ITERATIVE ALGORITHM FOR SPLIT GENERALIZED MIXED EQUILIBRIUM PROBLEM INVOLVING RELAXED MONOTONE MAPPINGS IN REAL HILBERT SPACES

UGOCHUKWU ANULOB\-O OSISIOGU 1, FRIDAY LAWRENCE ADUM 1, AND CHINEDU IZUCHUKWU 2

Received 28 January, 2018; accepted 3 August, 2018; published 15 October, 2018.

1DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EBONYI STATE UNIVERSITY, ABAKALIKI, NIGERIA.

uosisiogu@gmail.com, adumson2@yahoo.com

2SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF KWAZULU-NATAL, DURBAN, SOUTH AFRICA.

izuchukwuc@ukzn.ac.za, izuchukwu_c@yahoo.com

ABSTRACT. The main purpose of this paper is to introduce a certain class of split generalized mixed equilibrium problem involving relaxed monotone mappings. To solve our proposed problem, we introduce an iterative algorithm and obtain its strong convergence to a solution of the split generalized mixed equilibrium problems in Hilbert spaces. As special cases of the proposed problem, we studied the proximal split feasibility problem and variational inclusion problem.

Key words and phrases: Relaxed monotone mappings, split generalized mixed equilibrium problems, variational inclusion problems, variational inequality problems, maximal monotone mapping.

2010 Mathematics Subject Classification Primary 47H09, 47H10. Secondary 49J20, 49J40.
1. Introduction

Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$ and $S : C \to C$ be any nonlinear mapping. Then, $S$ is called $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$
\|Sx - Sy\| \leq L\|x - y\| \quad \forall \ x, y \in C,
$$

if $L = 1$, then $S$ is called nonexpansive. A point $x \in C$ is called a fixed point of $S$ if $Sx = x$. Throughout this paper, we shall denote the set of fixed points of $S$ by $\mathcal{F}(S)$. A mapping $S : C \to C$ is said to be

(i) monotone, if

$$
\langle Sx - Sy, x - y \rangle \geq 0, \quad \forall x, y \in C,
$$

(ii) $\mu$-strongly monotone, if there exists a constant $\mu > 0$ such that

$$
\langle Sx - Sy, x - y \rangle \geq \mu \|x - y\|^2, \quad \forall x, y \in C,
$$

(iii) $\mu$-inverse strongly monotone, if there exists a constant $\mu > 0$ such that

$$
\langle Sx - Sy, x - y \rangle \geq \mu \|Sx - Sy\|^2, \quad \forall x, y \in C,
$$

(iv) firmly nonexpansive, if

$$
\langle Sx - Sy, x - y \rangle \geq \|Sx - Sy\|^2, \quad \forall x, y \in C.
$$

A mapping $T : C \to H$ is said to be relaxed $\eta - \alpha$ monotone (see [3]), if there exists a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree $p$ (i.e., $\alpha(tx) = t^p \alpha(x)$ for all $t > 0$ and $x \in H$, where $p > 1$) such that

$$
\langle Tx - Ty, \eta(x, y) \rangle \geq \alpha(x - y) \quad \forall x, y \in C.
$$

In particular, if $\eta(x, y) = x - y$, $\forall x, y \in C$, $T$ is called relaxed $\alpha$-monotone. Furthermore, if $\eta(x, y) = x - y$, $\forall x, y \in C$ and $\alpha(z) = \mu \|z\|^p$, where $p > 1$ and $\mu > 0$ are constants, then $T$ is called $p$-monotone [12, 23]. In fact, if $p = 2$, then $T$ is called $\mu$-strictly monotone (see [24]). Clearly, every monotone mapping is relaxed $\eta - \alpha$ monotone with $\eta(x, y) = x - y \forall x, y \in C$ and $\alpha = 0$. Thus, inverse strongly monotone mappings are relaxed $\eta - \alpha$ monotone. The following is an example of a relaxed $\eta - \alpha$ monotone mapping.

**Example 1.1.** [7] Let $H = \mathbb{R}^2$ and $C = [0, 1] \times [0, 1]$. Define a mapping $T : C \to H$ by $T(x_1, x_2) = (x_1 + x_2) \forall (x_1, x_2) \in C$, $\alpha : H \to \mathbb{R}$ by $\alpha(x_1, x_2) = 3x_1^2 + 3x_2^2$ and $\eta : C \times C \to H$ by $\eta((x_1, x_2), (y_1, y_2)) = (4(x_1 - y_1), 4(x_2 - y_2)) \forall (x_1, x_2) \times (y_1, y_2) \in C \times C$. Then, $T$ is relaxed $\eta - \alpha$ monotone.

Recall that a mapping $F : C \to C$ is said to be averaged nonexpansive if $\forall x, y \in C$, $F = (1 - \beta)I + \beta S$ holds for a nonexpansive operator $S : C \to C$ and $\beta \in (0, 1)$. In this case, we say that $F$ is $\beta$-averaged. The term "averaged mapping" was coined by Biallon et al. [4]. Moreover, $F$ is firmly nonexpansive if and only if $F$ can be expressed as $F = \frac{1}{2}(I + S)$, where $S$ is nonexpansive (see [20]). Thus, we make the following remark which can be easily verified (see, also [13, 14]).

**Remark 1.1.** In a Hilbert space, $F$ is firmly nonexpansive if and only if it is averaged with $\beta = \frac{1}{2}$.

The metric projection $P_C$ is a map defined on $H$ onto $C$ which assigns to each $x \in H$, the unique point in $C$, denoted by $P_Cx$ such that

$$
\|x - P_Cx\| = \inf\{\|x - y\| : y \in C\}.
$$
It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x \rangle \leq 0$, $\forall z \in C$ and $P_C$ is a firmly nonexpansive mapping. Thus, $P_C$ is nonexpansive. For more information on metric projections, see [10, 5].

The Equilibrium Problem (EP) (in the sense of Blum and Oettli [1]) is to find $x \in C$ such that

\[(1.1) \quad \phi(x, y) \geq 0 \quad \forall y \in C,\]

where $\phi: C \times C \rightarrow \mathbb{R}$ is a bifunction. We denote the solution set of EP (1.1) by $G(\phi)$. To solve the EP, the bifunction $\phi$ is assumed to satisfy the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) $\phi$ is monotone; that is $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y \in C$, $\lim_{t \to 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y)$;
(A4) for all $x \in C$, $\phi(x, \cdot)$ is convex and lower semicontinuous.

