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EULER-MACLAURIN FORMULAS FOR FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. The first-order Euler-Maclaurin formula relates the sum of the values of a smooth function on an interval of integers with its integral on the same interval on $\mathbb{R}$. We formulate here the analogue for functions that are just of bounded variation.

Key words and phrases: Euler-Maclaurin; Bounded variation; Sums; Series.

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1. INTRODUCTION

The first order Euler-Maclaurin formula for a smooth function $f : [a, b] \to \mathbb{R}$ ($a < b$ in $\mathbb{Z}$) states that

$$\sum_{a\leq k<b} f(k) = \int_a^b f(t)\,dt - \frac{1}{2}\,[f]_a^b + R, \quad R = \int_a^b f'(t)B_1(t-[t])\,dt, \tag{1.1}$$

where $B_1(t) = t - \dfrac{1}{2}$ is the first Bernoulli polynomial. The formula is useful in the approximation of finite sums, and to relate the convergence of generalized integrals with that of numerical series: we refer to [1, 2, 3] for a survey on the subject. Since $|B_1| \leq \dfrac{1}{2}$ on $[0, 1]$ it follows that the remainder R is bounded above by $\dfrac{1}{2}\displaystyle\int_a^b |f'(t)|\,dt$, so that if f is monotonic one has

$$|R| \leq \frac{1}{2}|f(b) - f(a)|.$$

The proof is based on a simple, though smart, integration by parts and begins assuming that f is defined on $[0, 1]$: since $B_1' = 1$, writing that

$$\int_0^1 f(t)\,dt = \int_0^1 f(t)B_1'(t)\,dt = [fB_1]_0^1 - \int_0^1 f'(t)B_1(t)\,dt$$

yields

$$f(0) = \int_0^1 f(t)\,dt - \frac{1}{2}\,[f]_0^1 + \int_0^1 f'(t)B_1(t)\,dt,$$

which is (1.1) when $a = 0$ and $b = 1$.

In Theorem 3.1 we show that if f is just of bounded variation (BV) on $[a, b]$ then (1.1) holds with the exception that the remainder R is bounded above by $\dfrac{1}{2}\,\mathrm{pV}(f, [a, b])$. The proof of the result is elementary: indeed one can deal with monotonic function, and adapt the same arguments that are involved in the proof of the integral criterion for the convergence of a series with monotonic terms; part of the material arises from the thesis [4]. In the final part of Section 3 we obtain the results that follow the traditional Euler-Maclaurin formula for a smooth function, that is assumed here to be just BV: the approximation of the partial sums of the series $\displaystyle\sum_{0\leq k\leq N} f(k)$ in terms of $\displaystyle\sum_{0\leq k\leq N} f(k)$ ($n < N$), the existence of the Euler constant with a related asymptotic formula for $\displaystyle\sum_{0\leq k\leq N} f(k)$ as $n \to +\infty$ and a generalization to BV functions of the integral test for the convergence of a series.

The BV version of (1.1) is formulated in Theorem 4.3: the new formula takes into account the possible lack of continuity of the function f, and relates the sum of the averages of the left and right limits of f in an interval of integers with the Euler-Maclaurin first-order development $\displaystyle\int_a^b f(t)\,dt - \frac{1}{2}(f(b^-) - f(a^-))$. The remainder, the analogue of R in (1.1), is here the explicit integral of the mid-value modification of $B_1(t - [t])$, with respect to the Lebesgue-Stieltjes measure associated to f. Quite surprisingly, deducing Theorem 3.1 from the Euler-Maclaurin formula for BV functions as stated in Theorem 4.3 is not straightforward.

In Section 4 we prove a version of (1.1) based on a partial integration formula for BV functions; in this formula the measure theoretic variation of the function is involved, which may be smaller than the point variation for discontinuous functions, and to deduce (1.1) from it we need to explicit the formula that connects the two variations; this is done in Proposition 4.1.

We are not aware of other formulations of the Euler-Maclaurin formulas for BV functions in the spirit of Theorem 3.1. Instead, the approximation formula for the sum of a series (Corollary 3.4) was established in a more general setting in [5], [6, 4.1.5] for functions whose r-th derivative is BV. The methods involved there arise from Fourier analysis, far from our elementary approach. A recent extension, comparing in the multidimensional case the Fourier integral of a function of bounded variation and the corresponding trigonometric series with its Fourier coefficients was recently established in [7].

