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RELATIONS BETWEEN DIFFERENTIABILITY AND ONE−SIDED DIFFERENTIABILITY

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ABSTRACT. In this paper, we attempt to approach to the problem of connection between differentiation and one−side differentiation in a more simple and explicit way than in existing math literature. By replacing the condition of differentiation with one−sided differentiation, more precisely with right-hand differentiation, we give the generalization of a theorem having to do with Lebesgues integration of derivative of a function. Next, based on this generalized result it is proven that if a continuous function has bounded right−hand derivative, then this function is almost everywhere differentiable, which implies that the set of points where the function is not differentiable has measure zero.

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1. INTRODUCTION

The problems of relations between differentiability and one-sided differentiability of a function are so old and well known. The purpose of this paper is to bring a way of analysis for one of these problems, like the relation between derivative and right-hand derivative for function $f : [a, b] \to \mathcal{R}$, in the case when f is continuous in closed interval $[a, b]$ and the right derivative $f'_+(x)$ is bounded in $[a, b)$, where

$$f'_+(x) = \lim_{\substack{h \to 0 \\ y>0}} \frac{f(x+h) - f(x)}{h}$$

Let $\mathbf{B}$ be a Banach space (a complete normed space).

Lemma 1.1. *Let $f : [a, b] \to \mathbf{B}$ be a continuous function on the closed interval $[a, b]$. Let us suppose that for any point $x \in [a, b)$ the right-hand derivative $f'_+(x)$exists. If the right derivative f'_+ is bounded in $[a, b)$, that means exists any constant $K \geq 0$, such for every $x \in [a, b)$ we have $\|f'_+(x)\| \leq K$, then the inequality*

$$\|f(b) - f(a)\| \leq K(b - a) \tag{1.1}$$

is true.

A proof of this Lemma can be found in [2].

Note. Inequality (1.1) can be expressed also in the form:

$$\|f(b) - f(a)\| \leq (b - a) \sup_{t \in [a,b)} \|f'_+(x)\| \tag{1.2}$$

Definition 1.1. Let I be an interval, and let $f : I \to \mathcal{R}$ be a function. We say that f is a Darboux function provided that for any two points $p, q \in$ and any point y between $f(p)$ and $f(q)$, there is a point x between p and q such that $f(x) = y$ (i.e., for any subinterval J of I, $f(J)$ is an interval).

There are fairly simple functions that are Darboux but not continuous. For example, let

$$f(x) = \begin{cases} sin\frac{1}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0. \end{cases}$$

Bolzanos theorem or intermediate value theorem, the theorem that if a real function f is continuous on a closed bounded interval $[a, b]$, then it takes every value between $f(a)$ and $f(b)$ for at least one argument between a and b. This intermediate value property, which derivatives also possess by virtue of the Mean- Value Theorem, is also called the Darboux property. (Named after the Czech analyst Bernhard Bolzano (1781-1848).)

2. GENERALIZATION OF SOME RESULTS

Theorem 2.1. *If the function $f : [a, b] \to \mathbf{B}$ is continuous on the closed interval $[a, b]$ and has continuous right$-$hand derivative on $[a, b)$, then function f has continuous derivative on $[a, b)$.*

Proof. Let analyze a fix point $x_0 \in [a, b)$ and any point $x \in [a, b)$. If we write inequality (1.2) on closed interval $[x_0, x](x > x_0)$ for function

$$x \mapsto F(x) = f(x) - f'_+(x_0)(x - x_0) \tag{2.1}$$

we find inequality:

$$\|F(x) - F(x_0)\| \leq (x - x_0) \sup_{t\in[x_0,x]} \|F'_+(x)\| \tag{2.2}$$

or

$$\|f(x) - f(x_0) - f'_+(x_0)(x - x_0)\| \leq (x - x_0) \sup_{t\in[x_0,x]} \|f'_+(t) - f'_+(x_0)\| \tag{2.3}$$

