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GENERALIZED K-DISTANCE-BALANCED GRAPHS

AMIR HOSSEINI[1] AND MEHDI ALAEIYAN[2,*]

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[1]DEPARTMENT OF MATHEMATICS, KARAJ BRANCH, ISLAMIC AZAD UNIVERSITY, KARAJ, IRAN
amir.hosseini@kiau.ac.ir, hosseini.sam.52@gmail.com

[2] DEPARTMENT OF MATHEMATICS, IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, TEHRAN, IRAN.
alaeiyan@iust.ac.ir

ABSTRACT. A nonempty graph Γ is called *generalized k-distance-balanced,* whenever every edge ab has the following property: the number of vertices closer to a than to b, k times of vertices closer to b than to a, or conversely, $k \in N$ .In this paper we determine some families of graphs that have this property, as well as to prove some other result regarding these graphs.

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1. INTRODUCTION

Throughout of this paper let Γ be a finite, undirected graph with diameter d, and $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. The distance $d(a,b)$ between vertices $a, b \in V(\Gamma)$ is the length of a shortest path between $a, b \in V(\Gamma)$. For an edge ab of a graph Γ let W_{ab} be the set of vertices closer to a than to b, that is

$$W_{ab} = \{x \in V(\Gamma) | d(x,a) < d(x,b)\}.$$

In addition

$$_aW_b = \{x \in V(\Gamma) | d(x,a) = d(x,b)\}.$$

We also note that the sets W_{ab} appear in the chemical graph theory as well: The well-investigated Szeged index of a graph Γ as $S_z(\Gamma) = \sum_{ab \in E(\Gamma)} |W_{ab}|.|W_{ba}|$, cf. [2, 3].
To understand more about k-GDB graphs, we recall the following definition. We call a graph Γ *distance-balanced*, if $|W_{ab}|$=$|W_{ba}|$ hold for any edge ab of Γ. These graphs were, studied by Handa [4] who considered DB. Also in [6] were studied DB graphs in the framework of various kinds of graph products.
A graph Γ is called *nicely distance-balanced* whenever there exists a positive integer γ_Γ, such that for two adjacent vertices a, b of Γ, $|W_{ab}| = |W_{ba}| = \gamma_\Gamma$. These graphs were studied by Kutnar and Miklavic in [7].
A graph Γ is *strongly distance-balanced* if for every edge ab of Γ and every $i \geq 0$ the number of vertices at distance i from a and at distance $i+1$ from b is equal to the number of vertices at a distance $i+1$ from a and at distance i from b. These graphs were studied in [1, 7]. By definition, it is clear that every SDB graph is a DB graph. According to the above definition, a graph Γ is k-GDB whenever for every edge ab of Γ, $|W_{ab}| = k|W_{ba}|$ or $|W_{ba}| = k|W_{ab}|$.
If $k = 1$, then the graph Γ is DB graph. Throughout of this paper, we assume that

$$|W_{ab}| = k|W_{ba}|.$$

In this paper we gives some examples and results regarding such graphs.

2. EXAMPLE AND BASIC PROPERTIES

We first begin with an example of k-GDB graphs.

Example 2.1. *The complete bipartite graphs $K_{n,kn}$ are a family of k-GDB graphs.*

Proof. Suppose that $K_{n,kn}$ has two independent parts A and B. Pick adjacent vertices a and b of $K_{n,kn}$, such that $a \in A$, $b \in B$. According to $K_{n,kn}$ is bipartite and has diameter 2, vertex a contains $(kn-1)$ adjacent and vertex b contains $(n-1)$ adjacent. By definition W_{ab} we have, $|W_{ab}| = kn$ and $|W_{ba}| = n$. Thus $|W_{ab}| = k|W_{ba}|$. This show that $K_{n,kn}$ is k-GDB. ∎

Let ab be an arbitrary edge of Γ. For any two nonnegative integers i, j we assume that

$$D^i_j(a,b) = \{x \in V(\Gamma) | d(x,a) = i \; and \; d(x,b) = j\}.$$

We now suppose that Γ is k-GDB graph. since $|W_{ab}| = k|W_{ba}|$, we have

$|\{a\} \cup \bigcup_{i=1}^{d-1} D_{i+1}^{i}(a,b)| = k|\{b\} \cup \bigcup_{i}^{d-1} D_{i}^{i+1}(a,b)|.$