2. NOTATION

Our main reference for the basic facts and related notation on BV functions is [8]. Let us recall that a real valued function f defined on an interval I is of Bounded Variation (we often simply write BV) if the so-called *pointwise variation* $\mathrm{pV}(f, I)$ of f on I, given by

$$\mathrm{pV}(f, I) := \sup\left\{ \sum_{0\le i<n} |f(t_{i+1}) - f(t_i)| : \, t_i \in I, \, t_0 < t_1 < \cdots < t_n \right\}$$

is finite. In this case there exist two increasing and bounded functions $f_1, f_2 : I \to \mathbb{R}$ satisfying

$$f = f_1 - f_2, \qquad \mathrm{pV}(f, I) = \mathrm{pV}(f_1, I) + \mathrm{pV}(f_2, I). \tag{2.1}$$

In particular, every function of bounded variation is locally integrable. The left and right limit of a BV function f in c will be denoted, respectively, $f(c^-)$ and $f(c^+)$.

We find useful here to adopt the following sum notation that is quite common in the field of Discrete Calculus: if $a < b$ are natural numbers we set

$$\sum_{a\le k<b} f(k) := \sum_{k=a}^{b-1} f(k).$$

Moreover, we set $[f]_a^b = f(b) - f(a)$.

3. A EULER-MACLAURIN TYPE FORMULA FOR BV FUNCTIONS AND ITS CONSEQUENCES

3.1. A Euler-Maclaurin type formula.

Theorem 3.1 (Euler-Maclaurin type formula for BV functions)**.** *Let a, b in $\mathbb{Z}$ and $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then*

$$\sum_{a\le k<b} f(k) = \int_a^b f(x)\, dx - \frac{1}{2}\, [f]_a^b + R, \quad |R| \le \frac{1}{2}\, \mathrm{pV}(f, [a, b]). \tag{3.1}$$

Proof. Assume first that f is monotonic increasing. On every interval $[k, k + 1]$ ($k \in \mathbb{Z}$) contained in $[a, b]$ one has

$$f(k) \le f(t) \le f(k + 1) \qquad \forall t \in [k, k + 1],$$

from which it follows that

$$f(k) = \int_k^{k+1} f(k)\, dt \le \int_k^{k+1} f(t)\, dt \le \int_k^{k+1} f(k + 1)\, dt = f(k + 1).$$

Summing the terms of the foregoing inequalities, as k varies between a and $b-1$, one obtains

$$\sum_{a\le k<b} f(k) \le \int_a^b f(t)\,dt \le \sum_{a\le k<b} f(k) + f(b) - f(a).$$

Subtracting the term $\frac{1}{2}(f(b)-f(a))$ from the members of the preceding inequalities one finds

$$\sum_{a\le k<b} f(k) - \frac{1}{2}(f(b)-f(a)) \le \int_a^b f(t)\,dt - \frac{1}{2}(f(b)-f(a))$$
$$\le \sum_{a\le k<b} f(k) + \frac{1}{2}(f(b)-f(a)),$$

from which the conclusion follows.
If f is of bounded variation, let f_1, f_2 be as in (2.1): since

$$\sum_{a\le k<b} f_i(k) = \int_a^b f_i(x)\,dx - \frac{1}{2}\left[f_i\right]_a^b + R_i, \quad |R_i| \le \frac{1}{2}\,\mathrm{pV}(f_i,[a,b]) \quad (i=1,2)$$

by subtracting term by term we get

$$\sum_{a\le k<b} f(k) = \int_a^b f(x)\,dx - \frac{1}{2}\left[f\right]_a^b + R, \qquad R = R_1 - R_2,$$

so that

$$|R| \le |R_1| + |R_2| \le \frac{1}{2}\,\mathrm{pV}(f_1,[a,b]) + \frac{1}{2}\,\mathrm{pV}(f_2,[a,b]) = \frac{1}{2}\,\mathrm{pV}(f,[a,b]),$$

proving the claim. ∎

Remark 3.1. If f is monotonic on $[a,b]$ then $\mathrm{pV}(f,[a,b]) = |f(b)-f(a)|$, the remainder term R can be thus estimated by $\frac{1}{2}|f(b)-f(a)|$: this fact is well known as a consequence of the Euler-Maclaurin formula when f is monotonic or of class C^1 [1].