The Mixed Equilibrium Problem (MEP) is to find $x \in C$ such that

\[(1.2) \quad \phi(x, y) + \langle Tx, y - x \rangle + f(y) - f(x) \geq 0 \quad \forall y \in C,\]

where $\phi: C \times C \rightarrow \mathbb{R}$ is a bifunction, $T$ is some nonlinear mapping and $f: C \rightarrow (-\infty, +\infty]$ is a proper convex and lower semi continuous function. The solution set of (1.2) is denoted by $G(\phi, T, f)$.

Equilibrium problems and mixed equilibrium problems are known to be one of the most successful tools in many fields such as physics, economics, engineering, computer science, among others for solving problems like linear and nonlinear programming, variational inequality problems, fixed point problems, optimization problems and others (for example, see [3, 9, 17, 18]). The MEP have been studied widely by many authors in the case where $T$ is an inverse strongly monotone mapping (for example, see [3, 9] and the references therein). Since the introduction of the relaxed monotone mapping by Fang and Huang [8], authors are now beginning to study MEP for the case where $T$ is a relaxed monotone mapping. For instance, Wang et al. [24] introduced the following iterative algorithm for solving MEP (in the case where $f = 0$) and fixed point problem for a nonexpansive mapping in Hilbert space:

\[(1.3) \quad \begin{aligned}
x_1 & \in C \text{ chosen arbitrarily,} \\
\phi(u_n, y) + \langle Tu_n, \eta(y, u_n) \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 & \forall y \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n) \beta_n S x_n + (1 - \alpha_n)(1 - \beta_n) u_n, \\
C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \}, \\
Q_n = \cap_{j=1}^{n} C_j, \\
x_{n+1} = P_{Q_n} x_1, & n \geq 1,
\end{aligned}\]

where $\phi$ is a bifunction satisfying (A1)-(A4), $T$ is a relaxed $\eta$-$\alpha$ monotone mapping and $S: C \rightarrow C$ is nonexpansive. Under some conditions on the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, they obtained strong convergence of Algorithm (1.3) to a solution of the mixed equilibrium problem (in which $f = 0$), which is also a fixed point of $S$.

Recently, Chen et al. [7] studied the MEP with the relaxed monotone mapping in uniformly convex and uniformly smooth Banach space. They proposed the following algorithm to approximate a common solution of the MEP and fixed point problem for quasi-$\phi$ nonexpansive mapping:
where $\phi$ is a bifunction satisfying (A1)-(A4), $T$ is a relaxed $\eta$-monotone mapping, $f : C \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semi continuous function and $S$ is a quasi-$\phi$ nonexpansive mapping from $C$ to $C$. Under some certain assumptions on the parameter sequences $\{\alpha_n\}$ and $\{r_n\}$, they obtained strong convergence of (1.4) to common solution of MEP and fixed point problem for $S$.

Motivated by the works of Wang et al. [24] and Chen et al. [7], we introduce and study the following Split Generalized Mixed Equilibrium Problem (SGMEP) which involves relaxed monotone mappings:

(1.5) Find $x \in C_1$ such that $x \in G(\phi_1, T_1, f_1, F)$,

(1.6) and $Ax = y \in C_2$ such that $y \in (G(\phi_2, T_2, f_2) \cap F(S))$,

where $C_1$ and $C_2$ are nonempty closed and convex subsets of $H_1$ and $H_2$ respectively, $A : C_1 \to C_2$ is a bounded linear mapping, $\phi_1 : C_1 \times C_1 \to \mathbb{R}$ and $\phi_2 : C_2 \times C_2 \to \mathbb{R}$ are bifunctions, $T_1 : C_1 \to C_1$ and $T_2 : C_2 \to C_2$ are relaxed $\eta$-monotone mappings, $f_1 : C_1 \to (-\infty, +\infty]$ and $f_2 : C_2 \to (-\infty, +\infty]$ are proper convex and lower semicontinuous functions, $S : C_2 \to C_2$ is a nonlinear mapping and $F : C_1 \to C_1$ is a $\mu$-inverse strongly monotone mapping. Throughout this paper, we denote by $\Gamma$, the solution set of SGMEP (1.5)-(1.6). If we consider SGMEP (1.5)-(1.6) separately, then we denote by $G(\phi_1, T_1, f_1, F)$ the solution set of the problem: Find $x \in C$ such that

$$\phi(x, z) + \langle Tx - \eta(z, x) \rangle + f(z) - f(x) + \langle Fx, z - x \rangle \geq 0 \forall z \in C,$$

and by $G(\phi_1, T_1, f_1)$ the solution set of the problem: Find $y \in C$ such that

$$\phi(y, z) + \langle Ty - \eta(z, y) \rangle + f(z) - f(y) + \langle Fry, z - y \rangle \geq 0 \forall z \in C.$$

**Remark 1.2.** We observe that, to prove strong convergence results for MEP and other related optimization problems, the CQ (modified Haugazeau) algorithms are often used. In some other cases (where algorithms other than the CQ algorithm are used), some compactness conditions are assumed on the operators under consideration, or the proof maybe divided into two cases which may result to a very long proof (see, for example [7, 13, 14, 15, 16, 21, 24, 25, 26] and the references therein). On this note, Shehu and Iyiola [22] in 2017, proposed the following modified proximal split feasibility iterative algorithm:

**Algorithm 1.1.**

1. **Given the initial points** $x_1, u \in H_1$
2. **Set** $n = 1$ **and compute:**
3. $y_n = \alpha_n u + (1 - \alpha_n)x_n$
4. $\Theta(y_n) = \|A^*(I - \text{prox}_{\lambda g})Ay_n + (I - \text{prox}_{\lambda f})y_n\|$
5. $z_n = y_n - \rho_n \frac{h(y_n) + \langle y_n \rangle}{\Theta(y_n)} (A^*(I - \text{prox}_{\lambda g})Ay_n + (I - \text{prox}_{\lambda f})y_n)$
6. $x_{n+1} = (1 - \beta_n) y_n + \beta_n z_n.$
(7) If \( A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n \) and \( x_{n+1} = x_n \), then stop, otherwise

(8) set \( n = n + 1 \) and repeat step (3)-(6),

where \( h(y_n) := \frac{1}{2}|| (I - \text{prox}_{\lambda g})Ay_n ||^2 \), \( l(y_n) := \frac{1}{2}|| (I - \text{prox}_{\lambda f})y_n ||^2 \), and the sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\rho_n\} \) satisfy the following conditions:

(i) \( \{\alpha_n\} \subset (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \).

