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RELATION BETWEEN THE SET OF NON−DECREASING FUNCTIONS AND THE SET OF CONVEX FUNCTIONS

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ABSTRACT. In this article we address the problem of integral presentation of a convex function. Let I be an interval in $\mathcal{R}$. Here, using the Riemann or Lebesgues integration theory, we find the necessary and sufficient condition for a function $f : I \to \mathcal{R}$ to be convex in I.

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1. INTRODUCTION

For convex functions are written many articles. Here we will concentrate only on the aspect of their presentation by indefinite integral. In order to achieve this goal we have used three well-known facts about convex functions dealing with their continuity and differentiability (propositions (P1), (P2) and (P3)). To prove that the derivative of a convex function is Riemann integrable, we had to prove Lemma 2.1.

This problem is also considered in [1], but only in the case of Riemann's as the integration operator. Here, in addition, we reorganize this problem in a simpler, shorter and clearer way when as the integration operator is that of Riemann, and generalize it even when the integration operator is that of Lebesgue. This was made possible because of the fact that we have used the convex function defined by equation (1.2), unlike [7] and [1], where this meaning is given by equation (1.1).

Definition 1.1. The real function $f : I \to \mathcal{R}$ is called convex (from above) at an interval in I, if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \tag{1.1}$$

for every two points $x_1, x_2 \in I$ and every $t \in [0, 1]$.

If in the inequality (1.1) we substitute $t = \frac{1}{2}$ then we obtain the inequality (1.2)

$$f(\frac{x_1 + x_2}{2}) \leq \frac{f(x_1) + f(x_2)}{2} \tag{1.2}$$

for every two points $x_1, x_2 \in I$.

Conversely, from the validity of inequality (1.2) there is no validity of the inequality (1.1). For this it is sufficient to refer to [5] , where it is indicated that for function f defined by formula

$$f(x) = \begin{cases} x^2, \text{if } x \text{ is rational number} \\ 0, \text{if } x \text{ is irational number} \end{cases}$$

the inequality (1.2) is true, but not (1.1) .

Lemma 1.1. *If the function f is continuous in the interval I and satisfies the inequality (1.1) then it also satisfies the inequality (1.2), i.e. f is a convex function.*

An interesting proof of this fact is found in [6].

Characterization of convex functions ([7] App. III, theor. 2) is usually performed within Lebesgues integration theory, despite the fact that the involved integrands are non-decreasing (therefore Riemann integrable) functions. We transcribe its statement as it appears in the cited book:

Theorem 1.2. *The class of functions which are convex downward on the interval (a, b) coincides with the class of indefinite integrals of functions which are increasing on (a, b) and bounded on every $[p, q] \subset (a, b)$.*

The same result can be achieved in a similar way, by using only Riemann integrals. In order to remember the usual proof and show the simpler one, we prefer to adopt the following point of view.

Theorem 1.2 is an immediate corollary of the next result:

Theorem 1.3. *Let (a,b) be an interval and $x_0 \in (a,b)$. Let X be the space of non-decreasing functions on (a,b) and let Y be the space of convex functions on the same interval tending at x_0. Then the operators "indefinite integration from x_0" and "differentiation" are inverse to each other. So,*

$$\frac{d}{dx}\int_{x_0}^{x} f(t)dt = f(x), f \in X \tag{1.3}$$

and

$$\int_{x_0}^{x} F^{'}(t)dt = F(x), F \in Y \tag{1.4}$$

An elementary general result provides the suitable framework for this point of view.

Theorem 1.4. *Let $\Phi : X \to Y$ and $\Psi : Y \to X$ be two mappings such that*

$$\Psi \cdot \Phi = id_X \tag{1.5}$$

Then, they are inverse to each other, i.e.

$$\Phi \cdot \Psi = id_Y \tag{1.6}$$

if and only if one of the following conditions is satisfied, Ψ is one to one or Φ is onto.

However, Theorem 1.2 may be derived alternatively from the following (Riemann-ian) version of Theorem 1.3.

Theorem 1.5. *Let (a,b) be an interval and $x_0 \in (a,b)$. Let X be the space of right continuous non-decreasing functions on (a,b) and let Y be the space of convex functions on the same interval tending at x_0. Then the operators*

$$\Phi(x) = \int_{x_0}^{x} f(t)dt, f \in X$$

and

$$\Psi(F) = D^{+}F, F \in Y$$

are inverse to each other.