Corollary 3.2 (The approximation formula for finite sums)**.** *Let $f:[0,+\infty[\to\mathbb{R}$ be of bounded variation. For every $N\ge n$ the following **approximation formula** holds:*

$$\sum_{0\le k<N} f(k) = \sum_{0\le k<n} f(k) + \int_n^N f(x)\,dx - \frac{1}{2}\left[f\right]_n^N + \varepsilon_1(n,N),$$
$$|\varepsilon_1(n,N)| \le \frac{1}{2}\,\mathrm{pV}(f,[n,N]) \le \frac{1}{2}\,\mathrm{pV}(f,[n,+\infty[). \tag{3.2}$$

Proof. It is enough to remark that

$$\sum_{0\le k<N} f(k) - \sum_{0\le k<n} f(k) = \sum_{n\le k<N} f(k)$$

and to apply (3.1) with $a=n$ and $b=N$. ∎

3.2. **A generalization of the integral criterion for the convergence of a series.** Let $f : [0,+\infty[\to \mathbb{R}$ be locally integrable. We set

$$\gamma_n^f := \sum_{0\le k<n} f(k) - \int_0^n f(x)\,dx \qquad \forall n \in \mathbb{N}.$$

Notice that, if f is of bounded variation, then $f(\infty) := \lim_{x\to+\infty} f(x)$ exists and is finite.

Theorem 3.3 (The Euler constant). *Let $f : [0,+\infty[\to \mathbb{R}$ be of bounded variation. The Euler constant of f defined by $\gamma^f := \lim_{n\to+\infty} \gamma_n^f$ exists and is finite, and the following* ***estimate*** *of γ^f holds:*

$$\gamma^f = \gamma_n^f - \frac{1}{2}\,[f\,]_n^\infty + \varepsilon_1(n), \quad |\varepsilon_1(n)| \le \frac{1}{2}\,\mathrm{pV}(f,[n,+\infty[) \qquad \forall n \in \mathbb{N}. \tag{3.3}$$

Proof. Given $n, N \in \mathbb{N}$ with $N > n$, by Theorem 3.1 we have

$$\gamma_N^f - \gamma_n^f = \sum_{n\le k<N} f(k) - \int_n^N f(x)\,dx = -\frac{1}{2}\,[f\,]_n^N + R(n,N), \tag{3.4}$$

with $|R(n,N)| \le \dfrac{1}{2}\,\mathrm{pV}(f,[n,N])$.
Since the limits $\lim_{N\to+\infty} f(N)$ and $\lim_{N\to+\infty} \mathrm{pV}(f,[0,N]) = \mathrm{pV}(f,[0,+\infty[)$ are both finite, and $\mathrm{pV}(f,[n,N]) = \mathrm{pV}(f,[0,N]) - \mathrm{pV}(f,[0,n])$, it follows from the necessary part of the Cauchy convergence criterion that

$$\lim_{n,N\to+\infty} -\frac{1}{2}\,[f\,]_n^N + R(n,N) = 0.$$

The sufficiency part of the very same criterion thus implies that the limit $\lim_{n\to+\infty} \gamma_n^f$ exists and is finite. Passing to the limit in (3.4) we get

$$\gamma^f - \gamma_n^f = \sum_{n\le k<N} f(k) - \int_n^N f(x)\,dx = \frac{1}{2}(f(\infty) - f(n)) + \varepsilon_1(n),$$

where $\varepsilon_1(n) := \lim_{N\to+\infty} R(n,N)$ is dominated by $\dfrac{1}{2}\,\mathrm{pV}(f,[n,+\infty[)$. ∎

An immediate consequence of Theorem 3.3 is the following generalization of the well known integral criterion for the convergence of the series $\sum_{k=0}^{\infty} f(k)$ for bounded and monotonic functions.