(ii) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \), for all \( n \in \mathbb{N} \).

(iii) \( \liminf_{n \to \infty} \rho_n (4 - \rho_n) > 0 \).

Furthermore, Shehu and Iyiola [22] obtained strong convergence of Algorithm 1.1 to a solution of the following Proximal Split Feasibility Problem (PSFP): Find \( x \in H_1 \) such that

\[
\min_{x \in H_1} \{ f(x) + g(Ax) \},
\]

where \( A : H_1 \to H_2 \) is a bounded linear mapping, \( f : H_1 \to (-\infty, +\infty] \) and \( g : H_2 \to (-\infty, +\infty) \) are proper convex and lower semi-continuous functions.

**Remark 1.3.** As observed by Shehu and Iyiola [22], the termination test in the above algorithm (Algorithm 1.1) is justified by the fact that, if \( A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n \) and \( x_{n+1} = x_n \), then \( x_n \) solves (1.7). This is because \( A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n \) implies that \( y_n \) is a solution of (1.7). Also, from Algorithm 1.1, \( A^*(I - \text{prox}_{\lambda f})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n \) implies that \( z_n = y_n \) and \( x_{n+1} = y_n \). So that, if \( x_{n+1} = x_n \), then we get that \( x_n = y_n \) and hence, \( x_n \) is a solution of \( (1.7) \). Therefore, Algorithm 1.1 is well-defined.

Inspired by the above work of Shehu and Iyiola [22], we obtain strong convergence results for solving our proposed SGMEP (1.5)-(1.6) without using any of the methods mentioned in Remark 1.2 and the method of proof which we adopted appears to be more shorter and easier to read. Our results extends and improves the results of Wang et al. [24], Chen et al. [7], Shehu and Iyiola [22], and many other results in literature.

2. PRELIMINARIES

We state some useful results which will be needed in proving our main results.

**Lemma 2.1.** [5][11] Let \( H \) be a real Hilbert space, then for all \( x, y \in H \) and \( \alpha \in (0, 1) \), the following hold:

(i) \( 2(x, y) = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2 \);

(ii) \( ||x + y||^2 \leq ||x||^2 + 2(x, y) \);

(iii) \( ||\alpha x + (1 - \alpha)y||^2 = \alpha ||x||^2 + (1 - \alpha) ||y||^2 - \alpha(1 - \alpha) ||x - y||^2 \).

**Lemma 2.2.** [27] Let \( H \) be a real Hilbert space and \( T : H \to H \) be a nonlinear mapping, then \( T \) is nonexpansive if and only if \( I - T \) is \( \frac{1}{2} \)-inverse strongly monotone.

**Lemma 2.3.** [7] Let \( C \) be a nonempty, closed and convex subset of a real Hilbert space \( H \) and \( T : C \to H \) be a relaxed \( \eta - \alpha \) monotone mapping. Let \( \phi : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)-(A4) and \( f : C \to \mathbb{R} \cup \{+\infty\} \) be a proper convex function. For \( r > 0 \), define the resolvent mapping \( T_r : H \to C \) associated with \( \phi \), \( T \) and \( f \) by

\[
(\mathcal{T}) = \{ z \in C : \phi(z, y) + \langle Tz, \eta(y, z) \rangle + f(y) - f(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ y \in C \},
\]

for all \( x \in H \), and assume that

(i) \( \eta(x, y) + \eta(y, x) = 0 \ \forall \ x, \ y \in C \),

(ii) for any \( x, y \in C \), \( \alpha(x - y) + \alpha(y - x) \geq 0 \).
Then the following hold:
(1) $T_x$ is single-valued,
(2) $F(T_x) = G(\phi, T, f)$.

**Lemma 2.4.** [23] Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $S : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0 (i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x = Tx$).

**Lemma 2.5.** [19] Let $\{a_n\}$ be a sequence of non-negative numbers such that
$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \gamma_n,$$
where $\{\gamma_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then,
$$\limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} \gamma_n.$$

3. **Main results**

**Lemma 3.1.** Let $H$ be a real Hilbert space and $C$ be a nonempty closed and convex subset of $H$. Let $T : C \to C$ be a relaxed $\eta$-$\alpha$-monotone mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A2). Let $f : C \to (-\infty, +\infty)$ be a proper convex function and $F : C \to C$ be a $\mu$-inverse strongly monotone mapping. Assume that the following conditions are satisfied:

(i) $\eta(x, y) + \eta(y, x) = 0 \forall x, y \in C$,
(ii) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.

Then, for each $r > 0$,

(i) $T_r$ is nonexpansive,
(ii) $||T_r x - y||^2 + ||T_r x - x||^2 \leq ||x - y||^2 \forall x \in H$ and $y \in F(T_r)$,
(iii) for $0 < r \leq s$, we have that $||T_r x - T_s x|| \leq ||x - T_s x|| \forall x \in H$,
(iv) $z \in G(\phi, T, f, F)$ if and only if $z = T_r(I - rF)z$,
(v) for $r \in (0, 2\mu)$, $T_r(I - rF)$ is averaged.