The proof of Theorem 1.5 is obtained by applying Theorem 1.4.

2. WELL-KNOWN FACTS ABOUT CONVEX FUNCTIONS

If $f : I \to \mathcal{R}$ is a convex function, then the function f has the following properties:

(P1) There are partial derivatives $D^{-}f(x)$ and $D^{+}f(x)$, which are finite at each point $x \in I$. The functions $D^{-}f$ and $D^{+}f$ are non-decreasing in the interval I, meanwhile the right derivative is continuous from the right, while the left derivative is continuous from the left (see [2] or [8]).

(P2) The set of points where the function f is not derivable is computable (denumerable) (see [2] or [8]).

(P3) If $[a,b] \subset I$ and $M \leq max\{D^{+}f(x); D^{-}f(x)\}$, then for every two points x and y from $[a,b]$, it is true inequality

$$\mid f(x) - f(y) \mid = M \mid x - y \mid$$

which means that the function f satisfies the Lipschitz condition [8].

To prove that the derivative of a convex function is Riemann integrable, we had to prove Lemma 2.1.

Lemma 2.1. *If the function is convex, then the derivative $f^{'}$ is continuous in I, with an exceptional denumerable set of points in this interval.*

Proof. Due to the statement (P2) the derivative $f^{'}$ exists in I, with an exceptional denumerable set of points

$$E = \{x_1, x_2, ...\}.$$

For $x \in I \setminus E$ the equality $D^{-}f(x) = D^{+}f(x) = f^{'}(x)$ is true.

Since the function $D^{-}f$ is continuous from the right and the function $D^{+}f$ is continuous from the left (according to the statement (P1)), it follows that the derivative $f^{'}$ is a continuous function both from the left and from the right in $I \setminus E$, which means it is continuous in $I \setminus E$. ∎

3. INTEGRAL PRESENTATION OF A CONVEX FUNCTION

Let $f : I \to \mathcal{R}$ be defined at an interval I and $a \in I$ fixed point.

Theorem 3.1. *The necessary and sufficient condition that the function f is convex in the interval I is that for each $x \in I$, this function is presented in the form*

$$f(x) = C + (R) \int_a^x g(t)dt, \tag{3.1}$$

where g is a non-decreasing functions in I and C real constant (in fact, $C = f(a)$).

(The symbol (R) in front of the integer sign indicates Integration according to Riemann, which, during the proof, wont be written, but we will imply).

Proof. First, let proof that condition 3.1 is necessary (indispensable). Suppose that f is convex in the interval I. According to Lemma 2.1. the set of disconnection points of function $f^{'}$ has the mass (according to Lebesgue measure) zero. In the Riemann integral theory (see eg [3]) it is proven that:

- The derivative $f^{'}$ is integrable according to Riemann if and only if the set of disconnection points of $f^{'}$, which is of the type F_{σ}), have the mass zero according to Lebesgue.

- Each function $f : [a, b] \to \mathcal{R}$ that has an integrative derivative according to Riemann in segment $[a, b]$, is an indefinite integral of its derivative:

$$f(x) = f(a) + \int_a^x f^{'}(t)dt, \tag{3.2}$$

where $a \leq x \leq b$.

Formula 3.2 is also true when $b < a$, i.e. is true in a generalized segment $[a, b]$. Since each point $x \in I$ can be included in a generalized segment $[a, b]$, such that $[a, b] \subset I$, the equation 3.2 is true for each $x \in I$.

If in this formula we substitute derivative $f^{'}(t)$ with the right derivative $D^{+}f(t)$, formula 3.2 takes the form

$$f(x) = f(a) + \int_a^x D^{+}f(t)dt, (x \in \hat{E}) \tag{3.3}$$

or

$$f(x) = C + \int_a^x g(t)dt, (x \in \hat{E}) \tag{3.4}$$

where, according to the statement (P1), the function $g(t) = D^+ f(t)$ is non decreasing, meanwhile $C = f(a)$.

Now, let proof that condition 3.1 is sufficient. Suppose that for function f equation 3.2 is true. Since the indefinite integral

$$\int_{x_0}^{x} g(t)dt$$

is a uniformly continuous function, it follows that the function f is continuous. To prove that the function f is convex, it is enough to prove that $h(x) = \int_a^x g(t)dt$ is a convex function. For this, based on Lemma 1.2, it suffices that h proves the inequality:

$$h(\frac{x_1 + x_2}{2}) \leq \frac{h(x_1) + h(x_2)}{2} \tag{3.5}$$

A concise statement of 3.5 is found in [8] (page 40).