Corollary 3.4 (Integral criterion for series and approximation of its sum). *Let $f : [0,+\infty[\to \mathbb{R}$ be of bounded variation.*

(1) *The series $\sum_{k=0}^{\infty} f(k)$ and the generalized integral $\int_0^{+\infty} f(x)\,dx$ have the same behavior: both are either convergent or divergent.*

(2) *Assume that the series $\sum_{k=0}^{\infty} f(k)$ converges. For every $n \in \mathbb{N}$ the following* ***approximation*** *holds:*

$$\sum_{k=0}^{\infty} f(k) = \sum_{0\le k<n} f(k) + \int_n^{+\infty} f(x)\,dx - \frac{1}{2}\,[f\,]_n^{\infty} + \varepsilon_1(n),$$
$$(3.5)\qquad |\varepsilon_1(n)| \le \frac{1}{2}\,\mathrm{pV}(f,[n,+\infty[).$$

Proof. 1. We know from Theorem 3.3 that

$$\gamma^f = \lim_{n\to\infty}\left(\sum_{0\le k<n} f(k) - \int_0^n f(x)\,dx\right) \in \mathbb{R}.$$

Thus $\sum_{k=0}^{\infty} f(k)$ and the limit $\lim_{\substack{n\to+\infty\\ n\in\mathbb{N}}} \int_0^n f(x)\,dx$ have the same behavior. Since $f(\infty)$ belongs to $\mathbb{R}$, the value of $\lim_{\substack{n\to+\infty\\ n\in\mathbb{N}}} \int_0^n f(x)\,dx$ coincides with that of $\int_0^{+\infty} f(x)\,dx$: the conclusion follows.
2. It follows from (3.2) that for every $N \ge n$ we have

$$(3.6)\qquad \sum_{0\le k<N} f(k) = \sum_{0\le k<n} f(k) + \int_n^{N} f(x)\,dx - \frac{1}{2}\,[f\,]_n^{N} + \varepsilon_1(n,N),$$

with $|\varepsilon_1(n,N)| \le \frac{1}{2}\,\mathrm{pV}(f,[n,N]) \le \frac{1}{2}\,\mathrm{pV}(f,[n,+\infty[)$. From Point 1. we know that f is integrable in a generalized sense on $[0,+\infty[$. Passing to the limit for $N \to +\infty$ in (3.6) we deduce that $\varepsilon_1(n) := \lim_{N\to+\infty} \varepsilon_1(n,N)$ is finite, whence the validity of (3.5). ∎

Remark 3.2. The approximation formula (3.5) was established, for a wider class of functions and with an explicit form of the reminder, in [5], [6, 4.1.5] by means of Fourier analysis methods.

3.3. **Asymptotic formulas.**

Theorem 3.5 (Asymptotic formulas)**.** *Let $f : [0,+\infty[\to \mathbb{R}$ be a function.*

(1) *If f is of bounded variation, then for every $n \in \mathbb{N}$*

$$\sum_{0\le k<n} f(k) = \gamma^f + \int_0^n f(x)\,dx + \varepsilon_1'(n), \qquad |\varepsilon_1'(n)| \le \mathrm{pV}(f,[n,+\infty[).$$

(2) *If f is monotonic and unbounded then for every $n \in \mathbb{N}$ we have*

$$\sum_{0\le k<n} f(k) = \int_0^n f(x)\,dx + O\left(f(n)\right) \quad n \to +\infty;$$

Proof. 1. From (3.3) we obtain

$$\gamma_n^f = \gamma^f + \varepsilon_1'(n),$$

where $\varepsilon_1'(n) := \frac{1}{2}\,[f\,]_n^{\infty} - \varepsilon_1(n)$ and since $|\varepsilon_1(n)| \le \frac{1}{2}\,\mathrm{pV}(f,[n,+\infty[)$, the following estimate holds

$$|\varepsilon_1'(n)| = \left|\frac{1}{2}\,[f\,]_n^{\infty} - \varepsilon_1(n)\right| \le \mathrm{pV}(f,[n,+\infty[) :$$

the conclusion follows.

