', c_j\rangle + \alpha(u', c_j) + a_\beta(u, c_j) + \delta(\sin(u), c_j) = (f, c_j) + (u, c_j),\ \forall j \in \mathbb{N}, \\ &u(0) = u_0 \in V, \quad u'(0) = u_1 \in H, \end{aligned} \tag{2.8}$$

Thus

$$u'' + \alpha u' + A_\beta u + \delta \sin u = f + u, \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \tag{2.9}$$

which is understood in the sense of distributions on $(0, T)$ with the values in V'. For details see [3]. Two establish uniqueness of weak solution of (2.9), the following results are of critical importance.

Theorem 2.1. *Let* $w \in L^2(0, T; V),\ w' \in L^2(0, T; H)$ *and* $w'' + A_\beta w \in L^2(0, T; H)$. *Then, after a modification on the set of measure zero,* $w \in C([0, T]; V),\ w' \in C([0, T]; H)$ *and, in the sense of distributions on* $(0, T)$ *one has*

$$(w'' + A_\beta w, w') = \frac{1}{2}\frac{d}{dt}\{|w'|^2 + a_\beta(w, w)\} \tag{2.10}$$

For proof see [4].

Theorem 2.2. *(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on [0,T] which satisfies the integral inequality*

$$\xi(t) \le C_1 \int_0^t \xi(s)ds + C_2 \ \ for\ constants\ C_1\ , C_2 \ge 0 \tag{2.11}$$

almost everywhere t ∈ [0,T]. Then

$$\xi(t) \le C_2(1 + C_1 t e^{C_1 t})\ a.e.\ on\ 0 \le t \le T. \tag{2.12}$$

In particular, if

$$\xi(t) \le C_1 \int_0^t \xi(s)ds\ \ a.e.\ on\ 0 \le t \le T,\ then\ \ \xi(t)\ =\ 0\ \ a.e.\ on\ [0,\ T] \tag{2.13}$$

For proof see [2].

Theorem 2.3. *The solution of* (2.9) *is unique.*

For proof see [9].

Theorem 2.4. *Let $q = (\alpha, \beta(x), \delta) \in \mathcal{P}_{ad},\ u_0 \in V,\ u_1 \in H$ and $f \in L^2(0,T;H)$. Then*

(i). There exists a unique weak solution $u(t;q)$ of (1.1). This solution satisfies $u \in C([0,T];V) \cap W(0,T)$, $u' \in C([0,T];H)$, and

$$\max_{0\le t\le T}(\|u(t)\|^2 + |u'(t)|^2) + \|u''(t)\|^2_{L^2(0,T;V')} \le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \tag{2.14}$$

where C is a constant independent of $q \in \mathcal{P}_{ad}$. The approximate solutions $u_m(t;q)$ also satisfy the energy estimate (2.14) with the same constant C.

(ii). The solution $u(t;q)$ and its approximations $u_m(t;q)$ satisfy the following convergence estimate

$$\begin{aligned} &|u'(t) - u'_m(t)|^2 + \|u(t) - u_m(t)\|^2 \le C_2(|u_1 - P_m u_1|^2 + \|u_0 - P_m u_0\|^2 \\ &+\|f - P_m f\|^2_{L^2(0,T;H)} + \int_0^t |\sin u(s;q) - P_m \sin u(s;q)|^2 ds) \end{aligned} \tag{2.15}$$

where C_2 is a constant independent of $q \in \mathcal{P}$.

(iii). Furthermore, $u_m \to u$ in $C([0,T];V)$ and $u'_m \to u'$ in $C([0,T];H)$ as $m \to \infty$.

Proof. Proof of this theorem is an analog of the one we developed in [9]. However, special attention will be given for the variable diffusion coefficient $\beta(x) \in L^\infty(\Omega)$ throughout the proof. From the priori estimate outlined in [9] we have,

$$\max_{0\le t\le T}(\|u_m(t)\|^2 + |u'_m(t)|^2) + \|u''_m(t)\|^2_{L^2(0,T;V')} \le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right], \tag{2.16}$$

where C is a constant independent of $q \in \mathcal{P} = \{q = (\alpha, \beta(x), \delta) \in [\alpha_{min}, \alpha_{max}] \times \mathcal{B} \times [\delta_{min}, \delta_{max}]\}$.

