CLOSEDNESS AND SKEW SELF-ADJOINTNESS OF NADIR’S OPERATOR
MOSTEFA NADIR¹ AND ABDELLATIF SMATI²

Received 8 July, 2017; accepted 10 January, 2018; published 23 March, 2018.

¹,²DEPARTMENT OF MATHEMATICS UNIVERSITY OF MSILA 28000 ALGERIA.
¹mostefanadir@yahoo.fr
²smatilotfi@gmail.com

ABSTRACT. In this paper, we present some sufficient conditions which ensure the compactness, the normality, the positivity, the closedness and the skew self-adjointness for bounded and unbounded Nadir’s operator on a Hilbert space. We get also when the measurement of its adjointness is null and other related results are also established.

Key words and phrases: Nadir’s operator; Closedness; Skew self-adjointness of operators.

2000 Mathematics Subject Classification. Primary 05C38, 15A15, 05A15 and 15A18.
1. Introduction

Let $H$ be a complex Hilbert space. We denote by $B(H)$ the algebra of bounded linear operators on $H$, $C(H)$ the set of densely defined closed linear operators on $H$, and by $D(A)$ the domain of unbounded operator $A$, $A^*$ is the adjoint of $A$. Bounded operators are assumed to be defined on the whole Hilbert space. An unbounded operator $N$ is called a Nadir’s operator if $N = AB^* - BA^*$ where $A$ and $B$ are bounded or unbounded densely defined operators.

Recall that the unbounded operator $A$, defined on Hilbert space $H$, is said to be invertible if there exists an everywhere defined bounded operator $B$ such that $BA \subset AB = I$

where $I$ is usual identity operator.

An unbounded operator $A$ is said to be closed if it’s graph is closed , Self-adjoint (resp. Skew self-adjoint) if $A^* = A$ (resp. $A^* = -A$), normal if it is closed and $AA^* = A^*A$.

The standard and the known definition of the sum and the product of two operators $A$ and $B$ with domains respectively $D(A)$ and $D(B)$ is

- $(A + B)x = Ax + Bx$ for $x \in D(A + B) = D(A) \cap D(B)$
- $(AB)x = A(Bx)$ for $x \in B^{-1}(D(A))$

The study of unbounded Nadir’s operator is difficult because, if we talk about adjoints, then the results are not better. For instance, the adjoint of $AB$ if both operators are unbounded does not always equal to $B^*A^*$, and the adjoint of $A + B$ does not equal to $A^* + B^*$, $A^{**}$ also does not equal to $A$ unless $A$ is closed operator, the product $AB$ and the sum $A + B$ of two closed operators does not always closed, and it’s possible to $A$ is densely defined but $A^*$ and $AB$ does not, look at [1], [2], [3], [6], [8].

Lemma 1.1. Let $A$ be a densely defined operator, if $A$ is invertible then it is closed.

Theorem 1.2. $A$ is invertible if and only if $A^*$ is invertible and $(A^*)^{-1} = (A^{-1})^*$

Lemma 1.3. The product $AB$ of two closed operators is closed if

- $A$ is invertible
- $B$ is bounded

Remark 1.1. If $A$ and $B$ are two unbounded and invertible operators, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proposition 1.4.

- $A$ is left and right invertible simultaneously, then $A$ is invertible.
- $A$ right or left invertible unbounded self adjoint operator, then $A$ is invertible.
- $A$ right or left invertible unbounded normal operator, then $A$ is invertible.

Lemma 1.5. If $A$ and $B$ are densely defined and $A$ is invertible with inverse $A^{-1}$ in $B(H)$, then $(BA)^* = A^*B^*$.

Theorem 1.6. Let $A$ and $B$ be two unbounded normal and invertible operators then

$AB = BA \Rightarrow AB^* = B^*A; \quad BA^* = A^*B$.

Let $A$ and $B$ be two unbounded, normal and invertible operators, if $AB = BA$, then $AB^*$ and $BA^*$ are normal.
2. Main results

2.1. Boundedness of Nadir’s operator. In this case, we study the Nadir’s operator \( N = AB^* - BA^* \), with \( A \) and \( B \) are two bounded operators. If \( A \) or \( B \) is a compact operator then the operator \( N = AB^* - BA^* \) is also compact one.

The operator \( N = AB^* - BA^* \) is a Skew self-adjoint operator and \( N^2 \) is a negative one. Indeed, we have

\[
N^* = (AB^* - BA^*)^* = (AB^* - (AB^*)^*)
= (AB^*)^* - AB^* = BA^* - AB^* = -N.
\]

On the other hand

\[
N^2 = NN = -N^*N,
\]

the operator \( N^*N \) is a positive one

\[
\langle N^*Nx, x \rangle = \|Nx\|^2 \geq 0.
\]

The eigenvalues of the operator \( N \) are purely imaginary and the operator \( I - N \) is invertible. Let \( T \in B(H) \) be an operator has a unique representation \( T = R + iS \) where \( R \) and \( S \) are self-adjoint operators \cite{7}, so

\[
N = AB^* - BA^* = AB^* - (AB^*)^*
= (R + iS) - (R - iS) = 2iS.
\]

Therefore \( I - N \) is invertible.