∎

Theorem 3.2. *The necessary and sufficient condition that the function f to be convex in the interval I is that for each $x \in I$, this function is presented in the form*

$$f(x) = C + (L)\int_a^x g(t))dt, \tag{3.6}$$

where g is a non-decreasing functions in $[a, b]$ and C real constant (in fact, $C = f(a)$).

(The symbol (L) in front of the integer sign indicates integration according to Lebesgue, which, during the proof, wont be written, but we will imply).

Proof. First, let proof that condition 3.6 is necessary (indispensable). Suppose that f is convex in the interval I. According to the statement (P3) it follows that the function f is absolutely continuous in every segment $[a, b] \subset I$. Let be $x \in I$, then there is a segment $[a, b]$, such that $x \in [a, b] \subset I$. Based on the Lebesgue theorem ([4], page 345), the function f can be expressed as an indefinite integral (according to Lebesgue) of its derivative in the form

$$f(x) = f(a) + \int_a^x f'(t)dt \tag{3.7}$$

If we substitute derivative $f'(t)$ with the right derivative $D^+ f(t)$, formula 3.7 takes the form

$$f(x) = f(a) + \int_a^x D^+ f(t)dt \tag{3.8}$$

or

$$f(x) = C + \int_a^x g(t)dt \tag{3.9}$$

where, according to the statement (P2), the function $g(t) = D^+ f(t)$ is non decreasing function, while $C = f(a)$.

Now, let proof that condition 3.6 is sufficient. Suppose that for function f equation 3.6 is true. Since the indefinite integral

$$\int_{x_0}^{x} g(t)dt$$

is a absolutely continuous function ([4], pg. 344) it follows that the function f is continuous. To prove that the function f is convex, it is enough to prove that $h(x) = \int_a^x g(t)dt$ is a convex function. The other part of proof proceeds in the same way as the proof of the sufficient condition of Theorem 3.1. ∎

Theorem 3.3. *Fix a point $a \in I$. By the integration operators according to Riemann or according to Lebesgue, can establish a biunivoke correspondence (one by one) between the set $\Phi = \{\varphi\}$ of non-decreasing functions in an interval $I \subset \mathcal{R}$ and the set $F = \{f\}$ of the convex functions (from above) in I, for which $f(a) = 0$, according to the formula*

$$f(x) = \int_a^x \varphi(t)dt, (x \in I) \tag{3.10}$$

Proof. - Let $f \in F$. According to the Theorems 3.1 and 3.2 we can write the equation

$$f(x) = f(a) + \int_a^x D^+ f(t)dt = \int_a^x \varphi(t)dt,$$

where $x \in I$ and $\varphi = D^+ f$ is a monotone non-decreasing functions in I. Thus, the function $f \in F$ responds to the function $\varphi = D^+ f \in \Phi$.

- Let $\varphi \in \Phi$. Since f is a non-decreasing function in the interval I, then it is integrable according to Lebesgue, even according to Riemann. We will build the function $f(x) = \int_a^x \varphi(t)dt$, where $\int_a^x$ is the integration operator according to Riemann or according to Lebesgue. From the reasoning we did in Theorems 3.1 and 3.2, it follows that the function f is convex in the interval I. Thus we established a correspondence between sets F and Φ.

It remains to be proven that this correspondence is biunivoke (one by one).

- Let f_1 and f_2 be two different functions from F, it means $f_1 \neq f_2$. Let's mark it

$$\varphi_1 = \frac{d^+}{dx}(f_1) \text{ and } \varphi_2 = \frac{d^+}{dx}(f_2).$$

Let prove that $\varphi_1 \neq \varphi_2$. In opposite, we would have $\varphi_1 = \varphi_2$, which means $D^+ f_1 = D^+ f_2$. In [9] it is proved that if the right derivative of a continuous function is zero at an interval, then this function is constant at that interval. Thus, in our case, since $f_1 - f_2$ is continuous and

$$\frac{d^+}{dx}(f_2 - f_1) = 0, \text{in } I$$

it follows that $f_1 - f_2 = c(constant)$ in I. Meanwhile, since we have $f_1(a) - f_2(a) = 0$, it follows that for each $x \in I$, we have $f_1(x) - f_2(x) = 0$, which means $f_1 = f_2$, which contradicts the assumption.

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