Existence and convergence:

Estimate (2.16) shows that for any $q \in \mathcal{P}_{ad}$ and $m \in \mathbb{N}$ the approximate solutions $u_m(q)$ belong to same bounded convex ball $\|w\|_W \le C$ of $W(0,T)$ for the same $C > 0$. Fix a $q \in \mathcal{P}_{ad}$. Since $W(0,T)$ is a reflexive space, there exists a subsequence u_{m_k} of u_m that converges weakly to a function $z \in W(0,T)$. According to the energy estimate (2.16) we see that the sequence

$\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;V)$, $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H)$, and $\{u_m^{''}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;V^{'})$, where $V^{'}$ is the dual space of V. Since $L^2(0,T;V)$, $L^2(0,T;H)$, and $L^2(0,T;V^{'})$ are reflexive spaces, there exist a subsequence $\{u_{m_k}\}_{k=1}^{\infty} \subset \{u_m\}_{k=1}^{\infty}$ and $z \in L^2(0,T;V)$, $d^1 \in L^2(0,T;H)$, $d^2 \in L^2(0,T;V^{'})$ such that

$$\begin{aligned} u_{m_k} &\rightharpoonup z, \quad \text{in} \quad L^2(0,T;V),\\ u_{m_k}^{'} &\rightharpoonup d^1, \quad \text{in} \quad L^2(0,T;H),\\ u_{m_k}^{''} &\rightharpoonup d^2, \quad \text{in} \quad L^2(0,T;V^{'}), \end{aligned} \tag{2.17}$$

where $\rightharpoonup$ indicates the weak convergence. Since the convergence in $W(0,T)$ is the distributional convergence, we have

$$\begin{aligned} u_{m_k}^{'} &\rightharpoonup z^{'}, \quad \text{in} \quad L^2(0,T;H),\\ u_{m_k}^{''} &\rightharpoonup z^{''} \quad \text{in} \quad L^2(0,T;V^{'}) \quad \text{as} \quad k \to \infty. \end{aligned} \tag{2.18}$$

But the weak limit is unique when it exists. So $d^1 = z^{'}$ and $d^2 = z^{''}$. Energy estimate (2.16) also implies that $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;V)$ and the sequence $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;H)$. By the Alaoglu Theorem [10], we can find subsequences $\{u_{m_k}\}_{m=1}^{\infty}$ and $\{u_{m_k}^{'}\}_{m=1}^{\infty}$ of $\{u_m\}_{m=1}^{\infty}$ and $\{u_m^{'}\}_{m=1}^{\infty}$ respectively such that

$$\begin{aligned} u_{m_k} &\rightharpoonup z \quad \text{weak star in} \quad L^{\infty}(0,T;V),\\ u_{m_k}^{'} &\rightharpoonup z^{'} \quad \text{weak star in} \quad L^{\infty}(0,T;H). \end{aligned} \tag{2.19}$$

Now we show that z is a weak solution. Since V is compactly imbedded in H, then by the classical compactness theorem [4] $u_{m_k} \to z$ in $L^2(0,T;H)$. Using Cauchy Schwartz inequality, $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \leq \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \|w_k\|_{L^2(0,T;H)}$. Since $\{w_k\}_{k=1}^{\infty}$ is orthonormal in H the sequence $\{w_k\}_{k=1}^{\infty}$ is bounded in $L^2(0,T;H)$.
Thus $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \leq \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \to 0$ as $k \to \infty$. Hence $\sin(u_{m_k}) \to \sin(z)$ in $L^2(0,T;H)$. Thus we have,

$$\begin{aligned} &\langle u_m^{''}, w_j\rangle + \alpha(u_m^{'}, w_j) + a_\beta(u_m, w_j) + \delta(P_m \sin(u_m), w_j)\\ &= (P_m f, w_j) + (u_m, w_j),\\ &u_m(0) = P_m u_0, \quad u_m'(0) = P_m u_1 \quad \text{for} \quad j = 1,2,...,m. \end{aligned} \tag{2.20}$$

We pass to the limit in (2.20) to obtain

$$\begin{aligned} &\langle z^{''}, w_j\rangle + \alpha(z^{'}, w_j) + a_\beta(z, w_j) + \delta(\sin(z), w_j) = (f, w_j) + (z, w_j)\\ &z(0) = u_0, \quad z'(0) = u_1 \quad \text{for} \quad j = 1,2,...,m. \end{aligned} \tag{2.21}$$