2.2. Unboundedness of Nadir’s operator. In this case, we study the Nadir’s operator \( N = AB^* - BA^* \), with \( A \) and \( B \) are densely defined operators.

Let \( A \) be unbounded operator, then \( A^*B^* \subset (BA)^* \) for any unbounded operator \( B \) and if \( BA \) is densely defined. Besides \( (BA)^* = A^*B^* \) if \( B \) is bounded.

**Proposition 2.1.** Let \( A \) be unbounded and invertible operator, then \( A^n \) is a closed operator. Moreover if \( A^n \) is densely defined for all \( n \in \mathbb{N}^* \), then \( (A^n)^* = (A^*)^n \). Besides if \( A \) is a normal operator then \( A^n \) is also normal.

Indeed, the operator \( A \) is a closed because it’s densely defined and invertible and the same for \( A^2 = AA \) because \( A \) is closed and invertible and so on \( A^n \). For \( n = 1 \), \( A^* = A^* \) is true, \( n = 2 \), \((A^2)^* = (AA)^* = A^*A^* = (A^*)^2 \) is true. Assume it is true for \( n \), \( (A^n)^* = (A^*)^n \) this implies

\[
(A^{n+1})^* = (A^nA)^* = A^* (A^n)^* = A^* (A^*)^n = (A^*)^{n+1}.
\]

Hence \( (A^n)^* = (A^*)^n \) for all \( n \in \mathbb{N}^* \)

The operator \( A \) is a normal then

\[
(AA^*)^n = A^n (A^*)^n = A^n (A^n)^*
= (A^*A)^n = (A^*)^n A^n = (A^n)^* A^n.
\]
3. **Closedness and Skew self-adjointness of Nadir’s operator**

Recall that the unbounded operator $A$, defined on Hilbert space $H$, is said to be Hermitian if $A \subset A^*$, and said to be Skew-Hermitian if $A \subset -A^*$.

**Theorem 3.1.** Let $A$ and $B$ be two unbounded and invertible operators such that $AB^*$ is densely defined. If the Nadir’s operator $N$ is densely defined then it is Skew-Hermitian operator.

Since $A$ and $B$ are two unbounded and invertible operators and $AB^*$ is densely defined, then $BA^*$ is also densely defined. It is clear that

$$(BA^*)^* - (AB^*)^* \subset (BA^* - AB^*)^*.$$ 

Next $(AB^*)^* = BA^*$ and $(BA^*)^* = AB^*$, then we may write

$$N = AB^* - BA^* = (BA^*)^* - (AB^*)^* \subset (BA^* - AB^*)^* = -(AB^* - BA^*)^* = -N^*.$$ 

Hence $N$ is skew-Hermitian operator.

**Proposition 3.2.** Let $A$ and $B$ be two unbounded and invertible operators such that $R(B^*) \subset D(A)$. If the Nadir’s operator $N$ is densely defined then it is skew-Hermitian operator.

Indeed, since $A$ and $B$ are two unbounded and invertible operators and $R(B^*) \subset D(A)$ then $AB^*$ and $BA^*$ are densely defined operators.

Let $S = AB^*$ be unbounded, densely defined, normal and invertible operator. If $D(S^*S^{-1}) \subset D(S)$ then

- $N = S - S^*$ is closed operator on $D(S^*)$.
- If $N = S - S^*$ is densely defined then he is Skew Self-adjoint operator.

The hypothesis, $D(S^*S^{-1}) \subset D(S)$ cannot merely be dropped. As counter example, let $S$ be densely defined, self-adjoint and invertible operator with domain $D(S) \subsetneq H$ where $H$ is a complex Hilbert space. Then $S - S^* = 0$ on $D(S)$ is not closed. Moreover, $S$ is normal operator (because he is Self-adjoint operator) but

$$D(S^*S^{-1}) = D(SS^{-1}) = D(I) = H \nsubseteq D(S).$$

4. **Conclusion**

Let $A$ and $B$ be two unbounded operators such that $D(BA^*(AB^*)^{-1}) \subset D(AB^*)$ then the Nadir’s operator $N = AB^* - BA^*$ is closed on $D(AB^*)$ if $B$ is invertible, $AB^*$ is normal and invertible operator, or $A$ and $B$ are two normal and left (or right) invertible operators such that $AB^*$ is densely defined and $AB = BA$. If in addition $N$ is densely defined operator, then the Nadir’s operator $N$ is Skew self-adjoint (normal operator).

**References**


