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CLOSEDNESS AND SKEW SELF-ADJOINTNESS OF NADIR'S OPERATOR

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ABSTRACT. In this paper, we present some sufficient conditions which ensure the compactness, the normality, the positivity, the closedness and the skew self-adjointness for bounded and unbounded Nadir's operator on a Hilbert space. We get also when the measurement of its adjointness is null and other related results are also established.

Key words and phrases: Nadir's operator; Closedness; Skew self-adjointness of operators.

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1. INTRODUCTION

Let H be a complex Hilbert space. We denote by $B(H)$ the algebra of bounded linear operators on H, $C(H)$ the set of densely defined closed linear operators on H, and by $D(A)$ the domain of unbounded operator A, A^* is the adjoint of A. Bounded operators are assumed to be defined on the whole Hilbert space. An unbounded operator N is called a Nadir's operator if $N = AB^* - BA^*$ where A and B are bounded or unbounded densely defined operators.

Recall that the unbounded operator A, defined on Hilbert space H, is said to be invertible if there exists an *everywhere defined* bounded operator B such that

$$BA \subset AB = I$$

where I is usual identity operator.

An unbounded operator A is said to be closed if it's graph is closed , Self-adjoint (resp. Skew self-adjoint) if $A^* = A$ (resp. $A^* = -A$), normal if it is closed and $AA^* = A^*A$.

The standard and the known definition of the sum and the product of two operators A and B with domains respectively $D(A)$ and $D(B)$ is

- $(A + B)x = Ax + Bx$ for $x \in D(A + B) = D(A) \cap D(B)$
- $(AB)x = A(Bx)$ for $x \in B^{-1}(D(A))$

The study of unbounded Nadir's operator is difficult because, if we talk about adjoints, then the results are not better. For instance, the adjoint of AB if both operators are unbounded does not always equal to B^*A^*, and the adjoint of $A + B$ does not equal to $A^* + B^*$, A^{**} also does not equal to A unless A is closed operator, the product AB and the sum $A + B$ of two closed operators does not always closed, and it's possible to A is densely defined but A^* and AB does not, look at [1], [2], [3], [6], [8].

Lemma 1.1. *Let A be a densely defined operator, if A is invertible then it is closed.*

Theorem 1.2. *A is invertible if and only if A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$*

Lemma 1.3. *The product AB of two closed operators is closed if*

- A is invertible
- B is bounded

Remark 1.1. If A and B are two unbounded and invertible operators, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proposition 1.4.

- A is left and right invertible simultaneously, then A is invertible.
- A right or left invertible unbounded self adjoint operator, then A is invertible.
- A right or left invertible unbounded normal operator, then A is invertible.

Lemma 1.5. *If A and B are densely defined and A is invertible with inverse A^{-1} in $B(H)$, then $(BA)^* = A^*B^*$.*

Theorem 1.6. *Let A and B be two unbounded normal and invertible operators then*

$$AB = BA \Rightarrow AB^* = B^*A; \qquad BA^* = A^*B.$$

Let A and B be two unbounded, normal and invertible operators, if $AB = BA$, then $AB^*$and BA^* are normal.

2. MAIN RESULTS

2.1. **Boundedness of nadir's operator.** In this case, we study the Nadir's operator $N = AB^* - BA^*$, with A and B are two bounded operators. If A or B is a compact operator then the operator $N = AB^* - BA^*$ is also compact one.

The operator $N = AB^* - BA^*$ is a Skew self-adjoint operator and N^2 is a negative one.

Indeed, we have

$$\begin{aligned} N^* &= (AB^* - BA^*)^* = (AB^* - (AB^*)^*) \\ &= (AB^*)^* - AB^* = BA^* - AB^* = -N. \end{aligned}$$

On the other hand

$$N^2 = NN = -N^*N,$$

the operator N^*N is a positive one

$$\langle N^*Nx, x\rangle = \|Nx\|^2 \geqslant 0.$$

The eigenvalues of the operator N are purely imaginary and the operator $I - N$ is invertible.

Let $T \in B(H)$ be an operator has a unique representation $T = R + iS$ where R and S are self adjoint operators [7], so

$$\begin{aligned} N &= AB^* - BA^* = AB^* - (AB^*)^* \\ &= (R + iS) - (R - iS) = 2iS. \end{aligned}$$

Therefore $I - N$ is invertible.