Thus z is a weak solution of (1.1). It satisfies the energy estimate

$$\max_{0\leq t\leq T} [\|z(t)\|^2 + |z(t)^{'}|^2] + \|z(t)^{''}\|^2_{L^2(0,T;V^{'})} \leq C_1[\|u_0\|^2 + |u_1|^2 + \|f\|_{L^2(0,T;H)}],$$

where C_1 is a constant independent of $q \in \mathcal{P}_{ad} = \{q = (\alpha,\beta,\delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]$. By Lemma (2.3) the solution z is unique. Therefore $u_m \to z$ as $m \to \infty$ in $L^2(0,T;H)$ for the entire sequence. Hence (2.9) can be rewritten as $z^{''} + A_\beta z = f + z - \alpha z^{'} - \delta \sin\ z$. Hence $z^{''} + A_\beta z \in L^2(0,T;H)$. Similarly approximate solution can be rewritten as $u_m^{''} + A_\beta u_m = P_m f + u_m - \alpha u_m^{'} - \delta P_m \sin\ u_m$. Therefore $u_m^{''} + A_\beta u_m \in L^2(0,T;H)$. Subtract (2.20) from (2.21) to get

$$\begin{aligned} &(z - u_m)^{''} + A_\beta(z - u_m) = f - P_m f - \alpha(z - u_m)^{'}\\ &-\delta(\sin(z) - P_m \sin(u_m)) + (z - u_m) \in L^2(0,T;H). \end{aligned} \tag{2.22}$$

Therefore by Lemma (2.1) we have

$$\frac{1}{2}\frac{d}{dt}\{|z'-u'_m|^2+a_\beta(z-u_m,z-u_m)\}=((z-u_m)''+A_\beta(z-u_m),z'-u'_m))$$
$$=(f-P_mf-\alpha(z'-u'_m)-\delta(\sin(z)-P_m\sin(u_m))+z-u_m,z'-u'_m)$$
$$=(f-P_mf,z'-u'_m)-\alpha|z'-u'_m|^2-\delta(\sin(z)-P_m\sin(u_m),z'-u'_m)$$
$$+(z-u_m,z'-u'_m).$$

Integrating both sides over $[0,t]$ we get

$$|z'(t)-u'_m(t)|^2+a_\beta(z(t)-u_m(t),z(t)-u_m(t))\leq|u_1-P_mu_1|^2$$
$$+(u_0-P_mu_0,u_0-P_mu_0)+2\int_0^t|(f-P_mf)(z'-u'_m)|ds$$
$$+2|\alpha|\int_0^t|(z'-u'_m)|^2\,ds+2|\delta|\int_0^t|(\sin(z)-P_m\sin(u_m))(z'-z'_m)|ds$$
$$+\int_0^t|(z-u_m)(z'-u'_m)|ds.$$

Use $|ab|\leq\frac{a^2+b^2}{2}$ to get

$$|z'(t)-u'_m(t)|^2+\|z(t)-u_m(t)\|^2\leq|u_1-P_mu_1|^2+\|u_0-P_mu_0\|^2$$
$$+\|f-P_mf\|^2_{L^2(0,T;H)}+(2+|\alpha|+|\delta|)\int_0^t|z'-u'_m|^2(s)ds$$
$$+\int_0^t|z-u_m|^2(s)ds+\int_0^t|\sin(z)-P_m\sin(u_m)|^2(s)ds. \tag{2.23}$$

Since V is compactly embedded in H, (2.23) can be rewritten as

$$|z'(t)-u'_m(t)|^2+\|z(t)-u_m(t)\|^2\leq C[|u_1-P_mu_1|^2+\|u_0-P_mu_0\|^2$$
$$+\|f-P_mf\|^2_{L^2(0,T;H)}+\int_0^t|\sin(z)-P_m\sin(u_m)|^2(s)ds$$
$$+\int_0^t|z'-u'_m|^2(s)ds+\int_0^t\|z-u_m\|^2(s)ds] \tag{2.24}$$

where $C=max\{1,(2+|\alpha|+|\delta|),4K_1^2\}$.
Using Gronwall's Lemma we get

$$|z'(t)-u'_m(t)|^2+\|z(t)-u_m(t)\|^2\leq C[|u_1-P_mu_1|^2+\|u_0-P_mu_0\|^2$$
$$+\|f-P_mf\|^2_{L^2(0,T;H)}+\int_0^t|\sin(z)-P_m\sin(u_m)|^2(s)ds]. \tag{2.25}$$

Therefore $|z'(t)-u'_m(t)|^2+\|z(t)-u_m(t)\|^2\to 0$ as $m\to\infty$. This implies $u_m\to z$ in $L^\infty(0,T;V)$ and $u'_m\to z'$ in $L^\infty(0,T;H)$. But u_m, $u'_m\in C([0,T];V)$, being the solutions of the systems of ODEs. This implies $z\in C([0,T];V)$ and $z'\in C([0,T];H)$ after a modification on a set of measure zero on $[0,T]$. ∎

3. EXISTENCE OF OPTIMAL PARAMETERS

In this section we establish the continuity of the functional defined in (1.3) on compact subset of $\mathcal{B}$ defined in (1.2).