**Proof.** (i) Let $x, y \in H$, then we obtain from (2.1) that
$$\phi(T_r x, w) + \langle T(T_r x), \eta(w, T_r x) \rangle + f(w) - f(T_r x) + \frac{1}{r}\langle w - T_r x, T_r x - x \rangle \geq 0 \forall w \in C.$$ 

In particular, we have
$$\phi(T_r x, T_r y) + \langle T(T_r x), \eta(T_r y, T_r x) \rangle + f(T_r y) - f(T_r x) + \frac{1}{r}\langle T_r y - T_r x, T_r x - x \rangle \geq 0.$$ 

Similarly, we have that
$$\phi(T_r y, T_r x) + \langle T(T_r y), \eta(T_r x, T_r y) \rangle + f(T_r y) - f(T_r x) + \frac{1}{r}\langle T_r x - T_r y, T_r y - y \rangle \geq 0.$$ 

Adding both inequalities, and using assumption (i) and (A2), we obtain
$$\langle T(T_r x) - T(T_r y), \eta(T_r y, T_r x) \rangle + \frac{1}{r}\langle T_r y - T_r x, T_r x - x - T_r y + y \rangle \geq 0.$$ 

Since $T$ is relaxed $\eta$-$\alpha$ monotone, we obtain that
$$\langle T_r y - T_r x, (T_r x - x) - (T_r y - y) \rangle \geq r\langle T(T_r x) - T(T_r y), \eta(T_r y, T_r x) \rangle \geq r\alpha(T_r y - T_r x).$$

By exchanging $x$ and $y$ in (3.1), we obtain
$$\langle T_r x - T_r y, (T_r y - y) - (T_r x - x) \rangle \geq r\alpha(T_r x - T_r y).$$
Adding (3.1) and (3.2), and using assumption (ii), we obtain

\[ 2\langle T_r x - T_r y, (T_r y - y) - (T_r x - x) \rangle \geq 0. \]

That is,

\[ (3.3) \quad \langle T_r x - T_r y, T_r x - T_r y \rangle \leq \langle T_r x - T_r y, y - x \rangle, \]

which implies

\[ ||T_r x - T_r y||^2 \leq ||T_r x - T_r y|| ||x - y||, \]

and this gives that

\[ ||T_r x - T_r y|| \leq ||x - y||. \]

(ii) From (3.3), we obtain that

\[ ||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, y - x \rangle. \]

That is, \( T_r \) is firmly nonexpansive. Thus, for each \( x \in H, y \in F(T_r) \), we obtain from Lemma [2.1]i that

\[ ||T_r x - y||^2 \leq \langle T_r x - y, x - y \rangle \]

(3.4)

That is,

\[ ||T_r x - y||^2 + ||T_r x - x||^2 \leq ||y - x||^2. \]

(iii) Let \( z = T_r x \) and \( w = T_r x \), from (2.1), we have

\[ (3.5) \quad \phi(z, w) + \langle Az, \eta(w, z) \rangle + f(w) - f(z) + \frac{1}{r} \langle w - z, z - x \rangle \geq 0. \]

Similarly we obtain that

\[ (3.6) \quad \phi(w, z) + \langle Aw, \eta(z, w) \rangle + f(z) - f(w) + \frac{1}{s} \langle z - w, w - x \rangle \geq 0. \]

Adding equation (3.5) and (3.6), we obtain from assumption (i) that

\[ (3.7) \quad 2\phi(z, z) + \phi(w, z) + \langle Az - Aw, \eta(w, z) \rangle + \frac{1}{r} \langle w - z, z - x \rangle + \frac{1}{s} \langle z - w, w - x \rangle \geq 0. \]

Using condition (A2), we have

\[ (3.8) \quad \frac{1}{r} \langle w - z, z - x \rangle + \frac{1}{s} \langle z - w, w - x \rangle \geq \langle Aw - Az, \mu(w, z) \rangle \geq \alpha(w - z), \]

Observe that adding (3.5) and (3.6), and using assumption (i) and (A2), one can also get that

\[ \langle Aw - Az, \eta(z, w) \rangle + \frac{1}{r} \langle w - z, z - x \rangle + \frac{1}{s} \langle z - w, w - x \rangle \geq 0, \]

which by the definition of \( T \) implies

\[ (3.9) \quad \frac{1}{s} \langle z - w, w - x \rangle + \frac{1}{r} \langle w - z, z - x \rangle \geq \alpha(z - w). \]

Adding (3.8) and (3.9), and using condition (ii), we have

\[ (3.10) \quad 2 \left( \frac{1}{r} \langle w - z, z - x \rangle + \frac{1}{s} \langle z - w, w - x \rangle \right) \geq 0, \]

which implies that

\[ \langle x - z, z - w \rangle \geq \frac{r}{s} \langle x - w, z - w \rangle. \]
Thus, from Lemma 2.1 (i), we have that
\[(3.11)\]|x - w|^2 - |x - z|^2 - |z - w|^2 | \geq \frac{r}{s} \left( |x - w|^2 + |w - z|^2 - |x - z|^2 \right).

Since \( \frac{r}{s} \leq 1 \), we obtain that
\[
\left(1 + \frac{r}{s}\right) |z - w|^2 \leq \left(1 - \frac{r}{s}\right) |x - w|^2.
\]
So that
\[(3.12)\]|z - w|^2 \leq \left(\frac{s - r}{s + r}\right) |x - w|^2 \leq |x - w|^2.

Hence, \(|T_r x - T_s x| \leq |x - T_s x| \forall x \in H.

(iv) \(z \in G(\phi, T, f, F) \iff \phi(z, y) + \langle Tz - \eta(y, z) \rangle + f(y) - f(z) + \langle Fz, y - z \rangle \geq 0 \forall y \in C \)
\[
\iff \phi(z, y) + \langle Tz, \eta(y, z) \rangle + f(y) - f(z) + \frac{1}{r} \langle z - (z - rFz), y - z \rangle \geq 0
\iff \phi(z, y) + \langle Tz, \eta(y, z) \rangle + f(y) - f(z) + \frac{1}{r} \langle z - (I - rFz), y - z \rangle \geq 0
\iff z = T_r(I - rF)z.

(v) We first observe that for \( r \in (0, 2\mu) \), \( (I - rF) \) is \( \frac{1}{r\mu} \)-averaged. Also, since \( T_r \) is firmly nonexpansive, we have that \( T_r \) is averaged. Hence, the composition \( T_r(I - rF) \) is averaged for \( r \in (0, 2\mu) \).

Under the assumptions of Lemma 3.1, we make the following remark.

**Remark 3.1.** (i) Since every averaged mapping is nonexpansive, we have from Lemma 3.1 (v) that \( T_r(I - rF) \) is nonexpansive for \( r \in (0, 2\mu) \).