2. It follows from Theorem 3.1, together with Remark 3.1, that for every $n \in \mathbb{N}$

$$\sum_{0\le k<n} f(k) = \int_0^n f(x)\,dx - \frac{1}{2}(f(n)-f(0)) + R(n),$$

with $|R(n)| \le \frac{1}{2}|f(n)-f(0)|$. Since $\lim_{n\to+\infty} f(n) = \pm\infty$, then

$$f(n)-f(0) = O(f(n)) \qquad n\to+\infty,$$

whence $-\frac{1}{2}(f(n)-f(0)) + R(n) = O(f(n))$ for $n\to+\infty$: the conclusion follows. ∎

4. THE EULER-MACLAURIN FORMULA FOR BV FUNCTIONS: A MORE MEASURE THEORETIC LOOK

4.1. **Variation and point variation.** A function of locally bounded variation (i.e. of bounded variation on every bounded interval) $f:\mathbb{R}\to\mathbb{R}$ provides a finite signed measure μ_f on the $\sigma-$algebra of Borel subsets of any subinterval of $\mathbb{R}$ on which f is bounded, in particular on any bounded interval. Denoting by $f(x^-)$ (resp. $f(x^+)$) the left (resp. right) limit of f at a point x, the measures of bounded intervals with end-points $c<d$ are:

$$\mu_f\big(]c,d[\big) = f(d^-)-f(c^+),\ \mu_f\big([c,d]\big) = f(d^+)-f(c^-),$$

$$\mu_f\big([c,d[\big) = f(d^-)-f(c^-),\ \mu_f\big(]c,d]\big) = f(d^+)-f(c^+),$$

and for $c=d$ we have $\mu_f\big(\{c\}\big) = f(c^+)-f(c^-)$, the jump of f at c. As for every signed measure the *total variation measure* $|\mu_f|$ of the Borel set E is

$$|\mu_f|(E) = \sup\left\{\sum_{k=1}^m |\mu_f(A_k)| :\ A_1,\dots,A_m\subseteq E \text{ disjoint and Borel}\right\}.$$

When E is an interval one can prove that the same supremum is obtained if $A_1,\dots,A_m$ range only over subintervals of E, so that, if E is an interval

$$\begin{aligned}|\mu_f|(E) &= \sup\left\{\sum_{k=1}^m \big|\mu_f\big(]x_{k-1},x_k[\big)\big| + \sum_{k=0}^m \big|\mu_f\big(\{x_k\}\big)\big| :\ x_k\in E,\ x_0<\dots<x_m\right\}\\ &= \sup\left\{\sum_{k=1}^m |f(x_k^-)-f(x_{k-1}^+)| + \sum_{k=0}^m |f(x_k^+)-f(x_k^-)| :\ x_k\in E,\ x_0<\dots<x_m\right\}.\end{aligned}$$

If, moreover, E is open bounded then $|\mu_f|(E)$ coincides with the *variation* $V(f,E)$ of f on E [8], given by

$$V(f,E) := \sup\left\{\int_E f(x)\phi'(x)\,dx :\ \phi\in C^1_{\mathrm{c}}(E),\ |\phi|\le 1\right\}.$$

If $f:\mathbb{R}\to\mathbb{R}$ is locally BV it is convenient to introduce the function

$$\rho_f(x) := |f(x^+)-f(x)| + |f(x)-f(x^-)| - |f(x^+)-f(x^-)| \quad \forall x\in\mathbb{R}.$$

Notice that $\rho_f(x)$ equals twice the distance from $f(x)$ to the interval whose end-points are $f(x^-), f(x^+)$.

Here is how the pointwise variation of a BV function on a bounded *open* interval is related to its variation.

Proposition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally of bounded variation. Then for every bounded open interval E:*

$$\mathrm{pV}(f, E) = |\mu_f|(E) + \sum_{x \in E} \rho_f(x).$$

Proof. Given $\varepsilon > 0$ we can find $x_0 < x_1 < \cdots < x_m$ in E such that

$$\mathrm{pV}(f, E) - \varepsilon < \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|;$$

now for every $k \in \{0, \ldots, m\}$ we pick $x'_k,\, x''_k \in E$ such that

$$x'_0 < x_0;\; x_m < x''_m; \quad x_{k-1} < x''_{k-1} < x'_k < x_k$$

for every $k = 1, \ldots, m$. Consider now the set $\{x'_k, x_k, x''_k \,:\, k = 0, \ldots, m\}$; by the triangular inequality we get

$$\begin{aligned}\mathrm{pV}(f, E) - \varepsilon &< \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| \\ &\le \sum_{k=0}^{m} (|f(x_k) - f(x'_k)| + |f(x''_k) - f(x_k)|) + \sum_{k=1}^{m} |f(x'_k) - f(x''_{k-1})| \\ &\le \mathrm{pV}(f, E);\end{aligned}$$