2.2. **Unboundedness of nadir's operator.** In this case, we study the Nadir's operator $N = AB^* - BA^*$, with A and B are densely defined operators.

Let A be unbounded operator, then $A^*B^* \subset (BA)^*$ for any unbounded operator B and if BA is densely defined. Besides $(BA)^* = A^*B^*$ if B is bounded.

Proposition 2.1. *Let A be unbounded and invertible operator, then A^n is a closed operator. Moreover if A^n is densely defined for all $n \in \mathbb{N}^*$,then $(A^n)^* = (A^*)^n$. Besides if A is a normal operator then A^n is also normal.*

Indeed, the operator A is a closed because it's densely defined and invertible and the same for $A^2 = AA$ because A is closed and invertible and so on A^n. For $n = 1,\ A^* = A^*$ is true, $n = 2,\ (A^2)^* = (AA)^* = A^*A^* = (A^*)^2$ is true. Assume it is true for $n,\ (A^n)^* = (A^*)^n$ this implies

$$\left(A^{n+1}\right)^* = (A^nA)^* = A^*\left(A^n\right)^* = A^*\left(A^*\right)^n = (A^*)^{n+1}.$$

Hence $(A^n)^* = (A^*)^n$ for all $n \in \mathbb{N}^*$

The operator A is a normal then

$$\begin{aligned} (AA^*)^n &= A^n\left(A^*\right)^n = A^n\left(A^n\right)^* \\ &= (A^*A)^n = (A^*)^n A^n = (A^n)^* A^n. \end{aligned}$$

3. CLOSEDNESS AND SKEW SELF-ADJOINTNESS OF NADIR'S OPERATOR

Recall that the unbounded operator $A,$ defined on Hilbert space $H,$ is said to be *Hermitian* if $A \subset A^*$, and said to be *Skew-Hermitian* if $A \subset -A^*$.

Theorem 3.1. *Let A and B be two unbounded and invertible operators such that AB^* is densely defined . If the Nadir's operator N is densely defined then it is Skew-Hermitian operator.*

Since A and B are two unbounded and invertible operators and AB^* is densely defined, then BA^* is also densely defined. It is clear that

$$(BA^*)^* - (AB^*)^* \subset (BA^* - AB^*)^* .$$

Next $(AB^*)^* = BA^*$ and $(BA^*)^* = AB^*$, then we may write

$$\begin{aligned} N &= AB^* - BA^* \\ &= (BA^*)^* - (AB^*)^* \subset (BA^* - AB^*)^* = -(AB^* - BA^*)^* = -N^*. \end{aligned}$$

Hence N is skew-Hermitian operator.

Proposition 3.2. *Let A and B be two unbounded and invertible operators such that $R(B^*) \subset D(A)$. If the Nadir's operator N is densely defined then it is skew-Hermitian operator*

Indeed, since A and B be two unbounded and invertible operators and $R(B^*) \subset D(A)$ then AB^* and BA^* are densely defined operators.

Let $S = AB^*$ be unbounded , densely defined, normal and invertible operator. If $D(S^*S^{-1}) \subset D(S)$ then

- $N = S - S^*$ is closed operator on $D(S^*)$.
- If $N = S - S^*$ is densely defined then he is Skew Self-adjoint operator.

The hypothesis, $D(S^*S^{-1}) \subset D(S)$ cannot merely be dropped. As counter example, let S be densely defined, self-adjoint and invertible operator with domain $D(S) \subsetneqq H$ where H is a complex Hilbert space. Then $S - S^* = 0$ on $D(S)$ is not closed. Moreover, S is normal operator (because he is Self-adjoint operator) but

$$D(S^*S^{-1}) = D(SS^{-1}) = D(I) = H \nsubseteq D(S).$$

4. CONCLUSION

Let A and B be two unbounded operators such that $D(BA^*(AB^*)^{-1}) \subset D(AB^*)$ then the Nadir's operator $N = AB^* - BA^*$ is closed on $D(BA^*)$ if

B is invertible, AB^* is normal and invertible operator, or A and B are two normal and left (or right) invertible operators such that AB^* is densely defined and $AB = BA.$ If in addition N is densely defined operator, then the Nadir's operator N is Skew self-adjoint (normal operator).

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