Lemma 3.1. *Let $v\in V$. Then the mapping $\beta\to A_\beta v$ from $\mathcal{B}$ into V' is continuous.*

Proof. Suppose that $\beta_n \to \beta$ in $\mathcal{B}$ as $n \to \infty$. We denote $A = A_\beta$ and $A_n = A_{\beta_n}$. We claim that $\|(A_n - A)v\|_{V'} \to 0$ as $n \to \infty$. Let $w \in V$ with $\|w\| \le 1$. Then

$$\begin{aligned}|\langle (A_n - A)v, w\rangle|^2 &\le \left(\int_\Omega |\beta_n(x) - \beta(x)||\nabla v(x)||\nabla w(x)|dx\right)^2\\ &\le |\beta_n(x) - \beta(x)|^2 \int_\Omega |\nabla v(x)|^2 dx.\end{aligned}$$

For any positive constant C, let $\Omega_C = \{x \in \Omega : |\nabla(x)|^2 > C\}$. Since $|\nabla(x)|^2 \in L_1(\Omega)$ there exists $C > 0$ and $\epsilon > 0$ such that $\int_{\Omega_C} |\nabla(x)|^2 dx < \epsilon$. But we have,

$$\begin{aligned}&\int_\Omega |\beta_n(x) - \beta(x)|^2 \int_\Omega |\nabla v(x)|^2 dx\\ &= \int_{\Omega_M} |\beta_n(x) - \beta(x)|^2 \int_\Omega |\nabla v(x)|^2 dx + \int_{\Omega \setminus \Omega_M} |\beta_n(x) - \beta(x)|^2 \int_\Omega |\nabla v(x)|^2 dx\\ &\le 4M^2\epsilon + 2MC\|\beta_n - \beta\|_{L^1(\Omega)} \to 0 \ as \ n \to \infty.\end{aligned}$$

∎

Lemma 3.2. *Suppose that $\beta_n \to \beta$ in $\mathcal{B}$, and $v_n \rightharpoonup v$ weakly in V, as $n \to \infty$. Then $A_n v_n \rightharpoonup Av$ weakly in V'.*

Proof. Let $w \in V$, then

$$\begin{aligned}&|\langle A_n v_n, w\rangle - \langle Av, w\rangle| = |\langle A_n w, v_n\rangle - \langle Aw, v\rangle|\\ &\le |\langle (A_n - A)w, v_n\rangle| + |\langle Aw, v_n - v\rangle|.\end{aligned} \tag{3.1}$$

Since a weakly convergent sequence is bounded, we have

$$|\langle (A_n - A)w, v_n\rangle| \le \|A_n w - Aw\|_{V'}\|v_n\| \le c\|A_n w - Aw\|_{V'} \to 0$$

as $n \to \infty$ by Lemma 3.1. The second term $|\langle Aw, v_n - v\rangle| \to 0$ since $v_n \rightharpoonup v$. ∎

The weak solution of (1.1) $u(q)$ depends on $q \in \mathcal{P}_{ad}$. Next we show the solution map from $\mathcal{P}_{ad}$ into $C[0,T]; H)$ is continuous.

Lemma 3.3. *Let $q \in \mathcal{P}_{ad}$. Then the solution map $q \to u(q)$ from $\mathcal{P}_{ad}$ into $C([0,T]; H)$ is continuous.*