After dividing by $(x - x_0)$, inequality (2.3) becomes

$$\|\frac{f(x) - f(x_0)}{(x - x_0)} - f'_+(x_0)\| \leq \sup_{t\in[x_0,x]} \|f'_+(t) - f'_+(x_0)\| \tag{2.4}$$

It is obvious that inequality (2.4) maintains the same form even if we write inequality (1.2) for function F on interval $[x_0, x](x < x_0)$. If we pass to the limit when $x \to x_0(x \neq x_0)$ on the both sides of inequality (2.4), and take into consideration that function f'_+ is continuous at point x_0, we find

$$0 \leq \| \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'_+(x_0)\| \leq 0 \tag{2.5}$$

that means

$$\|f'(x_0) - f'_+(x_0)\| = 0, f'(x_0) = f'_+(x_0) \tag{2.6}$$

Since point x_0 is scalene, derives that the derivative $f'(x)$ exists for every $x \in [a, b)$ and that derivative is continuous on $[a, b)$, same as the right−hand derivative f'_+.

∎

Lemma 2.2. *Let $f : [a, b] \to \mathcal{R}$ be a continuous function on the closed interval $[a, b]$. Let us suppose that for any point $x \in [a, b)$ the right-hand derivative $f'_+(x)$ exists. Then in (a, b) exist points c and d such as*

$$f'_+(c) \leq \frac{f(b) - f(a)}{b - a} \leq f'_+(d) \tag{2.7}$$

A proof of this Lemma can be found in [3].

Theorem 2.3. *Let $f : [a, b] \to \mathcal{R}$ be a continuous function on the closed interval $[a, b]$, which has right−hand derivative $f'_+(t)$ at every point $t \in [a, b)$. If the function f'_+ is bounded on $[a, b)$, then it is Lebesgue integrable on every closed interval $[a, x] \subset [a, b)$ and the formula*

$$(L) \int_a^x f'_+(t)dt = f(x) - f(a) \tag{2.8}$$

is true.

Proof. First we extend f to the interval $[a, b + 1]$ by setting $f(t) = f(b)$ for $b < t \leq b + 1$. This implies that $f'_+(t) = 0$ for $b \leq t < b + 1$. Now, let us apply Lemma (2.2) for function f to the arbitrary interval $[u, v] \subset [a, b]$.

This implies that inequality

$$f'_+(c)(v - u) \leq f(v) - f(u) \leq f'_+(d)(v - u) \tag{2.9}$$

hold for some c, d in (u, v). By hypothesis the right−hand derivative is bounded on $[a, b)$; with our extension it exists and is bounded on $[a, x]$, that is, there exists $M > 0$ such that

$$-M \leq f_+^{'}(t) \leq M \text{ for all } t \in [a, b).$$

From (2.9) it follows that

$$-M(v-u) \leq f(v) - f(u) \leq M(v-u) \text{ and } \mid f(v) - f(u) \mid \leq M(v-u)$$

for any interval $[u, v] \subset [a, b]$. But this means that function f is absolutely continuous on $[a, x]$. According to Lebesgue theorem (see [4]. p.334-335) the derivative $f^{'}$ is integrable on $[a, x]$ and

$$(L)\int_a^x f^{'}(t)dt = f(x) - f(a)$$

In particular, $f_+^{'} = f^{'}$ a. e. and

$$(L)\int_a^x f_+^{'}(t)dt = f(x) - f(a)$$

∎

Corollary 2.4. *There is no function from $C_{[a,b]}$ whose right−hand derivative on $[a, b)$ equals the Dirichlè function*

$$\chi(x) = \begin{cases} 0, \text{ if } x \text{ is rational number} \\ 1, \text{ if } x \text{ is irational number} \end{cases}$$