(ii) For \( r \in (0, 2\mu) \), we obtain from Remark 1.1 and Lemma 3.1 (v) that \( T_r(I - rF) \) is firmly nonexpansive. Thus, for any \( x \in H \) and \( y \in F(T_r(I - rF)) \) with \( r \in (0, 2\mu) \), we have from Lemma 2.1 (i) that
\[
||T_r(I - rF)x - y||^2 \leq \langle T_r(I - rF)x - y, x - y \rangle
= \frac{1}{2} \left[ ||T_r(I - rF)x - y||^2 + ||x - y||^2 - ||T_r(I - rF)x - x||^2 \right],
\]
which implies
\[
||y - T_r(I - rF)x||^2 + ||x - T_r(I - rF)x||^2 \leq ||y - x||^2.
\]

**Lemma 3.2.** Let \( H \) be a real Hilbert space and \( C \) be a nonempty closed and convex subset of \( H \). Let \( T : C \rightarrow H \) be a relaxed \( \eta\alpha \)-monotone mapping and \( \phi : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A2). Let \( f : C \rightarrow (-\infty, +\infty] \) be a proper convex function and \( F : C \rightarrow H \) be any nonlinear mapping. Assume that the following conditions are satisfied:

(i) \( \eta(x, y) + \eta(y, x) = 0 \forall x, y \in C \),

(ii) for any \( x, y \in C \), \( \alpha(x - y) + \alpha(y - x) \geq 0 \).

Then, for \( 0 < r \leq s \), we have that \( ||T_r(I - rF)x - T_s(I - sF)x|| \leq ||x - T_s(I - sF)x|| \forall x \in H \).

**Proof.** Let \( z = T_r(I - rF)x \) and \( w = T_s(I - sF)x \), from (2.1), we have
\[
\phi(z, w) + \langle Az, \eta(w, z) \rangle + f(w) - f(z) + \frac{1}{r} \langle w - z, z - (I - rF)x \rangle \geq 0.
\]
Similarly, we obtain that

\[(3.14) \quad \phi(w, z) + \langle Aw, \eta(z, w) \rangle + f(z) - f(w) + \frac{1}{s}(z - w, w - (I - sF)x) \geq 0.\]

Thus, following the same line of arguments as in (3.7)-(3.10), we obtain that

\[2 \left( \frac{1}{r}(w - z, z - (I - rF)x) + \frac{1}{s}(z - w, w - (I - sF)x) \right) \geq 0.\]

That is,

\[\langle x - z - rFx, z - w \rangle - \frac{r}{s} \langle x - w - sFx, z - w \rangle \geq 0.\]

Hence,

\[\langle (x - rFx - z) - \left( \frac{r}{s}x - rFx - \frac{r}{s}w \right), z - w \rangle \geq 0,\]

which implies that

\[\langle x - z, z - w \rangle \geq \frac{r}{s} \langle x - w, z - w \rangle.\]

By the same line of arguments as in (3.11)-(3.12), we obtain the desired result.

Throughout this paper, we shall write \(T_r^{(1)}\) for the resolvent mapping associated with \(\phi_1,\ T_1\) and \(f_1\), and \(T_r^{(2)}\) for the resolvent mapping associated with \(\phi_2,\ T_2\) and \(f_2\). We also make the following assumptions

**Assumption 3.1.** Assume that \(\{\alpha_n\}, \{\beta_n\}\) and \(\{t_n\}\) are sequences of real numbers satisfying the following:

(i) \(\{\alpha_n\} \subset (0, 1)\) such that \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty.\)

(ii) \(0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1\), for all \(n \in \mathbb{N}.\)

(iii) \(\lim \inf_{n \to \infty} (2 - t_n) > 0.\)

Let \(h(x) := \frac{1}{2}||I - ST_r^{(2)}Ax||^2\) and \(l(x) := \frac{1}{2}||I - T_r^{(1)}(I - rF)x||^2.\) Then, we consider the following algorithm to study problem (1.5)-(1.6).

**Algorithm 3.1.**

1. Let \(\{\alpha_n\}, \{\beta_n\}\) and \(\{t_n\}\) be such that Assumption 3.1 is satisfied.

2. Given the initial point \(x_1 \in C_1\).

3. Set \(n = 1\) and compute:

4. \(y_n = \alpha_n g(x_n) + (1 - \alpha_n)x_n.\)

5. \(\Theta(y_n) := ||A^*(I - ST_r^{(2)})Ay_n + (I - T_r^{(1)}(I - rF))y_n||.\)

6. \(z_n = y_n - t_n \frac{h(y_n) + l(y_n)}{\Theta(y_n)} (A^*(I - ST_r^{(2)})Ay_n + (I - T_r^{(1)}(I - rF))y_n).\)

7. \(x_{n+1} = (1 - \beta_n)y_n + \beta_n z_n.\)

8. If \(A^*(I - ST_r^{(2)})Ay_n = 0 = (I - T_r^{(1)}(I - rF))y_n\) and \(x_{n+1} = x_n\), then stop, otherwise

9. set \(n = n + 1\) and repeat step (4)-(7).

We observe here that, by similar argument as in Remark 1.3, one can easily see that Algorithm 3.1 is well defined. Therefore, using Algorithm 3.1, we present in what follows, our strong convergence theorem for solving problem (1.5)-(1.6).

**Theorem 3.3.** Let \(C_1\) and \(C_2\) be nonempty closed and convex subsets of real Hilbert spaces \(H_1\) and \(H_2\) respectively, and \(A : C_1 \to C_2\) be a bounded linear mapping. Let \(\phi_1 : C_1 \times C_1 \to \mathbb{R},\ \phi_2 : C_2 \times C_2 \to \mathbb{R}\) be bifunctions satisfying (A1)-(A4) and \(T_1 : C_1 \to C_1, T_2 : C_2 \to C_2\) be \(\eta\)-hemicontinuous and relaxed \(\eta\)-monotone mappings. Let \(f_1 : C_1 \to (-\infty, +\infty], f_2 : C_2 \to (-\infty, +\infty]\) be proper convex and lower semicontinuous functions and \(F : C_1 \to C_1\) be a \(\mu\)-inverse strongly monotone mapping. Let \(S : C_2 \to C_2\) be a nonexpansive mapping and
\(g : C_1 \rightarrow C_1\) be a contraction with constant \(k\). Suppose that \(\Gamma \neq \emptyset\) and \(\{r_n\}\) is a real sequence such that \(0 < r \leq r_n \leq b < 2\mu\). Then, the sequence generated by Algorithm 3.1 converges strongly to \(z \in \Gamma\), where \(z = P_\Gamma g(z)\).