Proof. Let $q_n \to q$ in $\mathcal{P}_{ad}$ as $n \to \infty$. Since $u(t;q)$ is the weak solution of (1.1) for any $q \in \mathcal{P}_{ad}$, we have the following estimate

$$\begin{aligned}&\max_{0\le t\le T}(\|u(t;q_n)\|^2 + |u'(t;q_n)|^2) + \|u''(t;q_n)\|^2_{L^2(0,T;V')}\\ &\le C\left[\|u_0\|^2 + |u_1|^2 + \|f\|^2_{L^2(0,T;H)}\right],\end{aligned} \tag{3.2}$$

where C is a constant independent of $q \in \mathcal{P}_{ad}$. Estimate (3.2) shows that $u(t;q_n)$ is bounded in $W(0,T)$. Since $W(0,T)$ is reflexive, we can choose a subsequence $u(t;q_{n_k})$ weakly convergent to a function z in $W(0,T)$. The fact that $u(t;q_n)$ is bounded in $W(0,T)$ implies that $u(t;q_n)$ is bounded in $L^2(0,T;V)$, so $u(t;q_{n_k})$ weakly converges to a function z in $L^2(0,T;V)$. Since V is compactly imbedded in H, then by the classical compactness theorem in [8] $u(t;q_n) \to z$ in $L^2(0,T;H)$. Using Cauchy Schwartz inequality, $|(\sin(u_{m_k}) - \sin(z), w_k)_{L^2(0,T;H)}| \le \|\sin(u_{m_k}) - \sin(z)\|_{L^2(0,T;H)} \to 0$ as $k \to \infty$.

By (3.2) the derivatives $u'(t;q_{n_k})$ and z' are uniformly bounded in $L^\infty(0,T;H)$. Therefore functions $\{u(t;q_{n_k}), z\}_{k=1}^\infty$ are equicontinuous in $C([0,T];H)$. Thus $u(t;q_{n_k})z$ in $C([0,T];H)$.

In particular, $u(t;q_{n_k})z(t)$ in H and $u(t;q_{n_k}) \rightharpoonup z(t)$ weakly in V for any $t \in [0,T]$. By Lemma 3.2, $A_{n_k}u(t;q_{n_k}) \rightharpoonup Az(t)$ weakly in V'. Now we see that z satisfies equation (2.8), i.e. it is the weak solution $u(q)$. The uniqueness of the weak solutions implies that $u(q_n) \to u(q)$ as $n \to \infty$ in $C([0,T];H)$ for the entire sequence $u(q_n)$. Thus $u(t;q_n) \to u(q)$ in $C([0,T];H)$ as $q_n \to q$ in $\mathcal{P}_{ad}$ as claimed.
∎

Theorem 3.4. *Let $q \in \mathcal{P}_{ad}$. Then the solution maps $q \to u(q)$ from $\mathcal{P}_{ad}$ into $C([0,T];V)$ and $q \to u'(q)$ from $\mathcal{P}_{ad}$ into $C([0,T];H)$ are continuous.*

Proof. We prove this result for approximate solution u_m and then extend the proof for the weak solution u. Fix $m \in \mathbb{N}$. Suppose that $q_n \to q$ in $\mathcal{P}_{ad}$ as $n \to \infty$. Then we claim $u_m(q_n) \to u(q)$ in $C([0,T];V)$ and $u'_m(q_n) \to u'(q)$ in $C([0,T];H)$. The approximate solutions $u_m(q_n)$ and $u_m(q)$ satisfy

$$\begin{aligned} &u''_m(q_n) + A_n u_m(q_n) = P_m f + u_m(q_n) - \alpha_n u'_m(q_n) - \delta_n P_m \sin(u_m(q_n)),\\ &u''_m(q) + A u_m(q) = P_m f + u_m(q) - \alpha u'_m(q) - \delta P_m \sin(u_m(q)), \end{aligned} \tag{3.3}$$

Note that $A = A_\beta$ and $A_n = A_{\beta_n}$. Let $w = u_m(q_n) - u_m(q)$. Using (3.3) and taking H inner product we have,

$$\begin{aligned} &(w'' + A_n(w), w') = ((A - A_n)u_m(q), w') + (w, w') - \alpha_n |w'|^2\\ &+(\alpha - \alpha_n)(u'_m(q), w') - \delta_n(P_m(\sin(u_m(q_n)) - \sin(u_m(q))), w')\\ &+(\delta - \delta_n)(P_m \sin(u_m(q)), w'). \end{aligned} \tag{3.4}$$