Therefore,

$$\sum_{i=1}^{d-1} |D_{i+1}^{i}(a,b)| = k \sum_{i=1}^{d-1} |D_{i}^{i+1}(a,b)| + (k-1). \qquad \textbf{(1)}$$

As well as,

$$\sum_{i=1}^{d-1} |D_{i}^{i}(a,b)| = n - (|W_{ab}| + |W_{ba}|) = n - (k+1)|W_{ba}|. \qquad \textbf{(2)}$$

It follows from (**2**) that in the k-GDB bipartite graphs we have,

$n = |V(\Gamma)| = (k+1)t$,

where $t = |W_{ba}|$, because in bipartite graphs:

$\forall i \; D_{i}^{i}(a,b) = 0.$

Proposition 2.1. *Let Γ be a k-GDB bipartite graph with diameter 2. Then for every edge ab of Γ, deg(a)=$k$deg (b).*

Proof. It follows from (**1**) that for a k-GDB bipartite graph with diameter 2,

$|D_{2}^{1}(a,b)| = k|D_{1}^{2}(a,b)| + (k-1),$

for every edge ab of Γ. If $|D_{1}^{2}(a,b)| = t$, then

$|D_{2}^{1}(a,b)| = kt + (k-1).$

Therefore deg(b)=$t+1$ and deg(a)=$kt + k = k(t+1)$. So always deg(a)=$k$deg(b). ∎

Lemma 2.2. *Let Γ be a k-GDB bipartite graph with diameter 2. Then Γ is only $K_{n,kn}$.*

Proof. Let Γ be a k-GDB bipartite graph with diameter 2. We claim that Γ is a complete bipartite graph. Otherwise, it will not be diameter 2. It follows from Proposition 2.1 that ,

deg(a)=$k$deg(b).

Since Γ is complete bipartite graph, Γ must be $K_{n,kn}$. ∎

Proposition 2.3. *A connected bipartite graph Γ is k-GDB if and only if $S_z(\Gamma) = \frac{k\|\Gamma\|.|\Gamma|^2}{(k+1)^2}$. ($|\Gamma| = |V(\Gamma)|$ and $\|\Gamma\| = |E(\Gamma)|$).*

Proof. Suppose Γ is k-GDB. Then for any edge ab, $|W_{ab}| = k|W_{ba}|$ and since Γ is bipartite also $|W_{ab}| + |W_{ba}| = |\Gamma|$ holds. Therefore

$k^2|W_{ba}|^2 + |W_{ba}|^2 + 2k|W_{ba}|^2 = |\Gamma|^2.$

Hence $|W_{ba}|^2 = \frac{|\Gamma|^2}{(k+1)^2}$ and so,

$\sum_{ab\in E(\Gamma)} |W_{ab}|.|W_{ba}| = k \sum_{ab\in E(\Gamma)} |W_{ba}|^2 = k\frac{\|\Gamma\|.|\Gamma|^2}{(k+1)^2}.$

Conversely, suppose $S_z(\Gamma) = \frac{k\|\Gamma\|.|\Gamma|^2}{(k+1)^2}$ holds. Since Γ is bipartite, we have $|W_{ab}| + |W_{ba}| = |\Gamma|$. Hence $(k+1)|W_{ba}| = |\Gamma|$. As well as

$|W_{ab}|.|W_{ba}| = k\frac{|\Gamma|^2}{(k+1)^2}$,

and hence $|W_{ab}| = k|W_{ba}| = k\frac{|\Gamma|}{(k+1)}$. This show that $|\Gamma|$ is k-GDB. ∎

3. k-GDB ON THE PRODUCT GRAPHS

In this section we study the conditions under which the standard graph products produce a k-GDB graph. We first prove a Proposition that the cartesian product of two k-GDB graphs is a K-GDB graph. We start with the definition of this products. All of the graph product constructed from two graphs G and H have vertex set $V(G) \times V(H)$. In the cartesian product of G and H, denoted by $G\Box H$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or $h_1 = h_2$ and g_1, g_2 are adjacent in G. Note that the cartesian product is commutative and that

$$d_{G\Box H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2). \quad \textbf{(3)}$$

In the direct product $G \times H$, (g_1, h_1) and (g_2, h_2) are adjacent, if they are adjacent both coordinates. In the strong product $G \boxtimes H$, the edge set is $E(G\Box H) \cup E(G \times H)$. In the lexicographic product $G \circ H$, (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 = g_2$ and h_1, h_2 are adjacent in H, or g_1, g_2 are adjacent in G. See [6] for a more complete treatment of graph product.