**Proof.** Let \(z \in P_\Gamma g(z)\) and \(J_{r_n} = T_{r_n}(I - r_nF)\), then \(z = J_{r_n}z\) and \(Az = ST_{r_n}(2)(Az)\). Also, since \(0 < r \leq r_n \leq b < 2\mu\), we have from Remark 3.1(i) that \(J_{r_n}\) is nonexpansive. Again, from Lemma 3.1(i), we obtain that \(S \circ T_{r_n}(2)\) is nonexpansive. Thus, by Lemma 2.2, we obtain that

\[
\langle (I - ST_{r_n}(2))Ay_n, Ay_n - Az \rangle = \langle (I - ST_{r_n}(2))Ay_n - (I - ST_{r_n}(2))Az, Ay_n - Az \rangle
\]

\[
\geq \frac{1}{2} \| (I - ST_{r_n}(2))Ay_n - (I - ST_{r_n}(2))Az \|^2
\]

\[
= h(y_n).
\]

(3.15)

Similarly, we obtain that

\[
\langle (I - J_{r_n})y_n, y_n - z \rangle \geq l(y_n).
\]

(3.16)

From Lemma 2.1(i), (3.15), (3.16) and Algorithm 3.1 we obtain

\[
\|z_n - z\|^2 = \|y_n - z\|^2 - 2t_n \frac{h(y_n) + l(y_n)}{\Theta^2(y_n)} \langle A^\ast(I - ST_{r_n}(2))Ay_n + (I - J_{r_n})y_n, y_n - z \rangle
\]

\[
+ \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^2(y_n)} \| A^\ast(I - ST_{r_n}(2))Ay_n + (I - J_{r_n})y_n \|^2
\]

\[
= \|y_n - z\|^2 - 2t_n \frac{h(y_n) + l(y_n)}{\Theta^2(y_n)} \left[ \langle (I - ST_{r_n}(2))Ay_n, Ay_n - Az \rangle + \langle (I - J_{r_n})y_n, y_n - z \rangle \right]
\]

\[
+ \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^2(y_n)}
\]

\[
\leq \|y_n - z\|^2 - 2t_n \frac{h(y_n) + l(y_n)}{\Theta^2(y_n)} \left( h(y) + l(y_n) \right) + \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^2(y_n)}
\]

(3.47) \(\|y_n - z\|^2 - t_n(2 - t_n) \left[ \frac{(h(y_n) + l(y_n))^2}{\Theta^2(y_n)} \right] \).

Now, observe from Algorithm 3.1 that

\[
x_{n+1} - y_n = \beta_n(z_n - y_n).
\]

(3.18)

Thus, we obtain from Algorithm 3.1 that

\[
\|x_{n+1} - z\|^2 = \| (y_n - z) - \beta_n(y_n - z_n) \|^2
\]

\[
= \|y_n - z\|^2 - 2\beta_n(y_n - z, y_n - z_n) + \beta_n^2 \|y_n - z_n\|^2
\]

\[
\leq \|y_n - z\|^2 - \beta_n(1 - \beta_n) \|y_n - z_n\|^2
\]

(3.19)

\[
= \|y_n - z\|^2 - \frac{1}{\beta_n(1 - \beta_n)} \|x_{n+1} - y_n\|^2.
\]
From Algorithm [3.1], we obtain
\[
\|x_{n+1} - z\| = \|\alpha_n(g(x_n) - g(z)) + \alpha_n(g(z) - z) + (1 - \alpha_n)(x_n - z)\| \\
\leq \alpha_n k \|x_n - z\| + \alpha_n \|g(z) - z\| + (1 - \alpha_n) \|x_n - z\| \\
= (1 - \alpha_n(1 - k)) \|x_n - z\| + \alpha_n \|g(z) - z\| \\
\leq \max \left\{ \|x_n - z\|, \frac{\|g(z) - z\|}{1 - k} \right\} \\
\vdots \\
\leq \max \left\{ \|x_1 - z\|, \frac{\|g(z) - z\|}{1 - k} \right\}.
\]
Hence, \(\{x_n\}\) is bounded. So are \(\{y_n\}\) and \(\{z_n\}\). Now, from (3.18), we obtain
\[
(3.20) \quad \|z_n - y_n\|^2 = \frac{1}{\beta_n^2} \|x_{n+1} - y_n\|^2 = \frac{\alpha_n}{\beta_n^2} (\|x_{n+1} - y_n\|)^2.
\]
Also, from Algorithm [3.1] and Lemma 2.1(ii), we obtain
\[
\|y_n - z\|^2 = \|\alpha_n(g(x_n) - g(z)) + \alpha_n(g(z) - z) + (1 - \alpha_n)(x_n - z)\|^2 \\
\leq \|\alpha_n(g(x_n) - g(z)) + (1 - \alpha_n)(x_n - z)\|^2 + 2\alpha_n \|g(z) - z, y_n - z\| \\
\leq \alpha_n^2 k^2 \|x_n - z\|^2 + (1 - \alpha_n)^2 \|x_n - z\|^2 \\
+ 2\alpha_n \|1 - \alpha_n\| \|g(x_n) - g(z), x_n - z, x_n - z\| + 2\alpha_n \|g(z) - z, y_n - z\| \\
\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \|g(z) - z, y_n - z\| \\
+ 2\alpha_n (1 - \alpha_n) \|x_n - z\|^2 \\
= (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 - 2\alpha_n \|\langle g(z) - z, z - y_n\rangle - \frac{\alpha_n(1 + k^2)}{2} \|x_n - z\|^2\]
\[
(3.21) \quad (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 - 2\alpha_n \|\langle g(z) - z, z - y_n\rangle - \frac{\alpha_n(1 + k^2)}{2} \|x_n - z\|^2
\]
From (3.19) and (3.21), we obtain that
\[
\|x_{n+1} - z\|^2 \leq (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 - 2\alpha_n \|\langle g(z) - z, z - y_n\rangle - \frac{\alpha_n(1 + k^2)}{2} \|x_n - z\|^2\]
\[
- \frac{1}{\beta_n^2} \|x_{n+1} - y_n\|^2 \\
= (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 \\
(3.22) \quad -2\alpha_n \|\langle g(z) - z, z - y_n\rangle + \frac{1}{\beta_n^2 (1 - \alpha_n^2)} ||x_{n+1} - y_n\|^2 - \frac{\alpha_n(1 + k^2)}{2} ||x_n - z\|^2\]
Let \(\gamma_n = \langle g(z) - z, z - y_n\rangle + \frac{1}{\beta_n^2 (1 - \alpha_n^2)} ||x_{n+1} - y_n\|^2 - \frac{\alpha_n(1 + k^2)}{2} ||x_n - z\|^2\). Then, (3.22) becomes
\[
\|x_{n+1} - z\|^2 \leq (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 - 2\alpha_n \gamma_n \\
(3.23) \quad (1 - 2\alpha_n(1 - k)) \|x_n - z\|^2 + 2\alpha_n(1 - k)(-\gamma_n).
\]
Let \(\delta_n = 2\alpha_n(1 - k)\). Then, it follows from Assumption 3.1(i) that \(\sum_{n=1}^{\infty} \delta_n = \infty\). Also, we know that \(\{x_n\}\) is bounded below (so is \(\{y_n\}\)), thus \((-\gamma_n)\) is bounded above. Hence, applying
Lemma 2.5 in (3.23), we obtain that
\[
\limsup_{n \to \infty} ||x_n - z||^2 \leq \limsup_{n \to \infty} \gamma_n
\]
(3.24)
That is,
\[
\liminf_{n \to \infty} \gamma_n \leq -\limsup_{n \to \infty} ||x_n - z||^2.
\]
Thus, \( \liminf_{n \to \infty} \gamma_n \) exists. Also, by Assumption 3.1 (i), we obtain that
\[
\lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \left( g(z) - z, z - y_n \right) + \frac{1}{2\alpha_n \beta_n} (1 - \beta_n) ||x_{n+1} - y_n||^2.
\]
(3.25)
Hence, \( \lim\inf_{n \to \infty} \gamma_n = \lim\inf_{k \to \infty} \left( g(z) - z, z - y_n \right) + \frac{1}{2\alpha_n \beta_n} (1 - \beta_n) ||x_{n+1} - y_n||^2 \).

Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) that converges to a point \( x^* \in C_1 \), and
\[
\lim_{k \to \infty} \gamma_n = \lim_{k \to \infty} \left( g(z) - z, z - y_{n_k} \right) + \frac{1}{2\alpha_n \beta_n} (1 - \beta_n) ||x_{n+1} - y_{n_k}||^2.
\]
(3.25)
Hence, \( \left\{ \frac{1}{2\alpha_n \beta_n} (1 - \beta_n) ||x_{n+1} - y_{n_k}||^2 \right\} \) is bounded. Furthermore, Assumption 3.1 implies that there exists \( b \in (0, 1) \) such that \( \beta_n \leq b \leq 1 \). Thus,
\[
\frac{1}{2\alpha_n \beta_n} (1 - \beta_n) \geq \frac{1}{2\alpha_n \beta_n} (1 - b) > 0,
\]
which implies that \( \left\{ \frac{1}{2\alpha_n \beta_n} ||x_{n+1} - y_{n_k}||^2 \right\} \) is bounded. Also, Assumption 3.1 implies that there exists \( a \in (0, 1) \) such that \( 0 < a \leq \beta_n \). Thus, \( 0 < \frac{\alpha_n}{\beta_n} \leq \frac{\alpha_n}{a} \to 0 \), \( k \to \infty \). Hence, we obtain from (3.20) and the fact that \( \left\{ \frac{1}{\alpha_n \beta_n} ||x_{n+1} - y_{n_k}||^2 \right\} \) is bounded that
\[
\lim_{k \to \infty} ||z_{n_k} - y_{n_k}|| = 0.
\]
(3.26)
From Algorithm 3.1 and (3.26), we obtain that
\[
||x_{n+1} - y_n|| = \beta_n ||z_{n_k} - y_{n_k}|| \to 0, \ k \to \infty.
\]
(3.27)
Again, we obtain from Algorithm 3.1 that
\[
||y_n - x_n|| = \alpha_n ||g(x_n) - x_n|| \to 0, \ k \to \infty.
\]
(3.28)
From (3.27) and (3.28), we obtain that
\[
\lim_{k \to \infty} ||x_{n+1} - x_n|| = 0.
\]
(3.29)
Also, from (3.17) and (3.26), we obtain that
\[
t_{n_k} (2 - t_{n_k}) \left( \frac{h(y_{n_k}) + l(y_{n_k})}{\Theta^2(y_{n_k})} \right) \leq ||y_{n_k} - z||^2 - ||z_{n_k} - z||^2
\]
\[
\leq ||y_{n_k} - z_{n_k}||^2 + 2||y_{n_k} - z_{n_k}|| ||z_{n_k} - z|| \to 0, \ k \to \infty.
\]
By Assumption 3.1, we obtain that \( \lim_{k \to \infty} \frac{h(y_{n_k}) + l(y_{n_k})}{\Theta^2(y_{n_k})} = 0 \). Consequently, we obtain that
\[
\lim_{k \to \infty} h(y_{n_k}) + l(y_{n_k}) = 0 \iff \lim_{k \to \infty} h(y_{n_k}) = 0 \text{ and } \lim_{k \to \infty} l(y_{n_k}) = 0.
\]
That is,
\[
\lim_{k \to \infty} ||Ay_{n_k} - ST^{(2)}_{n_k} Ay_{n_k}|| = 0, \text{ and}
\]