We have $w(t) \in L^2(0,T;V)$, $w'(t) \in L^2(0,T;H)$ and $w'' + A_n(w) \in L^2(0,T;H)$. Integrating (3.4) from 0 to t we have,

$$\begin{aligned} &|w'(t)|^2 + \|w(t)\|^2 \le \int_0^t \|(A - A_n)u_m(q)\|^2_{V'} ds + \int_0^t |w'(s)|^2 ds\\ &+|\alpha - \alpha_n| \int_0^t |u'_m(s;q)|^2 ds + |\alpha - \alpha_n| \int_0^t |w'(s)|^2 ds\\ &+|\delta - \delta_n| \int_0^t \|u_m(s;q)\|^2 ds + |\alpha_n| \int_0^t |w'(s)|^2 ds + |\delta_n| \int_0^t \|w(s)\|^2 ds\\ &+|\delta_n| \int_0^t |w'(s)|^2 ds. \end{aligned} \tag{3.5}$$

In a finite dimensional normed space all norms are equivalent. Hence there exists a constant $C(m)$ such that $\|w'(s)\| \le C(m)|w'(s)|$ for any $s \in [0,T]$.

Now the Gronwall's inequality and the energy estimate (3.2) give

$$\begin{aligned} &|u'_m(t;q_n) - u'_m(t;q)|^2 + \|u_m(t;q_n) - u_m(t;q)\|^2\\ &\le c(m) \left(\int_0^T \|(A - A_n)u_m(s;q)\|^2_{V'} ds + |\alpha - \alpha_n| + |\delta - \delta_n| \right). \end{aligned} \tag{3.6}$$

By the assumption $q_n \to q$ in $\mathcal{P}_{ad}$, that is $\alpha_n \to \alpha$, $\delta_n \to \delta$ and $\beta_n \to \beta$ in $\mathcal{P}_{ad}$ as $n \to \infty$. The integral term in the right hand side of (3.6) approaches zero by Lemma 3.1 and the Lebesgue Dominated Convergence Theorem. Hence the required convergence $u_m(q_n) \to u_m(q)$ in $C([0,T];V)$ and $u'_m(q_n) \to u'_m(q)$ in $C([0,T];H)$ as $n \to \infty$ follows.

Note that the mapping $[0,T] \times \mathcal{P} \to H$ defined by $(s,q) \to u(s;q)$ is continuous, since $q \to u(q) \in C([0,T];H)$ is continuous by Lemma 3.3. Therefore the mapping $[0,T] \times \mathcal{P} \to H$

defined by $(s, q) \to \sin(u(s; q))$ is continuous. Thus it takes the compact set $[0, T] \times \mathcal{P}$ into a compact set in H, and the uniform convergence of the integrals in

$$\int_0^T |\sin(u(s;q)) - P_m \sin(u(s;q))|^2 ds \to 0, \quad m \to \infty \tag{3.7}$$

Therefore $u(q_n) \to u(q),\ m \to \infty$ in $C([0, T]; V)$ as claimed. Similar argument can be used for the convergence of the derivatives $u'(q_n) \to u'(q)$ in $C([0, T]; H)$. Thus the minimization problem in (1.4) has a solution if the minimization problem in restrected to compact subset of $\mathcal{P}_{ad}$.

∎

REFERENCES

[1] H. ATTOUCH, G. BUTAZZON and G. MICHAILLE, *Variational Analysis In Sobolev And BV Spaces, Applications to PDEs and Optimization*, SIAM, Philadelphia(2005).

[2] L. C. EVANS, Partial Differential Equations Graduate Studies in Mathematics, *American Mathematical Society*, Providence, Rhode Island. **19** (1998).

[3] M. LEVI, Beating modes in the Josephson junction, Chaos in Nonlinear Dynamical Systems, *SIAM*, **(**1984), pp. 56-73.

[4] J. L. LIONS, and E. MAGANES, *Existence and Regularity of Weak Solutions for Semilinear Second Order Evolution Equations*, Springer-Verlag, New York-Heildelberg, (1972).

[5] J. L. LIONS, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, (1971).

[6] J. HA and S. GUTMAN, Optimal parameters for damped sine-Gordon equations. *Korean Math. Soc.*, **46** (2009), pp. 1105-1117.

[7] J. HA and S. NAKAGIRI, Existence and regularity of weak solutions for semi-linear second order evolution equations, *Funkcial Ekvac.*, **41** (1998), pp. 1–24.

[8] R. TEMAM, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, Springer-Verlag, New York, (1988).

[9] N. THAPA, Optimal parameters for damped sine-Gordon equations with Neumann boundary conditions. *International Journal of Pure and Applied Mathematics*, **75** (2012), no. 1, pp. 69–91.

[10] L.V. KANTOROVICH and G. R. AKILOV, *Functional Analysis* Second Edition, Pergamon Press, 1982.