Proposition 3.1. *Let G and H be connected graphs. Then $G\Box H$ is k-GDB graph if and only if both G and H are k-GDB graphs.*

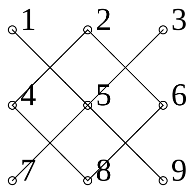
Proof. Set Y=$G\Box H$. Assume Y is k-GDB and e is an edge of Y. Without loss of generality it can be assumed $e = (x, u)(y, u)$ for $xy \in E(G)$. Then sets $W_{xy} \times V(H)$, $W_{yx} \times V(H)$ and ${}_xW_y \times V(H)$ form a partition of $V(Y)$. Assume that $(a, b) \in W_{xy} \times V(H)$. Then

$$d_Y((a, b), (x, u)) = d_G(a, x) + d_H(b, u) < d_G(a, y) + d_H(b, u) = d_Y((a, b), (y, u)).$$

Hence $(a, b) \in W_{(x,u)(y,u)}$. For $(a, b) \in W_{yx} \times V(H)$ (resp. $(a, b) \in {}_xW_y \times V(H)$) we similarly get $(a, b) \in W_{(y,u)(x,u)}$ (resp. $(a, b) \in {}_{(x,u)}W_{(y,u)}$). It follows that $W_{(x,u)(y,u)} = W_{xy} \times V(H)$ and $W_{(y,u)(x,u)} = W_{yx} \times V(H)$. Since Y is k-GDB, we have $|W_{(x,u)(y,u)}| = k|W_{(y,u)(x,u)}|$. Therefore $|W_{xy}| = k|W_{yx}|$. Hence G is k-GDB. The same argument applies for edges $e = (x, u)(x, v)$, and so H is k-GDB.

Conversely, let G be a k-GDB and $xy \in E(G)$. Then $|W_{xy} \times V(H)| = k|W_{yx} \times V(H)|$ and so $|W_{(x,u)(y,u)}| = k|W_{(y,u)(x,u)}|$. The same argument applies for edge $e = (x, u)(x, v)$, we have $|W_{(x,u)(x,v)}| = k|W_{(x,v)(x,u)}|$ and hence $G\Box H$ is k-GDB. ∎

The other three standard graph product, the direct, strong and lexicographic, do not preserve the property of being k-GDB. Let G and H are both $K_{1,2}$. Then the direct product of G and H (Figure 1) will not be 2-GDB.



(Figure **1**)

$W_{15} = \{1\}, W_{51} = \{5, 7, 8, 9\}$.
Similarly, it can be shown that the strong product and lexicographic product of two graphs G and H are not 2-GDB.

4. GENERALIZED NDB AND SDB GRAPHS

In a k-GDB graphs, for each ab of Γ, we have $|W_{ab}| = k|W_{ba}|$. If always for each arbitrary edge ab of Γ $|W_{ba}|$ is constant γ_Γ, then Γ is called Generalized k-nicely distance-balanced (k-GNDB).
Every k-GNDB graph is k-GDB , and every bipartite k-GDB graph is k-GNDB. We first begin with the obvious observation which follows from (**2**).

Lemma 4.1. *Let Γ be a connected k-GNDB graph with n vertices and diameter d. Then for every edge $ab \in E(\Gamma)$, there are exactly $n - (k+1)\gamma_\Gamma$ vertices of Γ, which are at the same distance from x and y. In other words,*

$$\sum_{i=1}^{d} |D_i^i(a,b)| = n - (k+1)\gamma_\Gamma.$$

Lemma 4.2. *Let Γ be a connected k-GNDB graph with diameter d. Then $d \leq k\gamma_\Gamma$.*

Proof. Pick vertices x_0 and x_d of Γ such that $d(x_0, x_d) = d$ and a shortest path

$$x_0, x_1, x_2, ..., x_d$$

between x_0 and x_d. We may assume without loss of generality that $|W_{x_1x_0}| = k|W_{x_0x_1}|$. Then $\{x_1, x_2, ..., x_d\} \subseteq W_{x_1x_0}$. Hence

$$|\{x_1, x_2, ..., x_d\}| \leq |W_{x_1x_0}| = k|W_{x_0x_1}|.$$

This show that $d \leq k\gamma_\Gamma$. ∎

Lemma 4.3. *Let G and H denote graphs. Then $\Gamma = G\Box H$ is k-GNDB if and only if both G and H are k-GNDB with $|V(H)|\gamma_G = |V(G)|\gamma_H$.*