AJMAA, Vol. 15, No. 2, Art. 13, pp. 1-16, 2018
From (3.32) and Lemma 3.1(iii), we obtain that
\[(3.33)\]
Also, \[(3.32)\]
(ii), we obtain that
\[(3.31)\]
Now, set \(v_n = T^{(2)}_n Ay_n\), then (3.30) becomes \[\lim_{k \to \infty} \|Ay_{n_k} - S v_{n_k}\| = 0.\] Thus, from Lemma 3.1 (ii), we obtain that
\[
\|Ay_{n_k} - v_{n_k}\|^2 \leq \|Ay_{n_k} - Az\|^2 - \|v_{n_k} - Az\|^2 \\
\leq \|Ay_{n_k} - Az\|^2 - \|S v_{n_k} - SAz\|^2 \\
\leq \|Ay_{n_k} - S v_{n_k}\|^2 + 2\|Ay_{n_k} - S v_{n_k}\|\|S v_{n_k} - SAz\| \to 0, \quad k \to \infty.
\]
That is,
\[(3.34)\]
\[
\lim_{k \to \infty} \|Ay_{n_k} - T^{(2)}_n Ay_{n_k}\| = 0.
\]
Also,
\[
\|Ay_{n_k} - S A y_{n_k}\| \leq \|Ay_{n_k} - S v_{n_k}\| + \|S v_{n_k} - S A y_{n_k}\| \\
\leq \|Ay_{n_k} - S v_{n_k}\| + \|v_{n_k} - A y_{n_k}\| \to 0, \quad k \to \infty.
\]
From (3.32) and Lemma 3.1(iii), we obtain that
\[
\|Ay_{n_k} - T^{(2)}_n Ay_{n_k}\| \leq \|Ay_{n_k} - T^{(2)}_n A y_{n_k}\| + \|T^{(2)}_n A y_{n_k} - T^{(2)}_n Ay_{n_k}\| \\
\leq 2\|Ay_{n_k} - T^{(2)}_n A y_{n_k}\| \to 0, \quad k \to \infty.
\]
Again, by Lemma 3.2, we obtain that
\[
\|y_{n_k} - J_r y_{n_k}\| \leq \|y_{n_k} - J_r y_{n_k}\| + \|J_r y_{n_k} - J r y_{n_k}\| \\
\leq 2\|y_{n_k} - J_r y_{n_k}\| \to 0, \quad k \to \infty.
\]
Since \(\{x_n\}\) converges weakly to \(x^* \in C_1\), we have from (3.28) that there exists a subsequence \(\{y_n\}\) of \(\{y_n\}\) such that \(\{y_n\}\) converges weakly to \(x^* \in C_1\). Also, since \(A\) is a bounded linear mapping, we have that there exists a subsequence \(\{Ay_{n_k}\}\) of \(\{Ay_n\}\) that converges weakly to \(Ax^* \in C_2\). It then follows from Lemma 2.4 (3.33), (3.34) and (3.35) that \(Ax^* \in (F(S) \cap F(T^{(2)}_r))\) and \(x^* \in F(J_r)\). Hence, \(x^* \in \Gamma\).

We now show that \(\{x_n\}\) converges strongly to \(x\). Now, from (3.25), (3.27) and by the property of the metric projection \(P_C\), we obtain
\[
\lim_{n \to \infty} \gamma_n = \lim_{k \to \infty} \langle g(z) - z, z - y_{n_k}\rangle = \langle g(z) - z, z - x^*\rangle \\
\geq 0.
\]
Thus, from (3.24), we obtain that \(\lim_{n \to \infty} \|x_n - z\|^2 \leq 0\). Hence, \(\lim_{n \to \infty} \|x_n - z\|^2 = 0\). Therefore, we conclude that \(\{x_n\}\) converges strongly to \(z\). ]

Consider the following Split Mixed Equilibrium Problem:
\[(3.36)\]

Find \(x \in C_1\) such that \(x \in G(\phi_1, T_1, f_1),\)
\[(3.37)\]

and \(Ax = y \in C_2\) such that \(y \in G(\phi_2, T_2, f_2),\)

where \(\phi_1, T_1, f_1, \phi_2, T_2, f_2\) are as defined in Theorem 3.3

As corollary of our main results, we can solve Problem (3.36)–(3.37) by setting \(S = I\) and \(F = 0\) in Algorithm 3.1. Also, by setting \(\phi_1 = \phi_2 = T_1 = T_2 = F = 0\) and \(S = I\) in Algorithm 3.1, we can apply Theorem 3.3 to solve the Proximal Split Feasibility Problem studied in [22].
4. Split Generalized Mixed Equilibrium Problem over the Solution Set of Variational Inclusions

Recall that a multivalued mapping \( M : H \to 2^H \) is called monotone, if
\[
\langle x - y, u - v \rangle \geq 0 \quad \forall x, y \in H, \ u \in M(x), \ v \in M(y),
\]
and maximal monotone if the graph \( G(M) \) of \( M \) defined by
\[
G(M) =: \{(x, y) \in H \times H : y \in M(x)\}
\]
is not properly contained in the graph of any other monotone mapping. The resolvent operator \( J^{M}_\lambda \) associated with a mapping \( M \) and \( \lambda \) is the mapping \( J^{M}_\lambda : H \to 2^H \) defined by
\[
J^{M}_\lambda (x) = (I + \lambda M)^{-1}x, \ x \in H, \lambda > 0.
\]
It is known that if the mapping \( M \) is monotone, then \( J^{M}_\lambda \) is single valued and firmly nonexpansive (see [2]).

Now, consider the following Monotone Variational Inclusion Problem (MVIP): Find
\[
x \in H \text{ such that } 0 \in M_1(x) + F_2(x),
\]
where \( M_1 : H \to 2^H \) is a multivalued mapping and \( F_2 : H \to 2^H \) is a single valued mapping. We shall denote the solution set of problem (4.2) by \((M_1 + F_2)^{-1}(0)\). In [20], Moudafi proved that \( x \in (M_1 + F_2)^{-1}(0) \) if and only if \( x = J^{M_1}_\lambda (I - \lambda F_2)(x), \ \forall \lambda > 0 \). It was also shown in [20] that, if \( F_2 \) is a \( \mu \)-inverse strongly monotone mapping and \( M_1 \) is a maximal monotone mapping, then \( J^{M_1}_\lambda (I - \lambda F_2) \) is averaged with \( 0 < \lambda < 2\mu \). Hence, \( J^{M_1}_\lambda (I - \lambda F_2) \) is a nonexpansive mapping with \( 0 < \lambda < 2\mu \).
Thus, by setting \( S = J^{M_1}_\lambda (I - \lambda F_2) \) in Algorithm 3.1, we can apply Theorem 3.3 to solve the following SGMEP over the solution set of MVIP:
\[
\text{Find } x \in C_1 \text{ such that } x \in G(\phi_1, T_1, f_1, F_1),
\]
\[
\text{and } Ax = y \in C_2 \text{ such that } y \in \left(G(\phi_2, T_2, f_2) \cap (M_1 + F_2)^{-1}(0)\right),
\]
where \( \phi_1, T_1, f_1, \phi_2, T_2, f_2 \) are as defined in Theorem 3.3.

References