taking limits in the preceding inequality as x'_k increases to x_k and x''_k decreases to x_k we get

$$\begin{aligned}\mathrm{pV}(f, E) - \varepsilon &< \sum_{k=0}^{m} (|f(x_k) - f(x_k^-)| + |f(x_k^+) - f(x_k)|) + \sum_{k=1}^{m} |f(x_k^-) - f(x_{k-1}^+)| \\ &\le \mathrm{pV}(f, E),\end{aligned}$$

which immediately yields

$$\begin{aligned}\mathrm{pV}(f, E) - \varepsilon &< \left(\sum_{k=1}^{m} |f(x_k^-) - f(x_{k-1}^+)| + \sum_{k=0}^{m} |f(x_k^+) - f(x_k^-)| \right) + \sum_{k=0}^{m} \rho_f(x_k) \\ &\le \mathrm{pV}(f, E);\end{aligned}$$

taking suprema on $\{x_0, \ldots, x_m\}$ this easily gives

$$\mathrm{pV}(f, E) - \varepsilon < |\mu_f|(E) + \sum_{x \in E} \rho_f(x) \le \mathrm{pV}(f, E),$$

and ends the proof. ∎

Remark 4.1. Notice that the claim of Proposition 4.1 does not hold, in general, if E is not open. It is easy to see that for a *compact* interval $[a, b]$ ($a < b$) we have

$$\begin{aligned}\mathrm{pV}(f, [a, b]) &= \mathrm{pV}(f,]a, b[) + |f(a) - f(a^+)| + |f(b) - f(b^-)| \\ &= |\mu_f|(]a, b[) + \sum_{x \in]a,b[} \rho_f(x) + |f(a) - f(a^+)| + |f(b) - f(b^-)|.\end{aligned}$$

This proves actually that $\mathrm{pV}(f, I)$ and $|\mu_f|(I)$ coincide for every bounded interval I if and only if f is continuous; thus $\mathrm{pV}(f, I)$ gives rise to a measure if and only if f is continuous.

4.2. **The Euler-Mac Laurin formula.** Let $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$. The *mid-value modification* f_m for f is the function defined by

$$f_m(x) := \frac{f(x^-) + f(x^+)}{2}.$$

The following version of the integration by parts formula for BV functions will be used in the sequel.

Lemma 4.2 (Integration by parts for BV functions)**.** *If $f, g : \mathbb{R} \to \mathbb{R}$ are locally of bounded variation then, for every $a < b$:*

$$\int_{[a,b[} g_m(x)\, d\mu_f(x) = g(b^-)f(b^-) - g(a^-)f(a^-) - \int_{[a,b[} f_m(x)\, d\mu_g(x). \tag{4.1}$$

Proof. By following the lines of the proof of [9, Theorem 3.36] one gets

$$\int_{[a,b[} g(x^-)\, d\mu_f(x) = g(b^-)f(b^-) - g(a^-)f(a^-) - \int_{[a,b[} f(x^+)\, d\mu_g(x),$$

$$\int_{[a,b[} g(x^+)\, d\mu_f(x) = g(b^-)f(b^-) - g(a^-)f(a^-) - \int_{[a,b[} f(x^-)\, d\mu_g(x).$$

The result is obtained by summing up term by term the members of the above equalities, and dividing by 2. ∎

The following Euler-Maclaurin formula for the sums $\sum_{a \le k < b} f_m(k)$ holds: differently from the classical one, the sums involve the mid-value modification of f, due to its possible discontinuities. The first Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$, restricted to $[0,1]$, is involved in the first-order Euler-Maclaurin formula for smooth functions [1, Theorem 12.27]; we will use here the mid-value modification of its extension by periodicity $\beta_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$\beta_1(x) := \begin{cases} B_1(x - [x]) & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 4.3 (First-order Euler-Maclaurin formula for BV functions)**.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is locally of bounded variation. Then, for any $a < b$ in $\mathbb{Z}$,*

$$\sum_{a \le k < b} f_m(k) = \int_a^b f(x)\, dx - \frac{1}{2}(f(b^-) - f(a^-)) + \int_{]a,b[} \beta_1(x)\, d\mu_f(x). \tag{4.2}$$