Proof. Pick adjacent vertices (g_1, h_1) and (g_2, h_2) of Γ. Then either h_1 and h_2 are adjacent in H and $g_1 = g_2$ or g_1 and g_2 are adjacent in G and $h_1 = h_2$. Assume first that h_1 and h_2 are adjacent in H and $g_1 = g_2$. It follows from (**3**) that for each $g' \in V(G)$, the vertices of Γ of the form

(g', x) which are closer to (g_1, h_1) than to (g_1, h_2) (resp. closer to (g_1, h_2) than to (g_1, h_1)) are exactly the vertices for each $x \in W^H_{h_1h_2}$ (resp. $x \in W^H_{h_2h_1}$). Therefore, the set $W^\Gamma_{(g_1,h_1)(g_1,h_2)}$ has $|V(G)||W^H_{h_1h_2}|$ elements, while $W^\Gamma_{(g_1,h_2)(g_1,h_1)}$ has $|V(G)||W^H_{h_2h_1}|$ elements. Suppose now that g_1 and g_2 are adjacent in G and $h_1 = h_2$. Similarly as above we obtain that the set $W^\Gamma_{(g_1,h_1)(g_2,h_1)}$ has $|V(H)||W^G_{g_1g_2}|$ elements, while the set $W^\Gamma_{(g_2,h_1)(g_1,h_1)}$ has $|V(H)||W^G_{g_2g_1}|$ elements.
Assume now that Γ is k-GNDB. By the above remark, for every $g_1g_2 \in E(G)$ and for $h_1h_2 \in E(H)$, we have

$$|V(H)||W^G_{g_1g_2}| = k|V(H)||W^G_{g_2g_1}| = |V(G)||W^H_{h_1h_2}| = k|V(G)||W^H_{h_2h_1}| = k\gamma_\Gamma, \qquad \textbf{(4)}$$

where in

$$\gamma_\Gamma = |V(H)||W^G_{g_2g_1}| = |V(G)||W^H_{h_2h_1}|.$$

It follow from (**4**) that both G and H are k-GNDB and that $|V(H)|\gamma_G = |V(G)|\gamma_H$ holds. Conversely, if both G and H are k-GNDB with $|V(H)|\gamma_G = |V(G)|\gamma_H$, by the above remark, Γ is k-GNDB. ∎

A graph Γ with diameter d is called k-GSDB whenever for every edge $ab \in E(\Gamma)$ and every $1 \le i \le d-1$,

$$|D^i_{i+1}(a, b)| = k|D^{i+1}_i(a, b)| + (k - 1).$$

For $k = 1$ this graph is also a SDB graph.

Example 4.1. *All complete bipartite graphs $K_{n,kn}$ are a family of k-GSDB graphs.*

Proof. Pick adjacent vertices a and b of $K_{n,kn}$. According to $K_{n,kn}$ is bipartite and has diameter 2, we have $D^1_2(a, b) = (kn - 1)$ and $D^2_1(a, b) = (n - 1)$. Therefore $D^1_2(a, b) = kD^2_1(a, b) + (k - 1)$. This show that $K_{n,kn}$ is k-GSDB. ∎

Lemma 4.4. *Let Γ denote a k-GSDB graph with diameter 2. Then Γ is k-GDB graph ($k \ge 2$).*

Proof. By (**1**), the graph Γ is k-GDB if and only if, for every edge $ab \in E(\Gamma)$ and every $1 \le i \le d-1$,

$$\sum_{i=1}^{d-1} |D^i_{i+1}(a, b)| = k \sum_{i=1}^{d-1} |D^{i+1}_i(a, b)| + (k - 1).$$

Assume now that Γ is k-GSDB. Then for every edge $ab \in E(\Gamma)$ and every $1 \le i \le d-1$, we have

$$|D^i_{i+1}(a, b)| = k|D^{i+1}_i(a, b)| + (k - 1).$$

Hence

$$\sum_{i=1}^{d-1} |D^i_{i+1}(a, b)| = k \sum_{i=1}^{d-1} |D^{i+1}_i(a, b)| + (k - 1)(d - 1).$$

If $d = 2$, then the last equality is:

$$|D^1_2(a, b)| = k|D^2_1(a, b)| + (k - 1).$$

This show that Γ is k-GDB. ∎

Note that, for $k = 1$, every SDB graph is DB.

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