Proof. Let us suppose that there exists a function $\in C_{[a,b]}$ such that

$$\forall x \in [a, b), g_+(x) = \chi(x)$$

Since conditions of Theorem (2.3) are satisfied, we can write formula (2.8) in the form:

$$\forall x \in [a, b), g(x) = g(a) + (L)\int_a^x g_+^{'}(t)dt \tag{2.10}$$

Since

$$(L)\int_a^x g_+^{'}(t)dt = 0,$$

the identity (2.10) becomes

$$\forall x \in [a, b), g(x) = g(a) \tag{2.11}$$

which means that

$$\forall x \in [a, b), g_+(x) = 0 \neq \chi(x),$$

a contradiction. ∎

Corollary 2.5. *If the right-hand derivative $f_+^{'}$ of the continuous function $f : [a, b] \to \mathcal{R}$ is bounded on the half−open interval $x \in [a, b)$, then the formula*

$$(L)\int_a^x f_+^{'}(t)dt = f(b) - f(a) \tag{2.12}$$

is true.

Proof. The formula (2.12) derives from the identity (2.8) , by taking the limit of both sides when $x \to b^-$. Formula (2.12) is a type of Leibnitz formula (see [6]). ∎

Corollary 2.6. *If the right−hand derivative $f_+^{'}$ of a continuous function $f : [a,b] \to R$ is bounded on the half−open interval $[a,b)$, then the function is almost everywhere differentiable on $[a,b)$.*

Proof. We know that derivative of the function $x \mapsto \Phi(x) = (L)\int_a^x f(t)dt$ where f is integrable on $[a,b]$, is almost everywhere equal with the function f. A proof of this fact can be found in [4]. If we take the derivative of both sides of (2.8) at the point $x \in [a,b)$, and apply the above theorem to the integral $(L)\int_a^x f_+^{'}(t)dt$, we have:

$$f_+^{'} = f^{'}(x)(\text{almost everywhere on } [a,b]) \tag{2.13}$$

which proves that Corollary (2.6) holds. ∎

Note. The question arises: Which conditions must be met by the right−hand derivative $f_+^{'}$ so that the equation (2.13) will be hold for every point $x \in [a,b)$? To give an answer to the above question we follow results from propositions (Lemma 3, Lemma 4), which can be found in [1].

Lemma 2.7. *If f is a measurable function, bounded and with the property Darboux on $[a,b]$, then for each closed subinterval $I = [p,q] \subset [a,b]$ there exists at least a point $\xi \in I$ such that*

$$(L)\int_p^q f(x)dx = f(\xi) \mid I \mid, (\mid I \mid = q - p) \tag{2.14}$$

The point ξ is called the mean point of I with respect to f.

Lemma 2.8. *Let f be a bounded function having the Darboux property on $[a,b]$. Then f is the derivative of a function in $[a,b]$ if and only if:*

1. f is measurable and

2. for every $x \in [a,b]$ and for each sequence of subintervals $I_n = [p_n,q_n] \subset [a,b]$ that converge to a point $x(I_n \to x)$ we have $f(x_n) \to f(x)$, where x_n is the mean point of I_n with respect to f.

Theorem 2.9. *If the right−hand derivative $f_+^{'}$ of a continuous function $f : [a,b] \to \mathcal{R}$ meets the following conditions:*

1. $f_+^{'}$ is bounded and has the Darboux property on the half−open interval $[a,b)$ and

2. for each $x \in [a,b]$ and for each sequence of subintervals $I_n = [p_n,q_n] \subset [a,b]$ that converge to a point $x(I_n \to x)$, we have $f_+^{'}(x_n) \to f_+^{'}(x)$, where x_n is the mean point of I_n with respect to f, then the function f is everywhere differentiable in $[a,b)$.

Proof. The function $f_+^{'}$ satisfies the conditions of Theorem (2.1), so the formula

$$(L)\int_a^x f_+^{'}(t)dt = f(x) - f(a) \tag{2.15}$$

is true.

Next, reasoning likewise on [1] (p. 244-245) , where Lemma (2.7) is proved , we find

$$x \in [a,b), f_+^{'}(x) = f^{'}(x). \tag{2.16}$$

∎

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