Proof. The proof of Theorem 4.3 goes formally as that of the classical first-order Euler-Maclaurin formula. Clearly β_1 is locally of bounded variation; plainly $\mu_{\beta_1} = \lambda_1 - \sum_{n \in \mathbb{Z}} \delta_n$, where λ_1 is the Lebesgue measure. Since $(\beta_1)_m = \beta_1$, applying formula (4.1) with $g = \beta_1$ we get

$$\begin{aligned}
\int_{[a,b[} \beta_1(x)\, d\mu_f(x) &= \beta_1(b^-)\, f(b^-) - \beta_1(a^-)\, f(a^-) - \int_{[a,b]} f_m(x)\, d\mu_{\beta_1}(x) \\
&= \frac{f(b^-) - f(a^-)}{2} - \int_a^b f(x)\, dx + \int_{[a,b[} f_m(x)\, d\left(\sum_{k \in \mathbb{Z}} \delta_k\right)(x) \\
&= \frac{f(b^-) - f(a^-)}{2} - \int_a^b f(x)\, dx + \sum_{a \le k < b} f_m(k),
\end{aligned}$$

which we can rewrite

$$\sum_{a\le k<b} f_m(k) = \int_a^b f(x)\,dx - \frac{f(b^-)-f(a^-)}{2} + \int_{[a,b[} \beta_1(x)\,d\mu_f(x);$$

since $\beta_1(a)=0$, we get $\displaystyle\int_{[a,b[} \beta_1(x)\,d\mu_f(x) = \int_{]a,b[} \beta_1(x)\,d\mu_f(x)$. ∎

Theorem 4.3 yields an alternative proof of (3.1).

Alternative proof of Theorem 3.1. To deduce (3.1) from the preceding theorem we rewrite

$$\sum_{a\le k<b} f(k) = \int_a^b f(x)\,dx - \frac{f(b)-f(a)}{2} + R,$$

$$\begin{aligned} R :&= \int_{]a,b[} \beta_1(x)\,d\mu_f(x) + \sum_{a\le k<b} f(k) - \sum_{a\le k<b} f_m(k) + \frac{1}{2}\,[f]_a^b - \frac{f(b^-)-f(a^-)}{2} \\ &= \int_{]a,b[} \beta_1(x)\,d\mu_f(x) + \frac{1}{2}\sum_{a<k<b} \big((f(k)-f(k^-)) + (f(k)-f(k^+))\big) + \\ &\qquad\qquad + \frac{1}{2}\big((f(a)-f(a^+)) + (f(b)-f(b^-))\big). \end{aligned}$$

so that

$$\begin{aligned} |R| \le \left|\int_{]a,b[} \beta_1(x)\,d\mu_f(x)\right| + \frac{1}{2}\sum_{a<k<b} \big(|f(k)-f(k^-)| + |f(k)-f(k^+)|\big) + \\ + \frac{1}{2}\big(|f(a)-f(a^+)| + |f(b)-f(b^-)|\big). \end{aligned} \tag{4.3}$$

Now, since $\displaystyle\int_{]a,b[} |\beta_1(x)|\,d|\mu_f|(x)$ lacks the contribution of the jumps of f on the integers and $|\beta_1| \le 1/2$,

$$\begin{aligned} \int_{]a,b[} |\beta_1(x)|\,d|\mu_f|(x) &\le \frac{1}{2}|\mu_f|\big(]a,b[\setminus\mathbb{Z}\big) \\ &= \frac{1}{2}|\mu_f|\big(]a,b[\big) - \frac{1}{2}\sum_{a<k<b} |f(k^+)-f(k^-)|. \end{aligned}$$

It follows from (4.3) and Proposition 4.1, taking account of Remark 4.1, that

$$\begin{aligned} |R| &\le \frac{1}{2}\Big(|\mu_f|\big(]a,b[\big) + \sum_{a<k<b} \rho_f(k)\Big) + \frac{1}{2}\big(|f(a)-f(a^+)| + |f(b)-f(b^-)|\big) \\ &\le \frac{1}{2}\,\mathrm{pV}(f,]a,b[) + \frac{1}{2}\big(|f(b)-f(b^-)| + |f(a^+)-f(a)|\big) = \frac{1}{2}\,\mathrm{pV}(f,[a,b]), \end{aligned}$$

proving the claim. ∎

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